

**GENERALIZED DISCRETE HALANAY INEQUALITIES
AND THE ASYMPTOTIC BEHAVIOR OF
NONLINEAR DISCRETE SYSTEMS**

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ABSTRACT. In this paper, some new generalized discrete Halanay inequalities are established. On the basis of these new established inequalities, we obtain the attracting set and the global asymptotic stability of the nonlinear discrete systems. Our results established here extend the main results in [R. P. Agarwal, Y. H. Kim, and S. K. Sen, *New discrete Halanay inequalities: stability of difference equations*, Commun. Appl. Anal. **12** (2008), no. 1, 83–90] and [S. Udpin and P. Niamsup, *New discrete type inequalities and global stability of nonlinear difference equations*, Appl. Math. Lett. **22** (2009), no. 6, 856–859].

1. Introduction

In 1966, Halanay proved the following famous result.

Halanay Inequality (Halanay [9, pp. 378–380]). *If*

$$f'(t) \leq -\alpha f(t) + \beta \sup_{[t-\tau, t]} f(s) \text{ for } t \geq t_0$$

and $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $K > 0$ *such that*

$$f(t) \leq K e^{-\gamma(t-t_0)} \text{ for } t \geq t_0.$$

Since then, Halanay inequality has widely been applied to the stability analysis of delay differential systems (see e.g. [3, 5, 7, 8, 9, 11, 12]). At the same time, various generalized Halanay inequalities have been presented and used by many authors (see e.g. [6, 10, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25]). In particular, in [1, 16], the authors consider the following discrete Halanay-type inequalities in order to study some discretized versions of functional differential equations.

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Theorem A (Udpin and Niamsup, [16, Theorem 2.1]). *Let $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^+$, $i = 1, \dots, r$, where $0 = h_0 < h_1 < \dots < h_r$ and $\sum_{i=0}^r q_i < p \leq 1$, and let $\{x_j\}_{j \in \mathbb{Z}^{-h_r}}$ be a sequence of real numbers satisfying the inequality*

$$(1.1) \quad \Delta x_n \leq -px_n + \sum_{i=0}^r q_i x_{n-h_i}, \quad n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$(1.2) \quad x_n \leq \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\} \lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover, λ_0 may be chosen as the smallest root of the polynomial

$$(1.3) \quad P(\lambda) = \lambda^{h_{r+1}} - (1 - p + q_0)\lambda^{h_r} - q_1\lambda^{h_r-h_1} - \dots - q_{r-1}\lambda^{h_r-h_{r-1}} - q_r$$

which lies in the interval $(0, 1)$.

Theorem B (Udpin and Niamsup, [16, Theorem 2.2]). *Let $p, \alpha_i, \beta_i \in \mathbb{R}^+$, $h_i \in \mathbb{Z}^+$, $i = 1, \dots, r$, where $0 = h_0 < h_1 < \dots < h_r$, $\sum_{i=0}^r \alpha_i = 1$ and $\prod_{i=0}^r \beta_i < p \leq 1$. Let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of real numbers such that $x_{n-h_i}^{\alpha_i}$ are defined for all $i = 1, \dots, r$; $n \in \mathbb{Z}^0$ which satisfies the inequality*

$$(1.4) \quad \Delta x_n \leq -px_n + (\prod_{i=0}^r \beta_i)(x_{n-h_i}^{\alpha_i}), \quad n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$(1.5) \quad x_n \leq \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\} \lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover, λ_0 can be chosen as the smallest root of the function

$$(1.6) \quad F(\lambda) = \lambda - (\prod_{i=0}^r \beta_i)(\lambda^{-\sum_{i=0}^r h_i \alpha_i}) + (p - 1)$$

which lies in the interval $(0, 1)$.

Theorem C (Agarwal, Kim, and Sen, [1, Theorem 2.2]). *Let $a_i, q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r - 1$; $a_r, q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \dots < h_r$. Let $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^r \alpha_i = 1$ and $[(1 - \delta)\prod_{i=0}^r \beta_i + \delta \sum_{i=0}^r q_i] < \sum_{i=0}^r a_i \leq 1$, where $0 \leq \delta \leq 1$ is a constant. Also, let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$(1.7) \quad \Delta x_n \leq \sum_{i=0}^r (\delta q_i x_{n-h_i} - a_i x_n) + (1 - \delta)(\prod_{i=0}^r \beta_i)(x_{n-h_i}^{\alpha_i}), \quad n \in \mathbb{Z}^0.$$

Then there exists a constant $\lambda_0 \in (0, 1)$ such that

$$(1.8) \quad x_n \leq \max\{0, x_0, x_{-h_1}, \dots, x_{-h_r}\} \lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover, λ_0 can be chosen as the root in the interval $(0, 1)$ of the equation

$$(1.9) \quad \lambda + \left(\sum_{i=0}^r a_i - 1\right) - (1 - \delta)(\prod_{i=0}^r \beta_i)(\lambda^{-\sum_{i=0}^r \alpha_i h_i}) - \delta \sum_{i=0}^r q_i \lambda^{-h_i} = 0.$$

Obviously, these discrete Halanay-type inequalities are important tools for investigating the stability of discrete systems. However, the equilibrium point sometimes does not exist in many real systems, especially in nonlinear dynamical systems. Therefore, an interesting subject is to discuss the attracting set of nonlinear discrete systems. However, the foregoing discrete Halanay-type inequalities are ineffective for studying the attracting sets of nonlinear discrete systems. With motivation from the above discussions, our main aim in the present paper is to improve the foregoing inequalities such that it is effective for studying the attracting sets of nonlinear discrete systems. We also illustrate the application of these inequalities.

2. Generalized discrete Halanay inequalities

Throughout this paper, unless otherwise specified, we use the following notations. Let \mathbb{R} denote the set of all real numbers, \mathbb{R}^+ the set of positive real numbers, \mathbb{R}_0^+ the set of nonnegative real numbers, \mathbb{Z} the set of integers, \mathbb{Z}^+ the set of positive integers, and $\mathbb{Z}^{-r} = \{z \in \mathbb{Z} : z \geq -r\}$. For a sequence of real number $\{x_n\}$, the difference operator Δ on x_n is defined as $\Delta x_n = x_{n+1} - x_n$.

In this section, we introduce some new generalized discrete Halanay inequalities which will be used to study the attracting set and the global asymptotic stability of the nonlinear discrete systems. We need the following lemma in the discussions of our main results.

Lemma 2.1 (Arithmetic-mean–geometric-mean inequality [4]). *For $x_i \geq 0$, $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$,*

$$\prod_{i=1}^n x_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i x_i,$$

the sign of equality holds if and only if $x_i = x_j$ for all $i, j \in \mathcal{N}$.

Theorem 2.2. *Let $a_i, q_i, \Upsilon \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r - 1$; $a_r, q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \dots < h_r$. Let $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^r \alpha_i = 1$ and $[(1 - \delta)\prod_{i=0}^r \beta_i + \delta \sum_{i=0}^r q_i] < \sum_{i=0}^r a_i \leq 1$, where $0 \leq \delta \leq 1$ is a constant. Also let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of real numbers satisfying the inequality*

$$(2.1) \quad \Delta x_n \leq \sum_{i=0}^r (\delta q_i x_{n-h_i} - a_i x_n) + (1 - \delta) (\prod_{i=0}^r \beta_i) (\sum_{i=0}^r \alpha_i x_{n-h_i}) + \Upsilon, \quad n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$(2.2) \quad x_n \leq \max\{0, x_0, x_{-h_1}, \dots, x_{-h_r}\} \lambda_0^n + \Lambda, \quad n \in \mathbb{Z}^0,$$

where $\Lambda = (\sum_{i=0}^r a_i - \delta \sum_{i=0}^r q_i - (1 - \delta) \prod_{i=0}^r \beta_i)^{-1} \Upsilon$. Moreover, λ_0 can be chosen as the root in the interval $(0, 1)$ of the equation

$$(2.3) \quad \lambda + (\sum_{i=0}^r a_i - 1) - (1 - \delta) (\prod_{i=0}^r \beta_i) (\sum_{i=0}^r \alpha_i \lambda^{-h_i}) - \delta \sum_{i=0}^r q_i \lambda^{-h_i} = 0.$$

Proof. Let y_n be a solution of the difference equation

$$(2.4) \quad \Delta y_n = \sum_{i=0}^r (\delta q_i y_{n-h_i} - a_i y_n) + (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i y_{n-h_i} \right) + \Upsilon, \quad n \in \mathbb{Z}^0.$$

Since $(1 - \sum_{i=0}^r a_i) \geq 0$, $q_i \in \mathbb{R}_0^+$, $\beta_i \in \mathbb{R}^+$, it is easy to prove that if x_n satisfies (2.1) and $x_n \leq y_n$ for $n = -h_r, \dots, 0$, then $x_n \leq y_n$ for all $n \in \mathbb{Z}^0$. For a given $K > 0$ and $\lambda \in (0, 1)$, the sequence $\{y_n\}$ defined by $y_n = K\lambda^n + \Lambda$ is a solution of (2.4) if and only if λ is a root of the polynomial (2.3). In fact,

$$\begin{aligned} & y_n = K\lambda^n + \Lambda \text{ is a solution of (2.4)} \\ \Leftrightarrow & K\lambda^{n+1} - K\lambda^n = \sum_{i=0}^r [\delta q_i (K\lambda^{n-h_i} + \Lambda) - a_i (K\lambda^n + \Lambda)] \\ & + (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i (K\lambda^{n-h_i} + \Lambda) \right) + \Upsilon \\ \Leftrightarrow & K\lambda^{n+1} - K\lambda^n \left(\sum_{i=0}^r a_i - 1 \right) - (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i K\lambda^{n-h_i} \right) \\ & - \sum_{i=0}^r \delta q_i (K\lambda^{n-h_i}) + \left(\sum_{i=0}^r a_i - \delta \sum_{i=0}^r q_i - (1-\delta)\prod_{i=0}^r \beta_i \right) \Lambda - \Upsilon = 0 \\ \Leftrightarrow & K\lambda^{n+1} - K\lambda^n \left(\sum_{i=0}^r a_i - 1 \right) - K\lambda^n (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i \lambda^{-h_i} \right) \\ & - K\lambda^n \sum_{i=0}^r \delta q_i (\lambda^{-h_i}) = 0 \\ \Leftrightarrow & \lambda + \left(\sum_{i=0}^r a_i - 1 \right) - (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i \lambda^{-h_i} \right) - \delta \sum_{i=0}^r q_i \lambda^{-h_i} = 0 \\ \Leftrightarrow & \lambda \text{ is a root of the polynomial (2.3).} \end{aligned}$$

Define a function by F by

$$(2.5) \quad F(\lambda) = \lambda + \left(\sum_{i=0}^r a_i - 1 \right) - (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i \lambda^{-h_i} \right) - \delta \sum_{i=0}^r q_i \lambda^{-h_i}.$$

Since

$$\begin{aligned} \lim_{\lambda \rightarrow 0^+} F(\lambda) &= \left(\sum_{i=0}^r a_i - 1 \right) - (1-\delta)(\prod_{i=0}^r \beta_i) \left(\sum_{i=0}^r \alpha_i \lim_{\lambda \rightarrow 0^+} \lambda^{-h_i} \right) \\ &\quad - \delta \sum_{i=0}^r q_i \lim_{\lambda \rightarrow 0^+} \lambda^{-h_i} < 0 \end{aligned}$$

and $F(1) = \sum_{i=0}^r a_i - (1-\delta)(\prod_{i=0}^r \beta_i) - \delta \sum_{i=0}^r q_i > 0$, it follows from continuity of F that there exists a real number $\lambda_0 \in (0, 1)$ such that $F(\lambda_0) = 0$. Thus for any $K \in \mathbb{R}_0^+$, the sequence $K\lambda_0^n + \Lambda$ is a solution of (2.4). Let $K_0 = \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\}$. Then, $\{y_n\} = K_0\lambda_0^n + \Lambda$ is a solution of (2.4) and obviously we have $x_n \leq y_n$ for $n = -h_r, \dots, 0$. Therefore, by using the first part of the proof, we conclude that $x_n \leq y_n = K_0\lambda_0^n + \Lambda, n \in \mathbb{Z}^0$. \square

Remark 2.3. When we take $\delta = 1$ and let $p = \sum_{i=0}^r a_i$. By Theorem 2.2 we have the following result.

Theorem 2.4. Let $q_i, \Upsilon \in \mathbb{R}_0^+, h_i \in \mathbb{Z}^+, i = 1, \dots, r$, where $0 = h_0 < h_1 < \dots < h_r$ and $\sum_{i=0}^r q_i < p \leq 1$, and let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of real numbers satisfying the inequality

$$(2.6) \quad \Delta x_n \leq -px_n + \sum_{i=0}^r q_i x_{n-h_i} + \Upsilon, n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$(2.7) \quad x_n \leq \max\{0, x_0, x_{-h_1}, \dots, x_{-h_r}\} \lambda_0^n + \Lambda, n \in \mathbb{Z}^0,$$

where $\Lambda = (p - \sum_{i=0}^r q_i)^{-1} \Upsilon$. Moreover, λ_0 may be chosen as the smallest root of the polynomial

$$(2.8) \quad P(\lambda) = \lambda^{h_{r+1}} - (1 - p + q_0)\lambda^{h_r} - q_1\lambda^{h_r-h_1} - \dots - q_{r-1}\lambda^{h_r-h_{r-1}} - q_r$$

which lies in the interval $(0, 1)$.

Remark 2.5. Suppose that $\Upsilon = 0$ in Theorem 2.4. Then we get Theorem A (Udpin and Niamsup, [16, Theorem 2.1]).

Remark 2.6. By Lemma 2.1 and Theorem 2.2, we can obtain the following theorem.

Theorem 2.7. Let $a_i, q_i, \Upsilon \in \mathbb{R}_0^+, h_i \in \mathbb{Z}^0, i = 0, \dots, r - 1; a_r, q_r \in \mathbb{R}^+, h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \dots < h_r$. Let $\alpha_i, \beta_i \in \mathbb{R}^+, \sum_{i=0}^r \alpha_i = 1$ and $[(1 - \delta)\prod_{i=0}^r \beta_i + \delta \sum_{i=0}^r q_i] < \sum_{i=0}^r a_i \leq 1$, where $0 \leq \delta \leq 1$ is a constant. Also let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of nonnegative real numbers satisfying the inequality

$$(2.9) \quad \Delta x_n \leq \sum_{i=0}^r (\delta q_i x_{n-h_i} - a_i x_n) + (1 - \delta)(\prod_{i=0}^r \beta_i)(x_{n-h_i}^{\alpha_i}) + \Upsilon, n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$(2.10) \quad x_n \leq \max\{0, x_0, x_{-h_1}, \dots, x_{-h_r}\} \lambda_0^n + \Lambda, n \in \mathbb{Z}^0,$$

where $\Lambda = (\sum_{i=0}^r a_i - \delta \sum_{i=0}^r q_i - (1 - \delta)\prod_{i=0}^r \beta_i)^{-1} \Upsilon$. Moreover, λ_0 can be chosen as the root in the interval $(0, 1)$ of the equation

$$(2.11) \quad \lambda + (\sum_{i=0}^r a_i - 1) - (1 - \delta)(\prod_{i=0}^r \beta_i) (\sum_{i=0}^r \alpha_i \lambda^{-h_i}) - \delta \sum_{i=0}^r q_i \lambda^{-h_i} = 0.$$

Proof. By (2.9) and Lemma 2.1, we get (2.1). Then, all the conditions of Theorem 2.2 are satisfied. By Theorem 2.2, we can obtain Theorem 2.7. \square

Remark 2.8. Suppose that $\Upsilon = 0$ in Theorem 2.7. Then we get the main result of Theorem C (Agarwal, Kim and Sen, [1]).

Remark 2.9. When we take $\delta = 0$. By Theorem 2.7 we have the following result.

Theorem 2.10. *Let $a_i, \Upsilon \in \mathbb{R}_0^+, h_i \in \mathbb{Z}^0, i = 0, \dots, r - 1; a_r \in \mathbb{R}^+, h_r \in \mathbb{Z}^+,$ where $0 = h_0 < h_1 < \dots < h_r.$ Let $\alpha_i, \beta_i \in \mathbb{R}^+, \sum_{i=0}^r \alpha_i = 1$ and $\prod_{i=0}^r \beta_i < \sum_{i=0}^r a_i \leq 1.$ Also let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$(2.12) \quad \Delta x_n \leq - \sum_{i=0}^r a_i x_n + (\prod_{i=0}^r \beta_i)(x_{n-h_i}^{\alpha_i}) + \Upsilon, \quad n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$(2.13) \quad x_n \leq \max\{0, x_0, x_{-h_1}, \dots, x_{-h_r}\} \lambda_0^n + \Lambda, \quad n \in \mathbb{Z}^0,$$

where $\Lambda = (\sum_{i=0}^r a_i - \prod_{i=0}^r \beta_i)^{-1} \Upsilon.$ Moreover, λ_0 can be chosen as the root in the interval $(0, 1)$ of the equation

$$(2.14) \quad \lambda + (\sum_{i=0}^r a_i - 1) - (\prod_{i=0}^r \beta_i)(\sum_{i=0}^r \alpha_i \lambda^{-h_i}) = 0.$$

Remark 2.11. Suppose that $\Upsilon = 0$ in Theorem 2.10, then we get the main result of Theorem B (Udpin and Niamsup, [16, Theorem 2.2]).

3. Asymptotic behavior of discrete systems

The inequalities obtained in Section 2 can be widely applied to research the asymptotic behavior of delay discrete dynamic systems. To illustrate the validity, consider the following discrete dynamic systems.

$$(3.1) \quad \Delta x_n = -px_n + f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}),$$

where $n, h_i \in \mathbb{Z}^+, i = 1, \dots, r \in \mathbb{Z}^+, p > 0.$ For any initial string $\{x_{-r}, x_{-r+1}, \dots, x_0\},$ (3.1) has a unique solution which can be explicitly calculated [1, 16]. However, it is difficult to obtain the attracting set and the global asymptotic stability using that form of solution. The following results give the attracting set and the global asymptotic stability of (3.1) by using the inequalities derived in Section 2.

Definition 3.1. The set $S \subset \mathbb{R}$ is called a global attracting set of (3.1), if for any initial string $\{x_{-r}, x_{-r+1}, \dots, x_0\},$ the solution $\{x_n\}$ satisfies

$$\text{dist}(x_n, S) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $\text{dist}(\phi, S) = \inf_{\psi \in S} \rho(\phi, \psi)$ for $\phi \in \mathbb{R}, \rho(\cdot, \cdot)$ is any distance in $\mathbb{R}.$

Theorem 3.2. Assume that there exist $q_i, \Upsilon \in \mathbb{R}_0^+, h_i \in \mathbb{Z}^+, q_r \in \mathbb{R}^+$, where $\sum_{i=0}^r q_i < p \leq 1$ such that

$$(3.2) \quad |f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \sum_{i=0}^r q_i |x_{n-h_i}| + \Upsilon$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$. Then, there exists $\lambda_0 \in (0, 1)$ such that every solution $\{x_n\}$ of (3.1) satisfies

$$(3.3) \quad |x_n| \leq \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n + \left(p - \sum_{i=0}^r q_i \right)^{-1} \Upsilon, \quad n \in \mathbb{Z}^0,$$

where λ_0 is chosen as in Theorem 2.4. As a consequence,

$$S = \left\{ \phi \in \mathbb{R} \mid |\phi| \leq \left(p - \sum_{i=0}^r q_i \right)^{-1} \Upsilon \right\}$$

is a positive attracting set of (3.1).

Proof. As in [2], it is straightforward to show that every solution $\{x_n\}$ of (3.1) can be written in the form

$$(3.4) \quad x_n = x_0(1-p)^n + \sum_{i=0}^{n-1} (1-p)^{n-i-1} f(i, x_i, x_{i-h_1}, \dots, x_{i-h_r}), \quad n \in \mathbb{Z}^0.$$

By using (3.2), we obtain

$$(3.5) \quad |x_n| \leq |x_0|(1-p)^n + \sum_{i=0}^{n-1} (1-p)^{n-i-1} \left(\sum_{j=0}^r q_j |x_{i-h_j}| + \Upsilon \right), \quad n \in \mathbb{Z}^0.$$

For each $n = -h_r, \dots, 0$, let $v_n = |x_n|$ and for each $n \in \mathbb{Z}^+$, we let

$$(3.6) \quad v_n = |x_0|(1-p)^n + \sum_{i=0}^{n-1} (1-p)^{n-i-1} \left(\sum_{j=0}^r q_j |x_{i-h_j}| + \Upsilon \right).$$

Then, we have $|x_n| \leq v_n, n \in \mathbb{Z}^{-h_r}$, and hence,

$$(3.7) \quad \Delta v_n = -pv_n + \sum_{i=0}^r q_i |x_{n-h_i}| + \Upsilon \leq -pv_n + \sum_{i=0}^r q_i v_{n-h_i} + \Upsilon, \quad n \in \mathbb{Z}^0.$$

Therefore, by Theorem 2.4, we obtain

$$(3.8) \quad |x_n| \leq v_n \leq \left(\max_{-h_r \leq i \leq 0} \{v_i\} \right) \lambda_0^n + \Lambda = \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n + \Lambda, \quad n \in \mathbb{Z}^0,$$

where $\Lambda = \left(p - \sum_{i=0}^r q_i \right)^{-1} \Upsilon$, and λ_0 is chosen as in Theorem 2.4. This completes the proof of the theorem. \square

Remark 3.3. Suppose that $\Upsilon = 0$ in Theorem 3.2. Then we get the following corollary.

Corollary 3.4 (Udpin and Niamsup [16, Theorem 3.1]). *Assume that there exist $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^+$, $q_r \in \mathbb{R}^+$, where $\sum_{i=0}^r q_i < p \leq 1$ such that*

$$(3.9) \quad |f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \sum_{i=0}^r q_i |x_{n-h_i}|$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$. Then, there exists $\lambda_0 \in (0, 1)$ such that every solution $\{x_n\}$ of (3.1) satisfies

$$(3.10) \quad |x_n| \leq \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where λ_0 is chosen as in Theorem 2.4.

Theorem 3.5. *Assume that $0 < p \leq 1$. Let $q_i, \Upsilon \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r-1$; $q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \dots < h_r$. Let $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^r \alpha_i = 1$ and $[(1-\delta)\prod_{i=0}^r \beta_i + \delta \sum_{i=0}^r q_i] < p \leq 1$, where $0 \leq \delta \leq 1$ is a constant. If*

$$(3.11) \quad |f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \sum_{i=0}^r \delta q_i |x_{n-h_i}| + (1-\delta)(\prod_{i=0}^r \beta_i) |x_{n-h_i}|^{\alpha_i} + \Upsilon$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, then there exists $\lambda_0 \in (0, 1)$ such that, for every solution $\{x_n\}$ of equation (3.1),

$$(3.12) \quad |x_n| \leq \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n + \Lambda, \quad n \in \mathbb{Z}^0,$$

where $\Lambda = (p - \delta \sum_{i=0}^r q_i - (1-\delta)\prod_{i=0}^r \beta_i)^{-1} \Upsilon$, and λ_0 is chosen as in Theorem 2.7. As a consequence, $S = \{\phi \in \mathbb{R} \mid |\phi| \leq \Lambda\}$ is a positive attracting set of (3.1).

Proof. As in [2], it is straightforward to show that every solution x_n of (3.1) can be written in the form

$$(3.13) \quad x_n = x_0(1-p)^n + \sum_{i=0}^{n-1} (1-p)^{n-i-1} f(i, x_i, x_{i-h_1}, \dots, x_{i-h_r}), \quad n \in \mathbb{Z}^0.$$

By using (3.11), we obtain

$$(3.14) \quad |x_n| \leq |x_0|(1-p)^n + \sum_{i=0}^{n-1} (1-p)^{n-i-1} \left(\sum_{j=0}^r \delta q_j |x_{i-h_j}| + (1-\delta)(\prod_{j=0}^r \beta_j) |x_{i-h_j}|^{\alpha_i} + \Upsilon \right), \quad n \in \mathbb{Z}^0.$$

For each $n = -h_r, \dots, 0$, let $v_n = |x_n|$ and for each $n \in \mathbb{Z}^+$, we let

$$(3.15) \quad v_n = |x_0|(1-p)^n$$

$$+ \sum_{i=0}^{n-1} (1-p)^{n-i-1} \left(\sum_{j=0}^r \delta q_j |x_{i-h_j}| + (1-\delta) (\prod_{j=0}^r \beta_j) |x_{i-h_j}|^{\alpha_i} + \Upsilon \right).$$

Then, we have $|x_n| \leq v_n$, $n \in \mathbb{Z}^{-h_r}$, and hence,

$$\begin{aligned} \Delta v_n &= -pv_n + \sum_{i=0}^r \delta q_i |x_{n-h_i}| + (1-\delta) (\prod_{i=0}^r \beta_i) |x_{n-h_i}|^{\alpha_i} + \Upsilon \\ (3.16) \quad &\leq -pv_n + \sum_{i=0}^r \delta q_i v_{n-h_i} + (1-\delta) (\prod_{i=0}^r \beta_i) (v_{n-h_i}^{\alpha_i}) + \Upsilon, \quad n \in \mathbb{Z}^0. \end{aligned}$$

Therefore, by Theorem 2.7, we obtain

$$(3.17) \quad |x_n| \leq v_n \leq \left(\max_{-h_r \leq i \leq 0} \{v_i\} \right) \lambda_0^n + \Lambda = \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n + \Lambda, \quad n \in \mathbb{Z}^0,$$

where $\Lambda = (p - \delta \sum_{i=0}^r q_i - (1-\delta) \prod_{i=0}^r \beta_i)^{-1} \Upsilon$, and λ_0 is chosen as in Theorem 2.7. This completes the proof of the theorem. \square

Remark 3.6. When we take $\delta = 0$. By Theorem 3.5 we have the following result.

Theorem 3.7. *Assume that $0 < p \leq 1$. Let $\Upsilon \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r-1$; $h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \dots < h_r$. Let $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^r \alpha_i = 1$ and $\prod_{i=0}^r \beta_i < p \leq 1$. If*

$$(3.18) \quad |f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq (\prod_{i=0}^r \beta_i) |x_{n-h_i}|^{\alpha_i} + \Upsilon$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, then there exists $\lambda_0 \in (0, 1)$ such that, for every solution $\{x_n\}$ of equation (3.1),

$$(3.19) \quad |x_n| \leq \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n + \Lambda, \quad n \in \mathbb{Z}^0,$$

where $\Lambda = (p - \prod_{i=0}^r \beta_i)^{-1} \Upsilon$, and λ_0 is chosen as in Theorem 2.10. As a consequence, $S = \{\phi \in \mathbb{R} \mid |\phi| \leq \Lambda\}$ is a positive attracting set of (3.1).

Remark 3.8. When we take $\Upsilon = 0$. By Theorem 3.5 we have the following result.

Theorem 3.9 ([1, Theorem 3.2]). *Assume that $0 < p \leq 1$. Let $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r-1$; $a_r, q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \dots < h_r$. Let $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^r \alpha_i = 1$ and $[(1-\delta) \prod_{i=0}^r \beta_i + \delta \sum_{i=0}^r q_i] < p \leq 1$, where $0 \leq \delta \leq 1$ is a constant. If*

$$(3.20) \quad |f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \sum_{i=0}^r \delta q_i |x_{n-h_i}| + (1-\delta) (\prod_{i=0}^r \beta_i) |x_{n-h_i}|^{\alpha_i}$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, then there exists $\lambda_0 \in (0, 1)$ such that, for every solution $\{x_n\}$ of equation (3.1),

$$(3.21) \quad |x_n| \leq \left(\max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0,$$

λ_0 is chosen as in Theorem 2.7. As a consequence, the trivial solution of the equation (3.1) is globally asymptotically stable.

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