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CURVES ON THE UNIT 3-SPHERE $S^3(1)$ IN EUCLIDEAN 4-SPACE $\mathbb{R}^4$

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Abstract. We show many examples of curves on the unit 2-sphere $S^2(1)$ in $\mathbb{R}^3$ and the unit 3-sphere $S^3(1)$ in $\mathbb{R}^4$. We study whether its curves are Bertrand curves or spherical Bertrand curves and provide some examples illustrating the resultant curves.

1. Introduction

Let C be a regular C^∞-curve in an Euclidean 3-space $\mathbb{R}^3$. We call curve C a C^∞-special Frenet curve if there exists the Frenet apparatus $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$, where $\mathbf{t}$ is the unit tangent vector field, $\mathbf{n}$ is the unit principal normal vector field, $\mathbf{b}$ is the unit binormal vector field, $\kappa(> 0)$ is the curvature function and $\tau(\neq 0)$ is the torsion function. A C^∞-special Frenet curve C is called a *Bertrand curve* if there exists another C^∞-special Frenet curve $\bar{C}$ and a C^∞-mapping $\varphi : C \to \bar{C}$ such that the principal normal lines of C and $\bar{C}$ at corresponding points coincide. Here, $\bar{C}$ is called a *Bertrand mate* of C. It is well-known that a C^∞-special Frenet curve C in $\mathbb{R}^3$ is a Bertrand curve if and only if there exists a linear relation $a\kappa(s) + b\tau(s) = 1$ for all $s \in I$, where a and b are non-zero constant real numbers.

Bertrand curves have attracted many authors [2, 3, 6, 8, 10, 13, 14]. We refer to [1, 2, 4, 5, 7] for the text book. Izumiya and Takeuchi [8] show that Bertrand curves can be constructed from spherical curves. Aminov [1] proved that a Bertrand curve does not exist in $\mathbb{R}^n$ if $n \geq 4$. Matsuda and Yorozu [10] gave a different proof for this. We wonder if Bertrand curves exist in S^3 or S^4. This is the motivation for the present paper. We note that Lucas and J. A. Ortega-Yagües [9] recently obtained a necessary and sufficient condition for a curve on $S^3(1)$ to be a Bertrand spherical curve. Our work is independent of theirs and provides a different proof of this fact from an entirely different viewpoint that we believe, taken in conjunction with the work of [9] deepens the understanding of Bertrand spherical curves. In addition, we present many

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concrete examples of curves on the unit 2-sphere $S^2(1)$ in $\mathbb{R}^3$ and the unit 3-sphere $S^3(1)$ in $\mathbb{R}^4$. We give Bertrand curves constructed from curves on $S^2(1)$ in $\mathbb{R}^3$ and we study whether curves on $S^3(1)$ are spherical Bertrand curves. We give Bertrand curves constructed from curves on $S^2(1)$ in $\mathbb{R}^3$ with respect to the method by Izumiya and Takeuchi [8]. Finally, we provide some examples illustrating the resultant curves; these yield concrete examples which do not appear in the literature.

2. Preliminaries

Let $\mathbb{R}^n$ be an Euclidean n-space with the inner product $\langle , \rangle$. For $\mathbf{x} \in \mathbb{R}^n$, that is $\mathbf{x} = (x^1, x^2, \dots, x^n)^t$ for $x^i \in \mathbb{R}$ with $1 \leq i \leq n$. Here "$(\cdots)^t$ " denotes to the transpose of $(\cdots)$. And we denote $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x^i y^i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left\{ \sum_{i=1}^n (x^i)^2 \right\}^{1/2}$ with the origin $\mathbf{o} = (0, 0, \dots, 0)$ in $\mathbb{R}^n$.

Let $S^{n-1}(1) := \{\mathbf{x} \mid \|\mathbf{x}\| = 1,\ \mathbf{x} \in \mathbb{R}^n\}$ be the $(n-1)$-sphere of radius 1, centered at the origin $\mathbf{o}$ in $\mathbb{R}^n$ and $T_{\mathbf{x}}(\mathbb{R}^n)$ be the tangent space of $\mathbb{R}^n$ at $\mathbf{x}$. And, for $\mathbf{x} \in S^{n-1}(1)$, let $T_{\mathbf{x}}(S^{n-1}(1))$ be the tangent space of $S^{n-1}(1)$ at $\mathbf{x}$. We can set $T_{\mathbf{x}}(\mathbb{R}^n) = T_{\mathbf{x}}(S^{n-1}(1)) \oplus \{\alpha \mathbf{x} \mid \forall \alpha \in \mathbb{R}\}$. Here, $T_{\mathbf{x}}(\mathbb{R}^n) \cong \mathbb{R}^n$. We define $P_{\mathbf{x}} : T_{\mathbf{x}}(\mathbb{R}^n) \to T_{\mathbf{x}}(S^{n-1}(1))$ as the orthogonal projection to the tangent space $T_{\mathbf{x}}(S^{n-1}(1))$ for $\mathbf{x} \in S^{n-1}(1)$.

Let $f : S^{n-1}(1) \to \mathbb{R}^n$ be a map given by $f(\mathbf{x}) = \mathbf{x}$ for in $\mathbf{x} \in S^{n-1}(1)$ in $\mathbb{R}^n$, and $g_{\mathbf{x}}(X, Y) := \langle f_*(X), f_*(Y) \rangle$ for $\mathbf{x} \in S^{n-1}(1)$ and $X, Y \in T_{\mathbf{x}}(S^{n-1}(1))$.

Let D be a covariant differentiation associated with the linear connection on $\mathbb{R}^n$, and ∇ be a covariant differentiation defined by

$$\nabla : \mathfrak{X}(S^{n-1}(1)) \times \mathfrak{X}(S^{n-1}(1)) \to \mathfrak{X}(S^{n-1}(1)) \quad ((X, Y) \mapsto \nabla_X Y)$$

on $S^{n-1}(1)$. Here, ∇ is the Levi-Civita connection associated the induced metric g on $S^{n-1}(1)$. Then, the formula of Gauss is

$$D_X Y = \nabla_X Y + h(X, Y)\mathbf{x},$$

where $\mathbf{x} \in S^{n-1}(1)$ is identified with the unit normal vector field at the point $\mathbf{x}$ of $S^{n-1}(1)$.

A regular C^∞-curve C in $\mathbb{R}^n$ is given by

$$\mathbf{x} : I \ni t \mapsto \mathbf{x}(t) \in \mathbb{R}^n,$$

where $t_0 \in I \subset \mathbb{R}$. The arc-length s from the point $\mathbf{x}(t_0)$ to a point $\mathbf{x}(t)$ is given by

$$s = \psi(t) = \int_{t_0}^{t} \| \dot{\mathbf{x}}(u) \| \ \mathrm{du},$$

then we get the inverse function φ of ψ so that we have $t = \varphi(s)$ with $s \in J \subset \mathbb{R}$.

The curve C is represented by arc-length parameter s as

$$\mathbf{c} : J \ni s \mapsto \mathbf{c}(s) \in \mathbb{R}^n,$$

where $\mathbf{c} = \mathbf{x} \circ \varphi$, that is, $\mathbf{c}(s) = \mathbf{x}(\varphi(s))$.

Here we put

$$\mathbf{c}'(s) = \left.\frac{\mathrm{d}\mathbf{c}}{\mathrm{ds}}\right|_s = \frac{\mathrm{d}\mathbf{c}(s)}{\mathrm{ds}}$$

and

$$\dot{\mathbf{x}}(\varphi(s)) = \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}\right|_{t=\varphi(s)}, \quad \ddot{\mathbf{x}}(\varphi(s)) = \left.\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{dt}^2}\right|_{t=\varphi(s)},$$

then we have

$$\varphi'(s) = \frac{\mathrm{d}\varphi(s)}{\mathrm{ds}} = \langle \dot{\mathbf{x}}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s))\rangle^{-1/2},$$
$$\varphi''(s) = -\langle \dot{\mathbf{x}}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s))\rangle^{-2}\langle \ddot{\mathbf{x}}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s))\rangle$$

and

$$\mathbf{c}'(s) = \varphi'(s)\cdot \dot{\mathbf{x}}(\varphi(s)),$$
$$\mathbf{c}''(s) = \varphi''(s)\cdot \dot{\mathbf{x}}(\varphi(s)) + (\varphi'(s))^2\cdot \ddot{\mathbf{x}}(\varphi(s)).$$

Hereafter, we shall work in the C^∞-category and refer to a special Frenet curve simply as a curve.

3. Examples of spherical curves on $S^2(1)$ in $\mathbb{R}^3$

Let $S^2(1)$ be the unit sphere in $\mathbb{R}^3$ with the origin $\mathbf{o}$ and let a, b and c be real numbers. We provide examples of spherical curves on $S^2(1)$ in $\mathbb{R}^3$ in this section.

Example 1. We consider a C^∞-curve $C_1(a,b,c)$ in $\mathbb{R}^3$ defined by

$$\mathbf{x}(t) = \begin{bmatrix} a\cos(t) - b\sin(3t) \\ a\sin(t) + b\cos(3t) \\ c(\cos(t)+\sin(t)) \end{bmatrix}, \quad t\in\mathbb{R}.$$

Then we obtain

$$\|\mathbf{x}(t)\|^2 = a^2+b^2+c^2+(c^2-2ab)\sin(2t).$$

The curve $C_1(a,b,c)$ is a curve on $S^2(1)$ if and only if $c^2=2ab$ and $(a+b)^2=1$. And we get

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -a\sin(t) - 3b\cos(3t) \\ a\cos(t) - 3b\sin(3t) \\ c(-\sin(t)+\cos(t)) \end{bmatrix}, \quad t\in\mathbb{R}.$$

Then we obtain

$$\|\dot{\mathbf{x}}(t)\|^2 = (a^2+9b^2+c^2)-(c^2+6ab)\sin(2t) = (a^2+9b^2+2ab)-(8ab)\sin(2t).$$

The curve $C_1(a,b,c)$ is a regular curve (i.e., $\|\dot{\mathbf{x}}(t)\|^2 \neq 0$) if and only if $(a^2+9b^2+2ab)-(8ab) = (a-3b)^2 > 0$, that is, $a\neq 3b$. Thus we have a family

$$\Gamma_1 = \left\{C_1(a,b,c) \mid a,b,c\in\mathbb{R}\ ;\ c^2=2ab,\ (a+b)^2=1,\ a\neq 3b\right\}.$$

This is a family of spherical, regular and C^∞-curves. For a curve C_1 in Γ_1, we calculate that the curvature $\kappa(t)$ and torsion $\tau(t)$ of C_1 in $\mathbb{R}^3$ are

$$\kappa(t) = \frac{\sqrt{A_1(t)}}{\|\dot{\mathbf{x}}(t)\|^3},$$
$$\tau(t) = \frac{24bc\{9b(\cos(t)-\sin(t)) - a(\cos(3t)+\sin(3t))\}}{A_1(t)},$$

where $A_1(t) = a^4+4a^3b+30a^2b^2+180ab^3+729b^4-24ab(a^2+2ab+33b^2)\sin(2t)+192a^2b^2\sin^2(2t)$.

Example 2. We consider a C^∞-curve $C_2(a,b,c)$ in $\mathbb{R}^3$ defined by

$$\mathbf{x}(t) = \begin{bmatrix} a\cos(t) - b\cos(3t) \\ a\sin(t) - b\sin(3t) \\ c\cos(t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

Then we obtain

$$\|\mathbf{x}(t)\|^2 = (a+b)^2 + (c^2-4ab)\cos^2(t).$$

The curve $C_2(a,b,c)$ is a curve on $S^2(1)$ if and only if $c^2 = 4ab$ and $(a+b)^2 = 1$. And we get

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -a\sin(t) + 3b\sin(3t) \\ a\cos(t) - 3b\cos(3t) \\ -c\sin(t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

Then we obtain

$$\|\dot{\mathbf{x}}(t)\|^2 = (a-3b)^2 + (c^2+12ab)\sin^2(t) = (a-3b)^2 + 16ab\sin^2(t).$$

The curve $C_2(a,b,c)$ is a regular curve if and only if $a \neq 3b$. Thus we have a family

$$\Gamma_2 = \{C_2(a,b,c) \mid a,b,c \in R\ ;\ c^2 = 4ab,\ (a+b)^2 = 1,\ a \neq 3b\}.$$

This is a family of spherical, regular and C^∞-curves. For a curve C_2 in Γ_2, we calculate that the curvature $\kappa(t)$ and torsion $\tau(t)$ of C_2 in $\mathbb{R}^3$ are

$$\kappa(t) = \frac{\sqrt{B_1(t)}}{\|\dot{\mathbf{x}}(t)\|^3},$$
$$\tau(t) = \frac{24bc(-9b\sin(t) + a\sin(3t))}{B_1(t)},$$

where $B_1(t) = (a-3b)^2((a+9b)^2 - 32ab) + 48ab(a-3b)(a-11b)\sin^2(t) + 768a^2b^2\sin^4(t)$. We know the elements of Γ_1 do not map isometrically to elements of Γ_2.

4. Bertrand curves in $\mathbb{R}^3$ constructed from spherical curves and their illustrations

We describe Bertrand curves in $\mathbb{R}^3$ constructed from spherical curves and their illustrations in this section.

Example 3. We consider a small circle C_3 on $S^2(1)$ defined by

$$\mathbf{c}(s) = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos(\sqrt{2}s) \\ \frac{1}{\sqrt{2}}\sin(\sqrt{2}s) \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad s \in \mathbb{R},$$

with respect to the arc-length parameter s.
By the direct computation, we get

$$\|\mathbf{c}(s)\| = 1, \qquad \|\mathbf{c}'(s)\| = 1,$$

$$\mathbf{c}'(s) = \begin{bmatrix} -\sin(\sqrt{2}s) \\ \cos(\sqrt{2}s) \\ 0 \end{bmatrix}, \quad \mathbf{c}''(s) = \begin{bmatrix} -\sqrt{2}\cos(\sqrt{2}s) \\ -\sqrt{2}\sin(\sqrt{2}s) \\ 0 \end{bmatrix}$$

for all $s \in \mathbb{R}$. Then, we have $\langle \mathbf{c}''(s), \mathbf{c}(s)\rangle = -1$ and $\|\mathbf{c}''(s)\|^2 = 2$. Thus, we obtain $\|\mathbf{c}''(s)\|^2 - 1 = 1$. Next we get

$$\mathbf{e}_2(s) = \mathbf{c}(s) \times \mathbf{e}_1(s) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\cos(\sqrt{2}s) \\ -\frac{1}{\sqrt{2}}\sin(\sqrt{2}s) \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

for all $s \in \mathbb{R}$. And also, we get

$$\mathbf{c}''(s) - \langle \mathbf{c}''(s), \mathbf{c}(s)\rangle \mathbf{c}(s) = \mathbf{c}''(s) + \mathbf{c}(s) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\cos(\sqrt{2}s) \\ -\frac{1}{\sqrt{2}}\sin(\sqrt{2}s) \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

for all $s \in \mathbb{R}$. Thus we have, for all $s \in \mathbb{R}$,

$$\begin{cases} k(s) = 1 \\ \nabla_s \mathbf{e}_1(s) = k(s)\mathbf{e}_2(s). \end{cases}$$

Next we compute

$$\mathbf{e}_2'(s) = \begin{bmatrix} \sin(\sqrt{2}s) \\ -\cos(\sqrt{2}s) \\ 0 \end{bmatrix}$$

for all $s \in \mathbb{R}$ and $\langle \mathbf{e}_2'(s), \mathbf{c}(s)\rangle = 0$. On the other hand,

$$\nabla_s \mathbf{e}_2(s) = \mathbf{e}_2'(s) = (-1)\mathbf{e}_1(s) = -k(s)\mathbf{e}_1(s)$$

for all $s \in \mathbb{R}$. Therefore we have

$$\begin{cases} \nabla_s \mathbf{e}_1(s) = k(s)\mathbf{e}_2(s) \\ \nabla_s \mathbf{e}_2(s) = -k(s)\mathbf{e}_1(s) \\ k(s) = 1 \end{cases}$$

for all $s \in \mathbb{R}$.

Now, we can construct a Bertrand curve from the curve C_3 by using Izumiya and Takeuchi's method [8]. The curve C_3 is depicted as in Figure 1 and we can draw a Bertrand curve from the curve C_3 as in Figure 2.

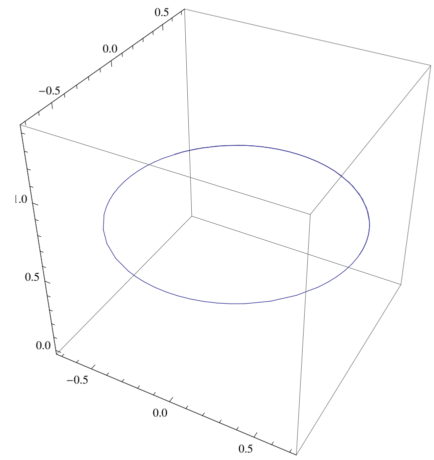

FIGURE 1

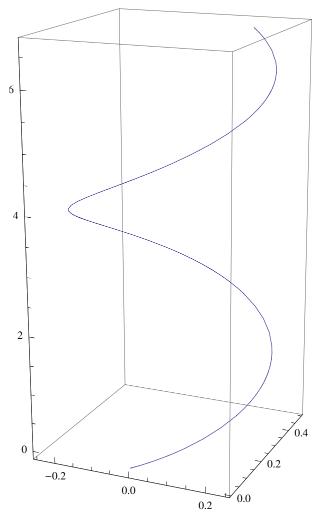

FIGURE 2

Example 1′. We consider a C^∞-curve $C_1(a,b,c)$ in $\mathbb{R}^3$ defined by

$$\mathbf{x}(t) = \begin{bmatrix} a\cos(t) - b\sin(3t) \\ a\sin(t) + b\cos(3t) \\ c(\cos(t) + \sin(t)) \end{bmatrix}, \quad t \in \mathbb{R}$$

in a family

$$\Gamma_1 = \left\{ C_1(a,b,c) \mid a,b,c \in \mathbb{R}\ ;\ c^2 = 2ab,\ (a+b)^2 = 1,\ a \neq 3b \right\}.$$

The curvature $k(s)$ of C_1 with respect to the arc-length parameter s is given by

$$k(s) = \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)],$$

and it is called the geodesic curvature of C_1. For $\mathbf{c}(s) = \mathbf{x}(\varphi(s))$, where $t = \varphi(s)$, we get

$$\mathbf{c}'(s) = \varphi'(s) \cdot \dot{\mathbf{x}}(\varphi(s)),$$
$$\mathbf{c}''(s) = \varphi''(s) \cdot \dot{\mathbf{x}}(\varphi(s)) + (\varphi'(s))^2 \cdot \ddot{\mathbf{x}}(\varphi(s))$$

and

$$\varphi'(s) = \langle \dot{\mathbf{x}}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s)) \rangle^{-1/2},$$
$$\varphi''(s) = -\langle \dot{\mathbf{x}}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s)) \rangle^{-2} \langle \ddot{\mathbf{x}}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s)) \rangle.$$

Thus we have

$$\det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] = (\varphi'(s))^3 \det[\mathbf{x}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s)), \ddot{\mathbf{x}}(\varphi(s))].$$

Thus, with respect to the parameter t, the curvature $k(t)$ at $\mathbf{x}(t)$ (i.e., $\mathbf{c}(s) = \mathbf{x}(\varphi(s))$) is given by

$$\frac{8bc\{3b(\cos(t) + \sin(t) + a(\cos(3t) - \sin(3t))\}}{\{a^2 + 9b^2 + 2ab - 8ab\sin(2t)\}^{3/2}},$$

where $c^2 = 2ab$ and $(a + b)^2 = 1$.

We can construct a Bertrand curve from the curve C_1 in Γ_1 by using Izumiya and Takeuchi's method [8]. In the case of $a = 1/4$, $b = 3/4$ and $c > 0$, the curve C_1 is depicted as in Figure 3 and we can draw a Bertrand curve from the curve C_1 as in Figure 4.

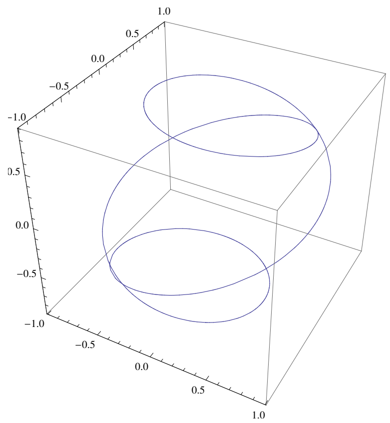

FIGURE 3

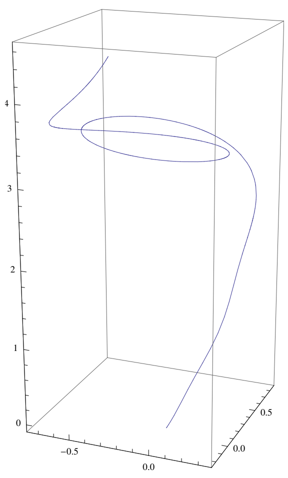

FIGURE 4

Example 2′. We consider a C^∞-curve $C_2(a,b,c)$ in $\mathbb{R}^3$ defined by

$$\mathbf{x}(t) = \begin{bmatrix} a\cos(t) - b\cos(3t) \\ a\sin(t) - b\sin(3t) \\ c\cos(t) \end{bmatrix}, \quad t \in \mathbb{R}$$

in a family

$$\Gamma_2 = \{C_2(a,b,c) \mid a,b,c \in R\,;\ c^2 = 4ab, (a+b)^2 = 1, a \neq 3b\}.$$

Since $k(s) = \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] = (\varphi'(s))^3 \det[\mathbf{x}(\varphi(s)), \dot{\mathbf{x}}(\varphi(s)), \ddot{\mathbf{x}}(\varphi(s))]$, then the curvature $k(t)$ at $\mathbf{x}(t)$ is given by

$$\frac{8bc(3b\cos(t) - a\cos(3t))}{((a-3b)^2 + 16ab\sin^2(t))^{\frac{3}{2}}},$$

where $c^2 = 4ab$ and $(a+b)^2 = 1$ with respect to the parameter t.

We can construct a Bertrand curve from the curve C_2 in Γ_2 by using Izumiya and Takeuchi's method [8]. In the case of $a = 1/4$, $b = 3/4$ and $c > 0$, the curve C_2 is depicted as in Figure 5 and we can draw a Bertrand curve from the curve C_2 as in Figure 6.

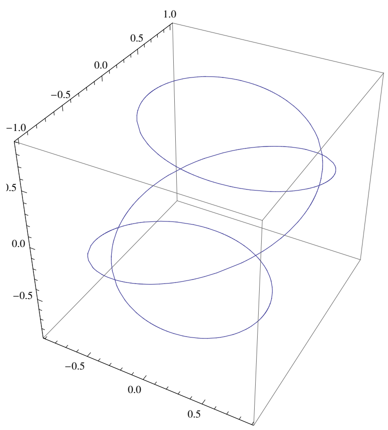

Figure 5

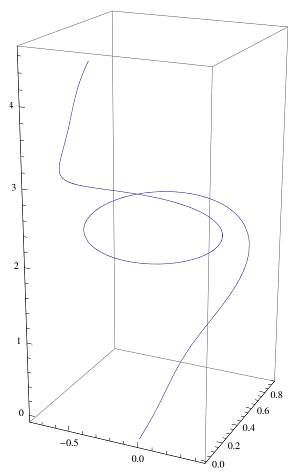

Figure 6

Next, a circle C_4 on $S^2(1)$ is given by

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{4} + \frac{1}{2}\cos(t) \\ \frac{1}{2}\sin(t) \\ \frac{1}{4}(11 - 4\cos(t))^{\frac{1}{2}} \end{bmatrix}, \quad t \in \mathbb{R}.$$

Here, C_4 is a curve which is the intersection of $S^2(1)$ and a circular cylinder

$$\left\{(x, y, z) \mid \left(x - \frac{1}{4}\right)^2 + y^2 = \frac{1}{4},\ z \geq 0\right\}.$$

This curve is depicted as in Figure 7.

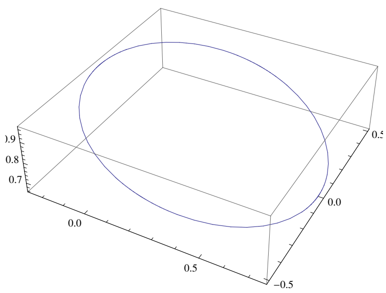

FIGURE 7

And, the curve C_5 on $S^2(1)$ is given by

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t)\sqrt{1 - \frac{1}{4}(\sin(2t))^2} \\ \sin(t)\sqrt{1 - \frac{1}{4}(\sin(2t))^2} \\ \frac{1}{2}\sin(2t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

This curve is depicted as in Figure 8 (see the next page).

5. Spherical Bertrand curves on $S^2(1)$ in $\mathbb{R}^3$

In this section, we study a spherical Bertrand curve on $S^2(1)$ and its spherical Bertrand curve. We recall that a curve C in $\mathbb{R}^n$ $(n = 2, 3)$ is called a Bertrand curve if there exists another curve $\hat{C}$, distinct from C, and a bijection f between C and $\hat{C}$ such that C and $\hat{C}$ have the same principal normal line at each pair of corresponding points under f. The curve $\hat{C}$ is called a Bertrand mate curve of C [7]. The following results are well-known.

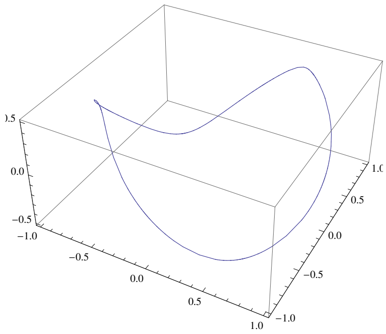

FIGURE 8

(A) Every curve in $\mathbb{R}^2$ is a Bertrand curve.
(B) A circular helix in $\mathbb{R}^3$ is a Bertrand curve.

We have the following definition:

Definition. A curve C on $S^{n-1}(1)$ in $\mathbb{R}^n$ is called a *spherical Bertrand curve* if there exists a curve $\tilde{C}$, distinct from C, and a bijection f between C and $\tilde{C}$ such that C and $\tilde{C}$ have the same principal normal great circle at each pair of corresponding points under f. The curve $\tilde{C}$ is called a *spherical Bertrand mate curve* of C.

A curve $\mathbf{c} : I \ni s \mapsto \mathbf{c}(s) \in S^2(1)$ is a parametrized curve C with arc-length parameter s. Thus we have $\| \mathbf{c}(s) \| = 1$ and $\| \mathbf{c}'(s) \| = 1$. We put $\mathbf{e}_1(s) = \mathbf{c}'(s) \in T_{\mathbf{c}(s)}(S^2(1))$ and $\mathbf{e}_2(s) = \mathbf{c}(s) \times \mathbf{c}'(s) \in T_{\mathbf{c}(s)}(S^2(1))$. We have

(a) $\langle \mathbf{c}(s), \mathbf{c}(s) \rangle = 1$, $\langle \mathbf{c}'(s), \mathbf{c}'(s) \rangle = 1$.
(b) $\langle \mathbf{c}(s), \mathbf{c}'(s) \rangle = 0$, $\langle \mathbf{c}(s), \mathbf{c}''(s) \rangle = -1$.
(c) $\langle \mathbf{c}(s) \times \mathbf{c}''(s), \mathbf{c}'(s) \rangle = \det [\mathbf{c}(s), \mathbf{c}''(s), \mathbf{c}'(s)] = -\det [\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)]$.
(d) $\| \mathbf{c}''(s) + \mathbf{c}(s) \|^2 = \| \mathbf{c}''(s) \|^2 - 1$, $\| \mathbf{c}(s) \times \mathbf{c}''(s) \|^2 = \| \mathbf{c}''(s) \|^2 - 1$.

Now, by (b), we have the $T_{\mathbf{c}(s)}(S^2(1))$-part of $\mathbf{e}_1'(s) = \mathbf{c}''(s)$ is given by $\mathbf{c}''(s)$ + $\mathbf{c}(s)$.

We denote $T_{\mathbf{c}(s)}(S^2(1))$- part of $\mathbf{e}_1'(s)$ by $\nabla_s \mathbf{e}_1(s)$. We have

$$\begin{aligned}
\langle \nabla_s \mathbf{e}_1(s), \mathbf{e}_2(s) \rangle &= \langle \mathbf{c}''(s) + \mathbf{c}(s), \mathbf{c}(s) \times \mathbf{c}'(s) \rangle \\
&= \langle \mathbf{c}''(s), \mathbf{c}(s) \times \mathbf{c}'(s) \rangle + \langle \mathbf{c}(s), \mathbf{c}(s) \times \mathbf{c}'(s) \rangle \\
&= \langle \mathbf{c}(s) \times \mathbf{c}'(s), \mathbf{c}''(s) \rangle + \langle \mathbf{c}(s) \times \mathbf{c}'(s), \mathbf{c}(s) \rangle \\
&= \det [\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] + \det [\mathbf{c}(s), \mathbf{c}'(s) \mathbf{c}(s)] \\
&= \det [\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] .
\end{aligned}$$

Thus we have

$$\nabla_s \mathbf{e}_1(s) = \det [\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] \cdot \mathbf{e}_2(s).$$

We remark that $\det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] > 0$ and $k(s) = \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] = \sqrt{\| \mathbf{c}''(s) \|^2 - 1}$. Then, we can take a constant number $\theta \in (0, 2\pi)$ such that

$$(\dagger) \qquad \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] \neq \cot(\theta)$$

for any s, we define a curve as:

$$\tilde{\mathbf{c}}(s) = \cos(\theta) \cdot \mathbf{c}(s) + \sin(\theta) \cdot \mathbf{e}_2(s), \; s \in J.$$

This curve is on $S^2(1)$. We have

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{c}}(s)}{\mathrm{ds}} &= \cos(\theta) \cdot \mathbf{c}'(s) + \sin(\theta) \cdot \mathbf{e}_2'(s) \\ &= \cos(\theta) \cdot \mathbf{c}'(s) + \sin(\theta) \cdot (\mathbf{c}(s) \times \mathbf{c}''(s)). \end{aligned}$$

Thus we have $\langle \frac{\mathrm{d}\tilde{\mathbf{c}}(s)}{\mathrm{ds}}, \mathbf{c}(s) \rangle = 0$ so that $\frac{\mathrm{d}\tilde{\mathbf{c}}(s)}{\mathrm{ds}} \in T_{\mathbf{c}(s)}(S^2(1))$. From (†), we have

$$\begin{aligned} \left\| \frac{\mathrm{d}\tilde{\mathbf{c}}(s)}{\mathrm{ds}} \right\|^2 &= (\cos(\theta))^2 \| \mathbf{c}'(s) \|^2 \\ &\quad + 2\sin(\theta)\cos(\theta) \langle \mathbf{c}'(s), \mathbf{c}(s) \times \mathbf{c}''(s) \rangle \\ &\quad + (\sin(\theta))^2 \| \mathbf{c}(s) \times \mathbf{c}''(s) \|^2 \\ &= (\cos(\theta))^2 - 2\sin(\theta)\cos(\theta) \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] \\ &\quad + (\sin(\theta))^2 (\| \mathbf{c}''(s) \|^2 - 1) \\ &= \{\cos(\theta) - \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] \sin(\theta)\}^2 > 0. \end{aligned}$$

Let $\tilde{s}$ be the arc-length parameter of $\tilde{C}$. We set $s = \Phi(\tilde{s})$, then we have

$$\left(\frac{\mathrm{d}\Phi(\tilde{s})}{\mathrm{d}\tilde{s}} \right)^2 = \{\cos(\theta) - \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s)] \sin(\theta)\}^{-2}.$$

The curve $\tilde{C}$ is represented as:

$$\tilde{\mathbf{c}}(\tilde{s}) = \cos(\theta) \cdot \mathbf{c}(\Phi(\tilde{s})) + \sin(\theta) \cdot \mathbf{e}_2(\Phi(\tilde{s})).$$

We put $\tilde{\mathbf{c}}^*(\tilde{s}) = \frac{\mathrm{d}\tilde{\mathbf{c}}}{\mathrm{d}\tilde{s}}(\tilde{s})$, that is, $*$ denotes the derivative with respect to $\tilde{s}$. We have

$$\begin{aligned} \tilde{\mathbf{c}}^*(\tilde{s}) &= \Phi^*(\tilde{s}) \{\cos(\theta) \cdot \mathbf{c}'(\Phi(\tilde{s})) + \sin(\theta) \cdot \mathbf{e}_2'(\Phi(\tilde{s}))\} \\ &= \Phi^*(\tilde{s}) \{\cos(\theta) \cdot \mathbf{c}'(\Phi(\tilde{s})) + \sin(\theta) \cdot (\mathbf{c}(\Phi(\tilde{s})) \times \mathbf{c}''(\Phi(\tilde{s})))\}. \end{aligned}$$

We put $\tilde{\mathbf{e}}_1(\tilde{s}) = \tilde{\mathbf{c}}^*(\tilde{s})$ and we have $\langle \tilde{\mathbf{e}}_1(\tilde{s}), \tilde{\mathbf{c}}(\tilde{s}) \rangle = 0$. We define $\tilde{\mathbf{e}}_2(\tilde{s}) = \tilde{\mathbf{c}}(\tilde{s}) \times \tilde{\mathbf{e}}_1(\tilde{s})$. For ease of writing, we omit $(\tilde{s})$ and $(\Phi(\tilde{s}))$ in the following. Then we have

$$\begin{aligned} \tilde{\mathbf{e}}_2 = &\; \{\cos(\theta) \cdot \mathbf{c} + \sin(\theta) \cdot \mathbf{e}_2\} \times \Phi^* \{\cos(\theta) \cdot \mathbf{c}' + \sin(\theta) \cdot \mathbf{e}_2'\} \\ = &\; \Phi^* \left\{ (\cos(\theta))^2 \cdot (\mathbf{c} \times \mathbf{c}') + \cos(\theta)\sin(\theta) \cdot (\mathbf{c} \times \mathbf{e}_2') \right. \\ &\quad \left. + \sin(\theta)\cos(\theta) \cdot (\mathbf{e}_2 \times \mathbf{c}') + (\sin(\theta))^2 \cdot (\mathbf{e}_2 \times \mathbf{e}_2') \right\} \\ = &\; \Phi^* \left\{ (\cos(\theta))^2 \cdot \mathbf{e}_2 + \cos(\theta)\sin(\theta) \cdot (\mathbf{c} \times (\mathbf{c} \times \mathbf{c}'')) \right. \end{aligned}$$

$$+ \sin(\theta)\cos(\theta) \cdot ((\mathbf{c} \times \mathbf{c}') \times \mathbf{c}') + (\sin(\theta))^2 \cdot ((\mathbf{c} \times \mathbf{c}') \times (\mathbf{c} \times \mathbf{c}''))\big\} .$$

By the below equalities:

$$\begin{aligned}
\mathbf{c} \times (\mathbf{c} \times \mathbf{c}'') &= -\{\langle \mathbf{c}, \mathbf{c}\rangle \cdot \mathbf{c}'' - \langle \mathbf{c}'', \mathbf{c}\rangle \cdot \mathbf{c}\} \\
&= -(\mathbf{c}'' + \mathbf{c}) = -\det[\mathbf{c}, \mathbf{c}', \mathbf{c}''] \cdot \mathbf{e}_2, \\
(\mathbf{c} \times \mathbf{c}') \times \mathbf{c}' &= \langle \mathbf{c}, \mathbf{c}'\rangle \cdot \mathbf{c}' - \langle \mathbf{c}', \mathbf{c}'\rangle \cdot \mathbf{c} = -\mathbf{c}, \\
(\mathbf{c} \times \mathbf{c}') \times (\mathbf{c} \times \mathbf{c}'') &= \langle \mathbf{c}, \mathbf{c} \times \mathbf{c}''\rangle \cdot \mathbf{c}' - \langle \mathbf{c}', \mathbf{c} \times \mathbf{c}''\rangle \cdot \mathbf{c} \\
&= -\langle \mathbf{c} \times \mathbf{c}'', \mathbf{c}'\rangle \cdot \mathbf{c} = \det[\mathbf{c}, \mathbf{c}', \mathbf{c}''] \cdot \mathbf{c},
\end{aligned}$$

thus we have

$$\begin{aligned}
\tilde{\mathbf{e}}_2 = {} & \Phi^* \cos(\theta)(\cos(\theta) - \det[\mathbf{c}, \mathbf{c}', \mathbf{c}''] \sin(\theta)) \cdot \mathbf{e}_2 \\
& - \Phi^* \sin(\theta)(\cos(\theta) - \det[\mathbf{c}, \mathbf{c}', \mathbf{c}''] \sin(\theta)) \cdot \mathbf{c}.
\end{aligned}$$

The principal normal great circle of $\tilde{C}$ at point $\tilde{\mathbf{c}}(\Phi(\tilde{s}))$ is given by

$$\{\cos(\alpha) \cdot \tilde{\mathbf{c}}(\tilde{s}) + \sin(\alpha) \cdot \tilde{\mathbf{e}}_2(\tilde{s}) \mid \alpha \in \mathbb{R}\} .$$

Since it holds, we omit $(\tilde{s})$ and $(\Phi(\tilde{s}))$ again,

$$\begin{aligned}
& \cos(\alpha) \cdot \tilde{\mathbf{c}} + \sin(\alpha) \cdot \tilde{\mathbf{e}}_2 \\
= {} & \{\cos(\alpha)\cos(\theta) - (\Phi^*) \sin(\alpha)\sin(\theta)(\cos(\theta) - \det[\mathbf{c}, \mathbf{c}', \mathbf{c}''] \sin(\theta))\} \cdot \mathbf{c} \\
& + \{\cos(\alpha)\sin(\theta) + (\Phi^*) \sin(\alpha)\cos(\theta)(\cos(\theta) - \det[\mathbf{c}, \mathbf{c}', \mathbf{c}''] \sin(\theta))\} \cdot \mathbf{e}_2,
\end{aligned}$$

curves C and $\tilde{C}$ have the same principal normal great circle at each pair of corresponding points. Thus we have the following theorem.

Theorem 1. *Every curve on $S^2(1)$ is a spherical Bertrand curve.*

6. Spherical Bertrand curves on $S^3(1)$ in $\mathbb{R}^4$

In this section, we discuss a spherical Bertrand curves on $S^3(1)$ in the same manner as in Section 5. The purpose of this section is to get conditions of a spherical Bertrand curve on $S^3(1)$ and represent it with $\mathbf{c}, \mathbf{c}'(s), \mathbf{c}''(s)$, etc. Let $\mathbf{c} : J \ni s \mapsto \mathbf{c}(s) \in S^3(1)$ in $\mathbb{R}^4$ be a curve C parametrized by the arc-length s. Then we have

$$\|\mathbf{c}(s)\| = 1, \qquad \|\mathbf{c}'(s)\| = 1.$$

Also we know

(a) $\langle \mathbf{c}(s), \mathbf{c}(s)\rangle = 1$, $\langle \mathbf{c}'(s), \mathbf{c}'(s)\rangle = 1$.
(b) $\langle \mathbf{c}'(s), \mathbf{c}(s)\rangle = 0$, $\langle \mathbf{c}''(s), \mathbf{c}'(s)\rangle = 0$.
(c) $\langle \mathbf{c}''(s), \mathbf{c}(s)\rangle = -1$, $\langle \mathbf{c}'''(s), \mathbf{c}(s)\rangle = 0$.
(d) $\langle \mathbf{c}'''(s), \mathbf{c}'(s)\rangle = -\|\mathbf{c}''(s)\|^2$, $\langle \mathbf{c}'''(s), \mathbf{c}''(s)\rangle = \frac{1}{2}(\|\mathbf{c}''(s)\|^2)'$.

And then, we have

$$\mathbf{c}'(s) \in T_{\mathbf{c}(s)}(S^3(1)).$$

We denote $\mathbf{e}_1(s) := \mathbf{c}'(s)$ for all $s \in I$. Since $\langle \mathbf{c}''(s) - \langle \mathbf{c}''(s), \mathbf{c}(s)\rangle \mathbf{c}(s), \mathbf{c}(s)\rangle = 0$, then we have

$$\mathbf{c}''(s) - \langle \mathbf{c}''(s), \mathbf{c}(s)\rangle \mathbf{c}(s) \in T_{\mathbf{c}(s)}(S^3(1)).$$

Then we obtain

$$P_{\mathbf{c}(s)}(\mathbf{e}_1'(s)) = \nabla_s \mathbf{e}_1(s) = \mathbf{c}''(s) - \langle \mathbf{c}''(s), \mathbf{c}(s)\rangle \mathbf{c}(s).$$

Therefore, we get

$$\nabla_s \mathbf{e}_1(s) = \mathbf{c}''(s) + \mathbf{c}(s).$$

And we have

$$\|\mathbf{c}''(s) + \mathbf{c}(s)\|^2 = \|\mathbf{c}''(s)\|^2 - 1.$$

We remark that $0 < \|\mathbf{c}''(s)\|^2 - 1 < +\infty$ and put $k(s) = \sqrt{\|\mathbf{c}''(s)\|^2 - 1}$ for all $s \in J$. Then, we denote

$$\mathbf{e}_2(s) := \frac{1}{k(s)}(\mathbf{c}''(s) + \mathbf{c}(s)).$$

Thus we have

$$\nabla_s \mathbf{e}_1(s) = k(s)\mathbf{e}_2(s)$$

and

$$\mathbf{e}_2'(s) = \left(\frac{1}{k(s)}\right)'(\mathbf{c}''(s) + \mathbf{c}(s)) + \frac{1}{k(s)}(\mathbf{c}'''(s) + \mathbf{c}'(s)).$$

Since $\langle \mathbf{e}_2'(s), \mathbf{c}(s)\rangle = 0$, then we have

$$\mathbf{e}_2'(s) \in T_{\mathbf{c}(s)}(S^3(1)).$$

So, we have

$$\nabla_s \mathbf{e}_2(s) = \left(\frac{1}{k(s)}\right)\mathbf{c}'''(s) + \left(\frac{1}{k(s)}\right)'\mathbf{c}''(s) + \left(\frac{1}{k(s)}\right)\mathbf{c}'(s) + \left(\frac{1}{k(s)}\right)'\mathbf{c}(s).$$

Then, we obtain

$$\begin{aligned}
&\nabla_s \mathbf{e}_2(s) + k(s)\mathbf{e}_1(s) \\
&= \left(\frac{1}{k(s)}\right)(\mathbf{c}'''(s) + \mathbf{c}'(s)) + \left(\frac{1}{k(s)}\right)'(\mathbf{c}''(s) + \mathbf{c}(s)) + k(s)\mathbf{c}'(s)
\end{aligned}$$

and

$$\|\nabla_s \mathbf{e}_2(s) + k(s)\mathbf{e}_1(s)\|^2 = \frac{1}{(k(s))^2}[\|\mathbf{c}'''(s)\|^2 - (k'(s))^2 - \{1 + (k(s))^2\}^2].$$

We denote $w(s)$ by

$$w(s) := \varepsilon \frac{1}{k(s)}\sqrt{\|\mathbf{c}'''(s)\|^2 - (k'(s))^2 - \{1 + (k(s))^2\}^2}.$$

Here, $\varepsilon = \pm 1$. We assume that $w(s) \neq 0$, since C is a C^∞-special Frenet curve. From $k(s) = (\langle \mathbf{c}''(s), \mathbf{c}''(s)\rangle - 1)^{1/2}$, then we obtain

$$k'(s) = \frac{\langle \mathbf{c}'''(s), \mathbf{c}''(s)\rangle}{(\langle \mathbf{c}''(s), \mathbf{c}''(s)\rangle - 1)^{1/2}}.$$

Thus we obtain

$$w(s) = \varepsilon \frac{1}{\|\mathbf{c}''(s)\|^2 - 1} \sqrt{\|\mathbf{c}'''(s)\|^2 - \frac{(\langle \mathbf{c}'''(s), \mathbf{c}''(s)\rangle)^2}{\|\mathbf{c}''(s)\|^2 - 1} - \|\mathbf{c}''(s)\|^4}.$$

Then we define

$$\begin{aligned}
\mathbf{e}_3(s) &= \frac{1}{w(s)}(\nabla_s \mathbf{e}_2(s) + k(s)\mathbf{e}_1(s)) \\
&= \left(\frac{1}{w(s)k(s)}\right)\mathbf{c}'''(s) + \left(\frac{1}{w(s)}\right)\left(\frac{1}{k(s)}\right)'\mathbf{c}''(s) \\
&\quad + \left(\frac{1+(k(s))^2}{w(s)k(s)}\right)\mathbf{c}'(s) + \left(\frac{1}{w(s)}\right)\left(\frac{1}{k(s)}\right)'\mathbf{c}(s).
\end{aligned}$$

Since we define that the orientation of the frame $\{\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$ at $\mathbf{c}(s)$ is positive, we have that

$$\det[\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)] = \frac{1}{w(s)(k(s))^2}\det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s), \mathbf{c}'''(s)] > 0,$$

$$w(s)\begin{cases} > 0 & \text{if} \quad \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s), \mathbf{c}'''(s)] > 0 \\ < 0 & \text{if} \quad \det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s), \mathbf{c}'''(s)] < 0. \end{cases}$$

and

$$\begin{aligned}
\varepsilon &= +1, \\
w(s) &= \frac{\det[\mathbf{c}(s), \mathbf{c}'(s), \mathbf{c}''(s), \mathbf{c}'''(s)]}{(k(s))^2},
\end{aligned}$$

by taking account of $\det[\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)] = 1$. Then, we obtain

$$\begin{aligned}
\mathbf{e}_3'(s) = &\left(\frac{1}{w(s)k(s)}\right)'\mathbf{c}'''(s) + \left(\frac{1}{w(s)k(s)}\right)\mathbf{c}''''(s) \\
&+ \left\{\left(\frac{1}{w(s)}\right)\left(\frac{1}{k(s)}\right)'\right\}'\mathbf{c}''(s) + \left(\frac{1}{w(s)}\right)\left(\frac{1}{k(s)}\right)'\mathbf{c}'''(s) \\
&+ \left(\frac{1+(k(s))^2}{w(s)k(s)}\right)'\mathbf{c}'(s) + \left(\frac{1+(k(s))^2}{w(s)k(s)}\right)\mathbf{c}''(s) \\
&+ \left\{\left(\frac{1}{w(s)}\right)\left(\frac{1}{k(s)}\right)'\right\}'\mathbf{c}(s) + \left(\frac{1}{w(s)}\right)\left(\frac{1}{k(s)}\right)'\mathbf{c}'(s)
\end{aligned}$$

and since $\langle \mathbf{e}_3'(s), \mathbf{c}(s)\rangle = 0$, then we obtain

$$\nabla_s \mathbf{e}_3(s) = \mathbf{e}_3'(s).$$

Also we obtain $\langle \mathbf{e}_3'(s), \mathbf{e}_1(s)\rangle = 0$. And, since $\langle \mathbf{e}_3(s), \mathbf{e}_3(s)\rangle = 1$, then we obtain $\langle \mathbf{e}_3'(s), \mathbf{e}_3(s)\rangle = 0$ for all $s \in I$. We have $\langle \mathbf{e}_3'(s), \mathbf{c}''(s)\rangle = -w(s)k(s)$ and

$\langle \mathbf{e}_3'(s), \mathbf{e}_2(s)\rangle = -w(s)$. Therefore, we obtain

$$\begin{cases} \nabla_s \mathbf{e}_1(s) = k(s)\mathbf{e}_2(s) \\ \nabla_s \mathbf{e}_2(s) = -k(s)\mathbf{e}_1(s) + w(s)\mathbf{e}_3(s) \\ \nabla_s \mathbf{e}_3(s) = -w(s)\mathbf{e}_2(s). \end{cases}$$

Now, to find a spherical Bertrand mate curve of C we consider a curve $\bar{C}$ defined by

$$\hat{\mathbf{c}}(s) = \cos(\theta(s)) \cdot \mathbf{c}(s) + \sin(\theta(s)) \cdot \mathbf{e}_2(s), \quad \sin(\theta(s)) \neq 0$$

for $s \in J$, because a spherical Bertrand mate curve is distinct from C. The curve $\bar{C}$ is on $S^3(1)$. Then, We have

$$\begin{aligned} \hat{\mathbf{c}}'(s) = &\left.\frac{\mathrm{d}\bar{\mathbf{c}}}{\mathrm{ds}}\right|_s \\ = &-\theta'(s)\sin(\theta(s)) \cdot \mathbf{c}(s) + \cos(\theta(s)) \cdot \mathbf{c}'(s) \\ &+ \theta'(s)\cos(\theta(s)) \cdot \mathbf{e}_2(s) + \sin(\theta(s)) \cdot \mathbf{e}_2'(s). \end{aligned}$$

Since $\langle \hat{\mathbf{c}}'(s), \mathbf{c}(s)\rangle = 0$ for all $s \in J$, we have $\theta'(s) = 0$, $s \in J$ so that $\theta(s) = \theta$ (θ is a constant number). Thus we have

$$\begin{aligned} \hat{\mathbf{c}}'(s) &= \cos(\theta) \cdot \mathbf{c}'(s) + \sin(\theta) \cdot P_{\bar{\mathbf{c}}(s)}(\mathbf{e}_2'(s)) \\ &= \cos(\theta) \cdot \mathbf{e}_1(s) + \sin(\theta) \cdot \nabla_s \mathbf{e}_2(s) \\ &= (\cos(\theta) - k(s)\sin(\theta)) \cdot \mathbf{e}_1(s) + (w(s)\sin(\theta)) \cdot \mathbf{e}_3(s). \end{aligned}$$

Let $\bar{s}$ be the arc-length parameter of $\bar{C}$ from $\hat{\mathbf{c}}(0)$ to $\hat{\mathbf{c}}(s)$. Then we get a function $\Phi : \bar{J} \to J$ such that $s = \Phi(\bar{s})$, and the curve $\bar{C}$ is represented by arc-length parameter $\bar{s}$, that is, $\bar{\mathbf{c}}(\bar{s}) = \hat{\mathbf{c}}(\Phi(\bar{s}))$. We have

$$\begin{aligned} \bar{\mathbf{c}}^*(\bar{s}) = &\left.\frac{\mathrm{d}\bar{\mathbf{c}}}{\mathrm{d\bar{s}}}\right|_{\bar{s}} \\ &= \Phi^*(\bar{s})\{(\cos(\theta) - k(\Phi(\bar{s}))\sin(\theta)) \cdot \mathbf{e}_1(\Phi(\bar{s})) + (w(\Phi(\bar{s}))\sin(\theta)) \cdot \mathbf{e}_3(\Phi(\bar{s}))\} \end{aligned}$$

and we have $\bar{\mathbf{c}}^*(\bar{s}) \in T_{\bar{\mathbf{c}}(\bar{s})}(S^3(1))$, that is, $\bar{\mathbf{e}}_1(\bar{s}) = \bar{\mathbf{c}}^*(\bar{s})$. Hereafter, we omit $(\bar{s})$ and $(\Phi(\bar{s}))$. We have

$$\begin{aligned} \bar{\mathbf{e}}_1^* &= \bar{\mathbf{c}}^{**} \\ &= \Phi^{**}\{(\cos(\theta) - k\sin(\theta)) \cdot \mathbf{e}_1 + (w\sin(\theta)) \cdot \mathbf{e}_3\} \\ &\quad + (\Phi^*)^2\{-(k'\sin(\theta)) \cdot \mathbf{e}_1 + (\cos(\theta) - k\sin(\theta)) \cdot \mathbf{e}_1' \\ &\quad + (w'\sin(\theta)) \cdot \mathbf{e}_3 + (w\sin(\theta)) \cdot \mathbf{e}_3'\}. \end{aligned}$$

Thus we have

$$\begin{aligned} \nabla_{\bar{s}}\bar{\mathbf{e}}_1 &= \Phi^{**}\{(\cos(\theta) - k\sin(\theta)) \cdot \mathbf{e}_1 + (w\sin(\theta)) \cdot \mathbf{e}_3\} \\ &\quad + (\Phi^*)^2\{-(k'\sin(\theta)) \cdot \mathbf{e}_1 + (\cos(\theta) - k\sin(\theta)) \cdot \nabla_s\mathbf{e}_1 \\ &\quad + (w'\sin(\theta)) \cdot \mathbf{e}_3 + (w\sin(\theta)) \cdot \nabla_s\mathbf{e}_3\} \end{aligned}$$

$$
\begin{aligned}
&= \{\Phi^{**}(\cos(\theta) - k\sin(\theta)) - (\Phi^*)^2 k' \sin(\theta)\} \cdot \mathbf{e}_1 \\
&\quad + \{(\Phi^*)^2(\cos(\theta) - k\sin(\theta))k - (\Phi^*)^2 w^2 \sin(\theta)\} \cdot \mathbf{e}_2 \\
&\quad + \{\Phi^{**} w \sin(\theta) + (\Phi^*)^2 w' \sin(\theta)\} \cdot \mathbf{e}_3.
\end{aligned}
$$

The principal normal great circle $\tilde{C}$ of $\bar{C}$ at $\bar{\mathbf{c}}(\bar{s})$ is given by

$$\tilde{\mathbf{c}} = \cos(\alpha) \cdot \bar{\mathbf{c}} + \sin(\alpha) \cdot \bar{\mathbf{e}}_2.$$

On the other hand, we have $\bar{\mathbf{e}}_2 = A^{-1} \cdot \nabla_{\bar{s}} \bar{\mathbf{e}}_1$, where A denotes the norm of $\nabla_{\bar{s}} \bar{\mathbf{e}}_1$. Here, we remark that α is a constant for the same reason of the case of θ. Then we have

$$
\begin{aligned}
\tilde{\mathbf{c}} &= (\cos(\alpha)\cos(\theta)) \cdot \mathbf{c} + (\cos(\alpha)\sin(\theta)) \cdot \mathbf{e}_2 \\
&\quad + \sin(\alpha) A^{-1} \{\Phi^{**}(\cos(\theta) - k\sin(\theta)) - (\Phi^*)^2 k' \sin(\theta)\} \cdot \mathbf{e}_1 \\
&\quad + \sin(\alpha) A^{-1} \{(\Phi^*)^2(\cos(\theta) - k\sin(\theta))k - (\Phi^*)^2 w^2 \sin(\theta)\} \cdot \mathbf{e}_2 \\
&\quad + \sin(\alpha) A^{-1} \{\Phi^{**} w \sin(\theta) + (\Phi^*)^2 w' \sin(\theta)\} \cdot \mathbf{e}_3.
\end{aligned}
$$

To get the same principal normal great circle, the components of $\mathbf{e}_1$ and $\mathbf{e}_3$ of the above equality must vanish. Thus we have

$$\Phi^{**}(\cos(\theta) - k\sin(\theta)) - (\Phi^*)^2 k' \sin(\theta) = 0$$

and

$$\Phi^{**} w \sin(\theta) + (\Phi^*)^2 w' \sin(\theta) = 0$$

so that

$$(\dagger) \qquad \frac{-k' \sin(\theta)}{\cos(\theta) - k\sin(\theta)} = \frac{w'}{w}.$$

By solving this differential equation, we have $\cos(\theta) - k\sin(\theta) = \nu w$, where ν is a constant number. The converse is easy to prove if we go reversely. Now we put $\lambda = \cos(\theta)$ and $\mu = \sin(\theta)$, then we have the following.

Theorem 2. *A C^∞-special Frenet curve on $S^3(1)$ is a spherical Bertrand curve if and only if there exist three constants λ, μ and ν such that $\lambda - \mu k(s) = \nu w(s)$, $\lambda^2 + \mu^2 = 1$, $\mu \neq 0$.*

Corollary 3. *A C^∞-special Frenet curve on $S^3(1)$ satisfying $k(s) = k_0$ (constant) and $w(s) = w_0$ (constant) is a spherical Bertrand curve.*

The following are the examples of spherical curves on $S^3(1)$.

Example 4. Let a, b, c and d be constant numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. A C^∞-curve C on $S^3(1)$ is defined by $\mathbf{c} : \mathbb{R} \to S^3(1)$;

$$\mathbf{c}(s) = \begin{bmatrix} a\cos(s) - b\sin(s) \\ b\cos(s) + a\sin(s) \\ c\cos(s) - d\sin(s) \\ d\cos(s) + c\sin(s) \end{bmatrix}$$

for all $s \in \mathbb{R}$.

By the direct computation, we get

$$\|\mathbf{c}(s)\| = 1, \qquad \|\mathbf{c}'(s)\| = 1,$$

$$\mathbf{c}'(s) = \begin{bmatrix} -a\sin(s) - b\cos(s) \\ -b\sin(s) + a\cos(s) \\ -c\sin(s) - d\cos(s) \\ -d\sin(s) + c\cos(s) \end{bmatrix}, \quad \mathbf{c}''(s) = \begin{bmatrix} -a\cos(s) + b\sin(s) \\ -b\cos(s) - a\sin(s) \\ -c\cos(s) + d\sin(s) \\ -d\cos(s) - c\sin(s) \end{bmatrix}$$

for all $s \in \mathbb{R}$. Since $\|\mathbf{c}''(s)\|^2 = 1$ and $\langle \mathbf{c}''(s), \mathbf{c}(s) \rangle = -1$, then we can calculate

$$\mathbf{c}''(s) \notin T_{\mathbf{c}(s)}(S^3(1)), \quad \mathbf{c}''(s) \in \{\alpha \cdot \mathbf{c}(s) \mid \alpha \in \mathbb{R}\}$$

and $k(s) = 0$ for all $s \in \mathbb{R}$. The curve C is a great circle on $S^3(1)$, but the curve C is not a C^∞-special Frenet curve on $S^3(1)$ since its curvature function vanishes.

Example 5. A C^∞-curve C on $S^3(1)$ is defined by $\mathbf{c} : \mathbb{R} \to S^3(1)$;

$$\mathbf{c}(s) = \begin{bmatrix} \dfrac{1}{\sqrt{3}} \cos\left(\dfrac{2\sqrt{2}}{\sqrt{3}} s\right) \\ \dfrac{1}{\sqrt{3}} \sin\left(\dfrac{2\sqrt{2}}{\sqrt{3}} s\right) \\ \dfrac{\sqrt{2}}{\sqrt{3}} \cos\left(\dfrac{1}{\sqrt{6}} s\right) \\ \dfrac{\sqrt{2}}{\sqrt{3}} \sin\left(\dfrac{1}{\sqrt{6}} s\right) \end{bmatrix}$$

for all $s \in \mathbb{R}$. This curve C is a spherical Bertrand curve.

By the direct computation, we get

$$\|\mathbf{c}(s)\| = 1, \qquad \|\mathbf{c}'(s)\| = 1,$$

$$\mathbf{c}'(s) = \begin{bmatrix} -\dfrac{2\sqrt{2}}{3} \sin\left(\dfrac{2\sqrt{2}}{\sqrt{3}} s\right) \\ \dfrac{2\sqrt{2}}{3} \cos\left(\dfrac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\dfrac{1}{3} \sin\left(\dfrac{1}{\sqrt{6}} s\right) \\ \dfrac{1}{3} \cos\left(\dfrac{1}{\sqrt{6}} s\right) \end{bmatrix}, \quad \mathbf{c}''(s) = \begin{bmatrix} -\dfrac{8}{3\sqrt{3}} \cos\left(\dfrac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\dfrac{8}{3\sqrt{3}} \sin\left(\dfrac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\dfrac{1}{3\sqrt{6}} \cos\left(\dfrac{1}{\sqrt{6}} s\right) \\ -\dfrac{1}{3\sqrt{6}} \sin\left(\dfrac{1}{\sqrt{6}} s\right) \end{bmatrix}$$

for all $s \in \mathbb{R}$. Since $\|\mathbf{c}''(s)\|^2 = \frac{43}{18}$ and $\langle \mathbf{c}''(s), \mathbf{c}(s) \rangle = -1$, then we obtain $\|\mathbf{c}''(s)\|^2 - 1 = \frac{25}{18}$ for all $s \in \mathbb{R}$. Thus, we get

$$k(s) = \frac{5}{3\sqrt{2}}$$

for all $s \in \mathbb{R}$. Next, we obtain

$$\mathbf{c}'''(s) = \begin{bmatrix} \frac{16\sqrt{2}}{9} \sin\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\frac{16\sqrt{2}}{9} \cos\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ \frac{1}{18} \sin\left(\frac{1}{\sqrt{6}} s\right) \\ -\frac{1}{18} \cos\left(\frac{1}{\sqrt{6}} s\right) \end{bmatrix}, \quad \|\mathbf{c}'''(s)\|^2 = \frac{683}{108}$$

for all $s \in \mathbb{R}$. Thus we set

$$\mathbf{e}_1(s) = \begin{bmatrix} -\frac{2\sqrt{2}}{3} \sin\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ \frac{2\sqrt{2}}{3} \cos\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\frac{1}{3} \sin\left(\frac{1}{\sqrt{6}} s\right) \\ \frac{1}{3} \cos\left(\frac{1}{\sqrt{6}} s\right) \end{bmatrix}, \quad \mathbf{e}_2(s) = \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \cos\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\frac{\sqrt{2}}{\sqrt{3}} \sin\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ \frac{1}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{6}} s\right) \\ \frac{1}{\sqrt{3}} \sin\left(\frac{1}{\sqrt{6}} s\right) \end{bmatrix},$$

and

$$|w(s)| = \frac{1}{k(s)} \sqrt{\|\mathbf{c}'''(s)\|^2 - (k'(s))^2 - \{1 + (k(s))^2\}^2} = \frac{2}{3}$$

for all $s \in \mathbb{R}$. Thus we get

$$w(s) = \frac{2}{3}\varepsilon, \qquad \varepsilon = \pm 1$$

for all $s \in \mathbb{R}$. Then we obtain

$$\frac{1}{k(s)w(s)} = \frac{9\sqrt{2}}{10}\varepsilon, \quad \frac{1 + (k(s))^2}{k(s)w(s)} = \frac{43\sqrt{2}}{20}\varepsilon$$

for all $s \in \mathbb{R}$. Therefore, we set

$$\mathbf{e}_3(s) = \varepsilon \begin{bmatrix} \frac{1}{3} \sin\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\frac{1}{3} \cos\left(\frac{2\sqrt{2}}{\sqrt{3}} s\right) \\ -\frac{2\sqrt{2}}{3} \sin\left(\frac{1}{\sqrt{6}} s\right) \\ \frac{2\sqrt{2}}{3} \cos\left(\frac{1}{\sqrt{6}} s\right) \end{bmatrix}$$

for all $s \in \mathbb{R}$. Thus we have

$$\det[\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)] = \varepsilon$$

for all $s \in \mathbb{R}$. Since $\det[\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)] = +1$, then we have $\varepsilon = +1$ for all $s \in \mathbb{R}$. Therefore, we obtain $k(s) = \frac{5}{3\sqrt{2}}$ and $w(s) = \frac{2}{3}$.

Example 6. A C^∞-curve C on $S^3(1)$ is defined by $\mathbf{c} : \mathbb{R} \to S^3(1)$;

$$\mathbf{c}(s) = \begin{bmatrix} \cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix}$$

for all $s \in \mathbb{R}$. This curve C is a spherical Bertrand curve.

By the direct computation, we get

$$\|\mathbf{c}(s)\| = 1, \qquad \|\mathbf{c}'(s)\| = 1,$$

$$\mathbf{c}'(s) = \begin{bmatrix} -\frac{1}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) - \frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\frac{1}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) + \frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) + \frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) - \frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix},$$

$$\mathbf{c}''(s) = \begin{bmatrix} -\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) + \frac{2\sqrt{2}}{3}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) - \frac{2\sqrt{2}}{3}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) + \frac{2\sqrt{2}}{3}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) - \frac{2\sqrt{2}}{3}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix}$$

for all $s \in \mathbb{R}$. Since $\|\mathbf{c}''(s)\|^2 = \frac{17}{9}$, then we obtain

$$k(s) = \frac{2\sqrt{2}}{3}$$

for all $s \in \mathbb{R}$. Next, we get

$$\mathbf{c}'''(s)=\begin{bmatrix} \frac{7}{3\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right)+\frac{5\sqrt{2}}{3\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \frac{7}{3\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right)-\frac{5\sqrt{2}}{3\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\frac{7}{3\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right)-\frac{5\sqrt{2}}{3\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\frac{7}{3\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right)+\frac{5\sqrt{2}}{3\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix}$$

and $\|\mathbf{c}'''(s)\|^2 = \frac{11}{3}$ for all $s \in \mathbb{R}$. Therefore, we set

$$\mathbf{e}_1(s) = \begin{bmatrix} -\frac{1}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right)-\frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\frac{1}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right)+\frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right)+\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right)-\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix},$$

$$\mathbf{e}_2(s) = \begin{bmatrix} \sin\left(\frac{1}{\sqrt{3}}s\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \sin\left(\frac{1}{\sqrt{3}}s\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \cos\left(\frac{1}{\sqrt{3}}s\right)\sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \cos\left(\frac{1}{\sqrt{3}}s\right)\cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix}$$

and

$$|w(s)| = \frac{1}{k(s)}\sqrt{\|\mathbf{c}'''(s)\|^2 - (k'(s))^2 - \{1+(k(s))^2\}^2} = \frac{1}{3}$$

for all $s \in \mathbb{R}$. Thus we obtain

$$w(s) = \frac{1}{3}\varepsilon, \qquad \varepsilon = \pm 1$$

for all $s \in \mathbb{R}$. Then we get

$$\frac{1}{k(s)w(s)} = \frac{9}{2\sqrt{2}}\varepsilon, \quad \frac{1+(k(s))^2}{k(s)w(s)} = \frac{17}{2\sqrt{2}}\varepsilon$$

for all $s \in \mathbb{R}$. Therefore, we set

$$\mathbf{e}_3(s) = \varepsilon \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \sin\left(\frac{1}{\sqrt{3}}s\right) \cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) - \frac{1}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{3}}s\right) \sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ \frac{\sqrt{2}}{\sqrt{3}} \sin\left(\frac{1}{\sqrt{3}}s\right) \sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) + \frac{1}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{3}}s\right) \cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\frac{\sqrt{2}}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{3}}s\right) \cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{1}{\sqrt{3}}s\right) \sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \\ -\frac{\sqrt{2}}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{3}}s\right) \sin\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{1}{\sqrt{3}}s\right) \cos\left(-\frac{\sqrt{2}}{\sqrt{3}}s\right) \end{bmatrix}$$

for all $s \in \mathbb{R}$. Thus we have

$$\det[\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)] = -\varepsilon$$

for all $s \in \mathbb{R}$. Since $\det[\mathbf{c}(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)] = +1$, then we have $\varepsilon = -1$ for all $s \in \mathbb{R}$. Therefore, we obtain $k(s) = \frac{2\sqrt{2}}{3}$ and $w(s) = -\frac{1}{3}$.

A C^∞-curve with constant curvature ratios is introduced in [12]. This curve is called a ccr-curve, briefly. The ccr-curve with constant extrinsic curvature ratios is proposed in the Euclidean space [12].

Example 7. A C^∞-special Frenet curve C on $S^3(1)$ is said to be a ccr-curve on $S^3(1)$ if its intrinsic curvature ratio $\frac{k}{w}$ is a constant number, where k and w are the curvature and the torsion of the curve C, respectively. Then we call the curve C "a ccr-curve on $S^3(1)$", briefly. The kind of ccr-curve on $S^3(1)$ is presented in [9] (Euler spirals or clothoids, n-clothoids, and generalized conical helices as the curvature and the torsion of each curve have some initial value). There are two cases that (1) ccr-curves on $S^3(1)$ have constant curvature k and torsion w, and (2) ccr-curves on $S^3(1)$ have non-constant curvature k and torsion w. In both cases, the ccr-curves on $S^3(1)$ are spherical Bertrand curves.

Let C^∞-curve C be a ccr-curve on $S^3(1)$ with constant intrinsic curvature ratio $\frac{k}{w}$ so that $k = cw$, where c is a constant number. If ccr-curve C is a spherical Bertrand curve, then the curve C satisfies the above differential equation $\frac{-k' \sin(\theta)}{\cos(\theta) - k\sin(\theta)} = \frac{w'}{w}$ (†), where θ is the non-zero constant angle between the spherical Bertrand curve and the pair curve. By the differential equation $\frac{-k' \sin(\theta)}{\cos(\theta) - k\sin(\theta)} = \frac{w'}{w}$ (†) satisfying $k = cw$, it is easy to get the following equation

$$k' \cos(\theta) = 0.$$

(1) If both $k' = 0$ and $\cos(\theta) = 0$, then k and $w(= k/c)$ are constant numbers and there exists a pair curve of the curve C with $\theta = \pi/2$ between the curve C and the pair curve of C. If $k' = 0$ and $\cos(\theta) \neq 0$, then k and w are constant numbers and there exist pair curves of the curve C with $\theta \neq \pi/2$ between the curve C and one of the pair curves

of C. Therefore, ccr-curve C on $S^3(1)$ with constant curvature k and torsion w is a spherical Bertrand curve.

(2) If $k' \neq 0$ and $\cos(\theta) = 0$, then k/w is a constant number and there exists a pair curve of the curve C with $\theta = \pi/2$ between the curve C and the pair curve of C. Therefore, the ccr-curve C on $S^3(1)$ with non-constant curvature k and torsion w is a spherical Bertrand curve.

In the case of n-clothoid : A C^∞-special Frenet curve C on $S^3(1)$ is said to be an n-clothoid with its curvature and torsion given by

$$k(s) = \alpha + \beta s^n, \qquad w(s) = \gamma + \delta s^n \qquad \text{for } \ s \geq 0,$$

where α, β, γ, and δ are positive constants and n is a positive integer [9]. For a positive constant c, if both $\gamma = c\alpha$ and $\delta = c\beta$, then it is trivial that the n-clothoid C is a ccr-curve on $S^3(1)$. We have $\lambda - \mu k(s) = \nu w(s)$ $(s \geq 0)$ with $\lambda = 0$, $\mu = 1$ and $\nu = -\frac{1}{c}$. Thus the curve C is a spherical Bertrand curve.

Now, we provide a spherical curve but not a spherical Bertrand curve.

Example 8. A C^∞-curve C on $S^3(1)$ is defined by $\mathbf{x} : \mathbb{R} \to S^3(1)$;

$$\mathbf{x}(t) = \begin{bmatrix} \cos(3t)\cos(t) \\ \cos(3t)\sin(t) \\ \sin(3t)\cos(2t) \\ \sin(3t)\sin(2t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

for all $t \in \mathbb{R}$. This curve C is not a spherical Bertrand curve.

By the direct computation, we get

$$\|\mathbf{x}(t)\| = 1,$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -3\sin(3t)\cos(t) - \cos(3t)\sin(t) \\ -3\sin(3t)\sin(t) + \cos(3t)\cos(t) \\ 3\cos(3t)\cos(2t) - 2\sin(3t)\sin(2t) \\ 3\cos(3t)\sin(2t) + 2\sin(3t)\cos(2t) \end{bmatrix}$$

and

$$\|\dot{\mathbf{x}}(t)\|^2 = 10 + 3\sin^2(3t).$$

Then we obtain

$$\ddot{\mathbf{x}}(t) = \begin{bmatrix} -10\cos(3t)\cos(t) + 6\sin(3t)\sin(t) \\ -10\cos(3t)\sin(t) - 6\sin(3t)\cos(t) \\ -13\sin(3t)\cos(2t) - 12\cos(3t)\sin(2t) \\ -13\sin(3t)\sin(2t) + 12\cos(3t)\cos(2t) \end{bmatrix}$$

and

$$\langle \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t) \rangle = 9\sin(3t)\cos(3t),$$
$$\langle \ddot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t) \rangle = 244 - 39\sin^2(3t).$$

We calculate

$$\begin{aligned}&\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle \langle \ddot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)\rangle - (\langle \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)\rangle)^2 - (\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle)^3 \\ &= 9\left(160 - 71\sin^2(3t) - 34\sin^4(3t) - 3\sin^6(3t)\right),\end{aligned}$$

then we get the curvature $k(t)$ is

$$\begin{aligned}k(t) &= \frac{\sqrt{\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle \langle \ddot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)\rangle - (\langle \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)\rangle)^2 - (\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle)^3}}{(\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle)^{\frac{3}{2}}} \\ &= \frac{3\sqrt{160 - 71\sin^2(3t) - 34\sin^4(3t) - 3\sin^6(3t)}}{(10 + 3\sin^2(3t))\sqrt{10 + 3\sin^2(3t)}}.\end{aligned}$$

Next, we obtain

$$\dddot{\mathbf{x}}(t) = \begin{bmatrix} 36\sin(3t)\cos(t) + 28\cos(3t)\sin(t) \\ 36\sin(3t)\sin(t) - 28\cos(3t)\cos(t) \\ -63\cos(3t)\cos(2t) + 62\sin(3t)\sin(2t) \\ -63\cos(3t)\sin(2t) - 62\sin(3t)\cos(2t) \end{bmatrix},$$

then we have

$$\det[\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \dddot{\mathbf{x}}(t)] = 18(14 + 23\sin^2(3t) + \sin^4(3t)).$$

Thus we get the curvature $w(t)$ is

$$\begin{aligned}w(t) &= \frac{\det[\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \dddot{\mathbf{x}}(t)]}{\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle \langle \ddot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)\rangle - (\langle \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)\rangle)^2 - (\langle \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle)^3} \\ &= \frac{2(14 + 23\sin^2(3t) + \sin^4(3t))(10 + 3\sin^2(3t))^3}{160 - 71\sin^2(3t) - 34\sin^4(3t) - 3\sin^6(3t)}.\end{aligned}$$

By the above differential equation $\frac{-k'\sin(\theta)}{\cos(\theta) - k\sin(\theta)} = \frac{w'}{w}$ (†), we conclude any constant angle θ does not exist. Therefore, this curve C is not a spherical Bertrand curve.

Problem. Are above curves $(1,3)$-Bertrand curves in $\mathbb{R}^4$?

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References

[1] Yu. Aminov, *Differential Geometry and Topology of Curves*, Gordon and Breach Science Publishers, Amsterdam, 2000.

[2] ———, *The Geometry of Submanifolds*, Gordon and Breach Science Publishers, Amsterdam, 2001.

[3] H. Balgetir, M. Bektas, and J.-I. Inoguchi, *Null Bertrand curves in Minkowski 3-space and their characterizations*, Note Mat. **23** (2004), no. 1, 7–13.

[4] P. A. Blaga, *Lectures on the Differential Geometry of Curves and Surfaces*, Napoca Press, 2005.

[5] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ. 1976.

[6] N. Ekmekci and K. Ilarslan, *On Bertrand curves and their characterization*, Differ. Geom. Dyn. Syst. **3** (2001), no. 2, 17–24.

[7] C.-C. Hsiung, *A First Course in Differential Geometry*, International Press, Cambridge, MA. 1997.

[8] S. Izumiya and N. Takeuchi, *Generic properties of helices and Bertrand curves*, J. Geom. **74** (2002), no. 1-2, 97–109.

[9] P. Lucas and J. A. Ortega-Yagües, *Bertrand curves in the three-dimensional sphere*, J. Geom. Phys. **62** (2012), no. 9, 1903–1914.

[10] H. Matsuda and S. Yorozu, *Notes on Bertrand curves*, Yokohama Math. J. **50** (2003), no. 1-2, 41–58.

[11] R. S. Millman and G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall Inc, 1977.

[12] J. Monterde, *Curves with constant curvature ratios*, Bol. Soc. Mat. Mexicana (3) **13** (2007), no. 1, 177–186.

[13] L. R. Pears, *Bertrand curves in Riemannian space*, J. London Math. Soc. (1935), no. 2, 180–183.

[14] W. K. Schief, *On the integrability of Bertrand curves and Razzaboni surfaces*, J. Geom. Phys. **45** (2003), no. 1-2, 130–150.

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