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COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF MEROMORPHIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In the present investigation, the author introduces two interesting subclasses of normalized meromorphic univalent functions $w = f(z)$ defined on $\tilde{\Delta} := \{z \in \mathbb{C} : 1 < |z| < \infty\}$ whose inverse $f^{-1}(w)$ is also univalent meromorphic in $\tilde{\Delta}$. Estimates for the initial coefficients are obtained for the functions in these new subclasses.

1. Introduction

Let Σ' denote the family of all meromorphic univalent functions of the form:

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{1.1}$$

defined on the domain $\tilde{\Delta} := \{z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}$ except for a simple pole at ∞ with residue 1. Let Σ'_0 be the subclass of Σ' for which $b_0 = 0$. It is well-known that every function $f \in \Sigma'$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \tilde{\Delta}),$$

and

$$f\left(f^{-1}(w)\right) = w \quad (M < |w| < \infty,\ M > 0).$$

If G is the inverse of a function $f \in \Sigma'$ (i.e., $G = f^{-1}$), then G has an expansion of the form

$$G(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} \tag{1.2}$$

in some neighborhood of $w = \infty$. A simple calculation shows that the function G, is given by

$$G(w) = f^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \cdots . \tag{1.3}$$

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Analogous to the bi-univalent analytic functions (for recent expository work on bi-univalent functions, see [2, 8, 9]), a function $f \in \Sigma'$ is said to be meromorphic bi-univalent in $\tilde{\Delta}$ if both f and G are univalent in $\tilde{\Delta}$. The class of all meromorphic bi-univalent functions is denoted by Σ'_b.

In literature, several authors were investigated the coefficient estimates of meromorphic univalent functions. For $f \in \Sigma'_0$, it follows from the area theorem that $|b_1| \leq 1$. Schiffer [5] obtained the sharp estimates $|b_2| \leq \frac{2}{3}$ for $f \in \Sigma'_0$. Duren [1] gave an elementary proof of the inequality $|b_n| \leq \frac{2}{n+1}$ for $f \in \Sigma'$ with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. For $G \in \Sigma'_0$, Springer [7] used variational methods to prove that

$$|B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2} \text{ and } |B_3| \leq 1$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \quad n = 3, 4, 5, \ldots.$$

Kubota [3] has proved that Springer conjecture is true for $n = 3, 4, 5$ by an elementary application of Grunsky's inequalities. Furthermore, for $G \in \Sigma'_0$, Schober [6] obtained sharp bounds for the coefficients $B_{2n-1},\ 1 \leq n \leq 7$.

The object of the present paper is to introduce two new subclasses of the function class Σ'_b and find estimates for the initial coefficients $b_0,\ b_1$ and b_2 for functions in these new subclasses.

We need the following lemma for our further investigation.

Lemma 1.1 ([4]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\Delta := \{z : z \in \mathbb{C}$ and $|z| < 1\}$ for which $\Re(h(z)) > 0$ where*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \Delta).$$

2. Coefficient bounds for the function class $M_{\Sigma'_b}(\alpha, \lambda)$

Definition 2.1. A function $f(z) \in \Sigma'_b$ given by (1.1) is said to be in the class $M_{\Sigma'_b}(\alpha, \lambda)$ if the following conditions are satisfied:

(2.1)
$$\left|\arg\left\{\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 1, z \in \tilde{\Delta})$$

and

(2.2)
$$\left|\arg\left\{\lambda\frac{wG'(w)}{G(w)} + (1-\lambda)\left(1 + \frac{wG''(w)}{G'(w)}\right)\right\}\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 1, w \in \tilde{\Delta}),$$

where the function G is given by (1.3).

For $\lambda = 1$, we denote the class $M_{\Sigma'_b}(\alpha, \lambda) = M_{\Sigma'_b}(\alpha)$.

We state and prove our main results.

Theorem 2.2. *Let* $f \in M_{\Sigma'_b}(\alpha, \lambda)$. *Then*

$$|b_0| \le \frac{2\alpha}{\lambda}, \tag{2.3}$$

$$|b_1| \le \frac{\alpha}{2\lambda - 1}\sqrt{(\alpha-2)^2 + \frac{4\alpha^2}{\lambda^2}}, \tag{2.4}$$

and

$$|b_2| \le \frac{2\alpha}{3(3\lambda-2)}\left[2\left\{\frac{6\alpha^2 - \lambda^2(\alpha^2 - 3\alpha + 2)}{3\lambda^2}\right\} + 3 - 2\alpha\right]. \tag{2.5}$$

Proof. Since $f \in M_{\Sigma'_b}(\alpha, \lambda)$, there exist two functions p and q such that

$$\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right) = (p(z))^\alpha \tag{2.6}$$

and

$$\lambda\frac{wG'(w)}{G(w)} + (1-\lambda)\left(1 + \frac{wG''(w)}{G'(w)}\right) = (q(w))^\alpha, \tag{2.7}$$

respectively, where $p(z)$ and $q(w)$ satisfy the inequalities $\Re(p(z)) > 0$ $(z \in \tilde{\Delta})$ and $\Re(q(w)) > 0$ $(w \in \tilde{\Delta})$.

Furthermore, the functions $p(z)$ and $q(w)$ have the forms:

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \tilde{\Delta})$$

and

$$q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \tilde{\Delta}).$$

By definition of f and G, we have

$$\begin{aligned}\lambda\frac{zf'(z)}{f(z)} &+ (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= 1 - \frac{\lambda b_0}{z} + \frac{\lambda b_0^2 + 2(1-2\lambda)b_1}{z^2} - \frac{\lambda b_0^3 - 3\lambda b_0 b_1 - 3(2-3\lambda)b_2}{z^3} + \cdots\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\lambda\frac{wG'(w)}{G(w)} + (1-\lambda)\left(1 + \frac{wG''(w)}{G'(w)}\right) \\ &= 1 + \frac{\lambda b_0}{w} + \frac{\lambda b_0^2 - 2(1-2\lambda)b_1}{w^2} + \frac{\lambda b_0^3 - 3(2-3\lambda)b_2 - 6(1-2\lambda)b_0 b_1}{w^3} + \cdots.\end{aligned} \tag{2.9}$$

A simple calculation shows

$$\begin{aligned}(p(z))^\alpha = 1 + \frac{\alpha c_1}{z} &+ \frac{\frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2}{z^2} \\ &+ \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3 + \alpha(\alpha-1)c_1c_2 + \alpha c_3}{z^3} + \cdots\end{aligned} \tag{2.10}$$

and

$$(q(w))^{\alpha} = 1 + \frac{\alpha d_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2}{w^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)d_1^3 + \alpha(\alpha-1)d_1d_2 + \alpha d_3}{w^3} + \cdots . \tag{2.11}$$

Using (2.8), (2.10) in (2.6) and (2.9), (2.11) in (2.7), we get

$$-\lambda b_0 = \alpha c_1, \tag{2.12}$$

$$\lambda b_0^2 + 2(1-2\lambda)b_1 = \frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2, \tag{2.13}$$

$$-\lambda b_0^3 + 3\lambda b_0 b_1 + 3(2-3\lambda)b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}c_1^3 + \alpha(\alpha-1)c_1c_2 + \alpha c_3, \tag{2.14}$$

$$\lambda b_0 = \alpha d_1, \tag{2.15}$$

$$\lambda b_0^2 - 2(1-2\lambda)b_1 = \frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2 \tag{2.16}$$

and

$$\lambda b_0^3 - 6(1-2\lambda)b_0b_1 - 3(2-3\lambda)b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}d_1^3 + \alpha(\alpha-1)d_1d_2 + \alpha d_3. \tag{2.17}$$

From (2.12) and (2.15), it follows that

$$b_0 = -\frac{\alpha c_1}{\lambda} = \frac{\alpha d_1}{\lambda} \quad (c_1 = -d_1) \tag{2.18}$$

and

$$b_0^2 = \frac{\alpha^2}{2\lambda^2}(c_1^2 + d_1^2). \tag{2.19}$$

As $\Re(p(z)) > 0$ in $\tilde{\Delta}$, the function $p(\frac{1}{z}) \in \mathcal{P}$. Similarly $q(\frac{1}{w}) \in \mathcal{P}$. So, the coefficients of $p(z)$ and $q(w)$ satisfy the inequality of Lemma 1.1. Applications of triangle inequality and followed by Lemma 1.1 in (2.19) give us the required estimates on b_0 as asserted in (2.3). Also, the estimates on b_0 follows from the direct consequence of (2.12).

By squaring and adding (2.13) and (2.16), using (2.19) in the computation leads to

$$b_1^2 = \frac{\alpha^2}{8(2\lambda-1)^2}\left[\frac{(\alpha-1)^2}{4}(c_1^4+d_1^4) + (c_2^2+d_2^2) + (\alpha-1)(c_1^2c_2 + d_1^2d_2) - \frac{\alpha^2}{2\lambda^2}(c_1^4 + d_1^4 + 2c_1^2d_1^2)\right],$$

which in turn yields the estimates on b_1 given in (2.4).

Finally, to determine the bounds on b_2, consider the sum of (2.14) and (2.17) with $c_1 = -d_1$, we have

$$b_0 b_1 = \frac{1}{3(5\lambda - 2)} \left[\alpha(\alpha - 1)c_1(c_2 - d_2) + \alpha(c_3 + d_3)\right]. \tag{2.20}$$

Subtracting (2.17) from (2.14) with $c_1 = -d_1$, we obtain

$$\begin{aligned} -6(3\lambda - 2)b_2 = 2\lambda b_0^3 + 3(3\lambda - 2)b_0 b_1 + \frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3 \\ + \alpha(\alpha - 1)c_1(c_2 + d_2) + \alpha(c_3 - d_3). \end{aligned} \tag{2.21}$$

Using (2.18) and (2.20) in (2.21) gives

$$\begin{aligned} \frac{6(3\lambda - 2)}{\alpha} b_2 = \frac{6\alpha^2 - \lambda^2(\alpha^2 - 3\alpha + 2)}{3\lambda^2} c_1^3 + \frac{4(1 - \alpha)(2\lambda - 1)}{5\lambda - 2} c_1 c_2 \\ + \frac{2\lambda(1 - \alpha)}{5\lambda - 2} c_1 d_2 - \frac{4(2\lambda - 1)}{5\lambda - 2} c_3 + \frac{2\lambda}{5\lambda - 2} d_3. \end{aligned}$$

Finally, an application of Lemma 1.1 for the above equation immediately yields the desired estimates on b_2 given by (2.5). The proof of Theorem 2.2 is thus completed. □

Taking $\lambda = 1$ in Theorem 2.2, we get the following results.

Corollary 2.3. *Let $f \in M_{\Sigma'_b}(\alpha)$. Then*

$$|b_0| \le 2\alpha,$$

$$|b_1| \le \alpha\sqrt{5\alpha^2 - 4\alpha + 4},$$

and

$$|b_2| \le \frac{10\alpha}{9}(2\alpha^2 + 1).$$

3. Coefficient bounds for the function class $\mathcal{T}_{\Sigma'_b}(\beta, \lambda)$

Definition 3.1. A function $f(z) \in \Sigma'_b$ given by (1.1) is said to be in the class $\mathcal{T}_{\Sigma'_b}(\beta, \lambda)$ if the following conditions are satisfied:

$$\Re\left\{\lambda \frac{zf'(z)}{f(z)} + (1 - \lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta \quad (0 \le \beta < 1,\ \lambda \ge 1,\ z \in \tilde{\Delta}) \tag{3.1}$$

and

$$\Re\left\{\lambda \frac{wG'(w)}{G(w)} + (1 - \lambda)\left(1 + \frac{wG''(w)}{G'(w)}\right)\right\} > \beta \quad (0 \le \beta < 1,\ \lambda \ge 1,\ w \in \tilde{\Delta}), \tag{3.2}$$

where the function G is the inverse of f given by (1.3).

For $\lambda = 1$, we use the notation:

$$\mathcal{T}_{\Sigma'_b}(\beta, \lambda) = \mathcal{T}_{\Sigma'_b}(\beta).$$

Theorem 3.2. *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{T}_{\Sigma'_b}(\beta,\lambda)$. Then*

$$|b_0| \le \frac{2(1-\beta)}{\lambda}, \tag{3.3}$$

$$|b_1| \le \frac{(1-\beta)}{2\lambda-1}\sqrt{1+\frac{4(1-\beta)^2}{\lambda^2}}, \tag{3.4}$$

and

$$|b_2| \le \frac{2(1-\beta)}{3(3\lambda-2)}\left[1+\frac{4(1-\beta)^2}{\lambda^2}\right]. \tag{3.5}$$

Proof. Let $f \in \mathcal{T}_{\Sigma'_b}(\beta,\lambda)$. Then, by definition of the class $\mathcal{T}_{\Sigma'_b}(\beta,\lambda)$,

$$\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)\left(1+\frac{zf''(z)}{f'(z)}\right) = \beta + (1-\beta)p(z) \tag{3.6}$$

and

$$\lambda\frac{wG'(w)}{G(w)} + (1-\lambda)\left(1+\frac{wG''(w)}{G'(w)}\right) = \beta + (1-\beta)q(w), \tag{3.7}$$

where p and q are as in Theorem 2.2.

Equating coefficients in (3.6) and (3.7) yield

$$-\lambda b_0 = (1-\beta)c_1, \tag{3.8}$$

$$\lambda b_0^2 + 2(1-2\lambda)b_1 = (1-\beta)c_2, \tag{3.9}$$

$$-\lambda b_0^3 + 3\lambda b_0 b_1 + 3(2-3\lambda)b_2 = (1-\beta)c_3, \tag{3.10}$$

and

$$\lambda b_0 = (1-\beta)d_1, \tag{3.11}$$

$$\lambda b_0^2 - 2(1-2\lambda)b_1 = (1-\beta)d_2, \tag{3.12}$$

$$\lambda b_0^3 - 3(2-3\lambda)b_2 - 6(1-2\lambda)b_0 b_1 = (1-\beta)d_3. \tag{3.13}$$

From (3.8) and (3.11), we get

$$c_1 = -d_1$$

and

$$b_0^2 = \frac{(1-\beta)^2}{2\lambda^2}(c_1^2+d_1^2). \tag{3.14}$$

An application of triangle inequality and Lemma 1.1 in (3.14) give the desired estimate on b_0 as asserted in (3.3). The estimate on b_0 also follows from the direct consequence of (3.8).

Next, to determine bound on b_1, squaring and adding (3.9) and (3.12), we obtain

$$8(1-2\lambda)^2 b_1^2 + 2\lambda^2 b_0^4 = (1-\beta)^2(c_2^2+d_2^2). \tag{3.15}$$

Using (3.14) in (3.15) gives

$$b_1^2 = \frac{1}{8(1-2\lambda)^2}\left[(1-\beta)^2(c_2^2+d_2^2) - \frac{(1-\beta)^4}{2\lambda^2\left(c_1^4+d_1^4+2c_1^2d_1^2\right)}\right].$$

An application of Lemma 1.1 in the above equation, yields the required estimate on b_1 as asserted in (3.4).

Finally, in order to obtain the bound on b_2, adding (3.10) and (3.13) yields

$$b_0b_1 = \frac{(1-\beta)}{3(5\lambda-2)}(c_3+d_3). \tag{3.16}$$

Subtracting (3.13) from (3.10), we obtain

$$-6(3\lambda-2)b_2 = 2\lambda b_0^3 + 3(3\lambda-2)b_0b_1 + (1-\beta)(c_3-d_3). \tag{3.17}$$

Using (3.8) and (3.16) in (3.17) lead to

$$b_2 = \frac{(1-\beta)}{3(3\lambda-2)}\left[\frac{(1-\beta)^2}{\lambda^2}c_1^3 - \frac{2(2\lambda-1}{5\lambda-2}c_3 + \frac{\lambda}{5\lambda-2}d_3\right], \tag{3.18}$$

which eventually leads to the desired estimates (3.5) on b_2. The proof of Theorem 3.2 is thus completed. □

Taking $\lambda = 1$ in Theorem 3.2, we the get the following result.

Corollary 3.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{T}_{\Sigma_b'}(\beta)$. Then*

$$|b_0| \le 2(1-\beta),$$
$$|b_1| \le (1-\beta)\sqrt{4\beta^2-4\beta+5}$$

and

$$|b_3| \le \frac{2(1-\beta)}{3}(4\beta^2-4\beta+5).$$

Remark 3.4. From the above discussion it is cleared that the estimates of b_0, b_1 and b_2 in Theorem 2.2 when $\alpha = 1$ is the same as the corresponding estimates in Theorem 3.2 when $\beta = 0$.

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