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Extensions of linearly McCoy rings

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Abstract. A ring R is called linearly McCoy if whenever linear polynomials $f(x),\ g(x) \in R[x]\backslash\{0\}$ satisfy $f(x)g(x) = 0$, there exist nonzero elements $r,\ s \in R$ such that $f(x)r = sg(x) = 0$. In this paper, extension properties of linearly McCoy rings are investigated. We prove that the polynomial ring over a linearly McCoy ring need not be linearly McCoy. It is shown that if there exists the classical right quotient ring Q of a ring R, then R is right linearly McCoy if and only if so is Q. Other basic extensions are also considered.

1. Introduction

All rings are associative with unity. For a ring R, the polynomial ring over R is denoted by $R[x]$ with x its indeterminate, and E_{ij} stands for the usual matrix unit (i.e., with 1 at (i,j)-entry and 0 elsewhere).

McCoy proved in 1942 [14] that if two polynomials annihilate each other over a commutative ring, then each polynomial has a nonzero annihilator in the base ring. Rege and Chhawchharia [16] and Nielsen [15] independently introduced the notion of a McCoy ring. A ring R is *right McCoy* if the equation $f(x)g(x) = 0$ with $f(x),\ g(x) \in R[x]\backslash\{0\}$ implies there exists a nonzero $r \in R$ such that $f(x)r = 0$; left McCoy rings are defined similarly. A ring R is called *McCoy* if it is both right and left McCoy. The class of McCoy rings contains the class of Armendariz rings (These rings are defined through the condition 'whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i,\ g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j [16]'). A ring R is *semi-commutative* provided $ab = 0$ implies $aRb = 0$ for $a, b \in R$. In [7] it was claimed that all semi-commutative rings were McCoy. However, Hirano's claim assumed that $R[x]$ is semi-commutative if R is semi-commutative, and this was shown to be false in [8]. In 2006, Nielsen [15] gave an example of semi-commutative ring which is not right McCoy. The concept of a linearly McCoy ring, which properly generalizes McCoy rings and semi-commutative rings, was introduced by Camillo and Nielsen [4] in 2008. Recall that a ring R

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is called (*right* or *left*) *linearly McCoy* if the McCoy condition holds for nonzero linear polynomials $f(x) = a_0 + a_1x,\ g(x) = b_0 + b_1x \in R[x]$. Related results on McCoy conditions can be found in [4, 5, 10, 12, 15, 16, 17], etc. Recently, the McCoy and the Armendariz conditions were extended to their module versions (see [3, 6]). Due to Lee and Wong [11], a ring R is called *weak Armendariz* (also called *linearly Armendariz* in literature) if for given $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x]$, $f(x)g(x) = 0$ implies that $a_ib_j = 0$ for each $i,\ j$. Weak Armendariz rings are clearly linearly McCoy; the falsity of the converse can be deduced from [9, Example 1.2(4)].

The polynomial extension property of rings plays an important role in ring theory. For rings that admit Armendariz or McCoy condition, it was proved in [1] (resp., [12]) that a ring R is Armendariz (resp., McCoy) if and only if $R[x]$ is Armendariz (resp., McCoy). But it is still an open question of whether the polynomial ring over a weak Armendariz ring is weak Armendariz (see [9]). In this paper, we show that the polynomial ring over a semi-commutative ring is not linearly McCoy. It is proved that if R is a linearly McCoy ring, then $R[x]/(x^n)$ is linearly McCoy for any integer $n \geq 1$; and if there exists a classical right quotient ring Q of a ring R, then R is right linearly McCoy if and only if Q is right linearly McCoy. Some other basic extensions of linearly McCoy rings are also considered.

2. Polynomial rings

In this section, we investigate the polynomial ring over a linearly McCoy ring. We first recall a fact in [4].

Lemma 2.1. *All semi-commutative rings are linearly McCoy.*

Anderson and Camillo [1] proved that a ring R is Armendariz if and only if $R[x]$ is Armendariz, and Lei et al. [12] showed the same property holds for McCoy rings. It is natural to consider whether polynomial rings over linearly McCoy rings are still linearly McCoy. Motivated by results in [15], we have the following result.

Theorem 2.2. *There exists a semi-commutative ring over which the polynomial ring is not linearly McCoy.*

Proof. Let $K = \mathbb{F}_2\langle a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3\rangle$ be the free associative algebra (with 1) over $\mathbb{F}_2$ generated by nine indeterminates (as labeled above). Let I be the ideal generated by the following relations:

$$\begin{aligned}\langle\ & a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1 + a_3b_0, a_2b_2 + a_3b_1, a_3b_2,\\ & a_0b_3 + a_4b_0, a_1b_3 + a_4b_1, a_2b_3 + a_4b_2, a_3b_3, a_4b_3, a_0a_j\ (0 \leq j \leq 4),\\ & a_3a_j\ (0 \leq j \leq 4), a_4a_j\ (0 \leq j \leq 4), a_1a_j + a_2a_j\ (0 \leq j \leq 4),\\ & b_ib_j\ (0 \leq i, j \leq 3), b_ia_j\ (0 \leq i \leq 3, 0 \leq j \leq 4)\rangle.\end{aligned}$$

Let $R = K/I$. Set $F(y) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4y$, $G(y) = b_0 + b_1x + b_2x^2 + b_3y$. Note that the first and the second rows of relations in I guarantee $F(y)G(y) = 0$ in $R[x][y]$.

The degree of each nonzero monomial $\alpha \in R$ is defined as the number of indeterminates in α, denote it by $\deg(\alpha)$; the degree of each element $\gamma = \sum_i r_i \in R$ is defined as $\deg(\gamma) = \max\{\deg(r_i) : i \in \mathbb{N}\}$, where r_i is the part of degree i in γ.

Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{F}_2$. Notice that H_n is finite for any n and the ideal I of R is homogeneous (i.e., if $\sum_{i=1}^{s} r_i \in I$ with $r_i \in H_i$, then $r_i \in I$). The proof will be divided into the following four claims.

Claim 1. Let $\gamma \in R$. Then

$$\begin{aligned}\gamma = f_0 &+ f_1(a_2)a_1 + f_2(a_2)a_2 + f_3(a_2)a_3 + f_4(a_2)a_4 + g(a_2)a_0\\ &+ [r_0 + r_1(a_2)a_1 + r_2(a_2)a_2 + r_3(a_2)a_3 + r_4(a_2)a_4]b_0\\ &+ [s_0 + s_1(a_2)a_1 + s_2(a_2)a_2 + s_3(a_2)a_3 + s_4(a_2)a_4]b_1\\ &+ [t_0 + t(a_2)a_4]b_2 + h_0b_3\end{aligned}$$

with $f_0, r_0, s_0, t_0, h_0 \in \mathbb{F}_2$ and $f_i, g, r_j, s_k, t \in \mathbb{F}_2[x]$ $(1 \le i, j, k \le 4)$, and the expression is unique.

Proof of Claim 1. We directly adopt the diamond lemma (see [2]), one may reduce any given monomial through the relations specified in the definition of I as follows.

Firstly, check whether the monomial we plan to reduce has any occurrence of a_0b_0, a_3b_2, a_3b_3, a_4b_3, a_0a_j, a_3a_j, a_4a_j, b_ib_j, b_ia_j. If so, then the monomial is zero. If not, repeatedly replace all occurrences of a_0b_1, a_0b_2, a_1b_2, a_2b_2, a_0b_3, a_1b_3, a_2b_3, a_1a_j with a_1b_0, $a_1b_1+a_2b_0$, $a_2b_1+a_3b_0$, a_3b_1, a_4b_0, a_4b_1, a_4b_2, a_2a_j, respectively. Then the resulting monomial will be in reduced form, and any element of R is just a sum of monomials. Hence, each element of R can be written uniquely as the form above. $\square$

Claim 2. The ring R is semi-commutative.

Proof of Claim 2. Let $\gamma,\ \gamma' \in R$ with $\gamma\gamma' = 0$, where $\gamma,\ \gamma'$ are written in the form of Claim 1. We write f_1 for $f_1(a_2)$, and do the same for other polynomials in the variable a_2. Then $\gamma = f_0 + f_1a_1 + f_2a_2 + \cdots + h_0b_3$, $\gamma' = f'_0 + f'_1a_1 + f'_2a_2 + \cdots + h'_0b_3$. Throughout we use the fact that I is a homogeneous ideal, so the sum of all monomials of any given degree in $\gamma\gamma'$ is zero.

For all $r \in R$, we prove that $\gamma r\gamma' = 0$. Clearly, it is true if either γ or γ' is zero. So we may assume that $\gamma,\ \gamma'$ are nonzero in R. $\gamma\gamma' = 0$ implies that $f_0f'_0 = 0$. So $f_0 = 0$ or $f'_0 = 0$. Suppose that $f_0 = 0$. Let $\lambda \neq 0$ be the sum of the (nonzero) terms of γ with lowest degree. Since I is homogeneous, $\lambda f'_0 = 0$, and so $f'_0 = 0$. Similarly, if $f'_0 = 0$ we obtain $f_0 = 0$. Thus, $f_0 = f'_0 = 0$.

Suppose that $\deg(r) = 1$. Notice that $b_i\gamma' = 0$ for $0 \le i \le 3$ since $f'_0 = 0$. Therefore $\gamma b_i\gamma' = 0$. So it suffices to check $\gamma a_j\gamma' = 0$ for $0 \le j \le 4$. By Claim 1, we get $\gamma a_j = (f_1 + f_2)a_2a_j$. So if $f_1 = f_2$, then we have $\gamma a_j\gamma' = 0$. In what follows, assume that $f_1 \neq f_2$. We show below that this contradicts the assumption $\gamma' \neq 0$.

Computing the reduced form of $\gamma\gamma'$ yields

$$\begin{aligned}
0 &= \gamma\gamma' \\
&= (f_1 + f_2)a_2(f'_1a_1 + f'_2a_2 + f'_3a_3 + f'_4a_4 + g'a_0) \\
&\quad + (r'_0f_1 + s'_0g + (f_1 + f_2)a_2r'_1)a_1b_0 + (r'_0f_2 + t'_0g + (f_1 + f_2)a_2r'_2)a_2b_0 \\
&\quad + (r'_0f_3 + t'_0f_1 + (f_1 + f_2)a_2r'_3)a_3b_0 + (r'_0f_4 + h'_0g + (f_1 + f_2)a_2r'_4)a_4b_0 \\
&\quad + (s'_0f_1 + t'_0g + (f_1 + f_2)a_2s'_1)a_1b_1 + (s'_0f_2 + t'_0f_1 + (f_1 + f_2)a_2s'_2)a_2b_1 \\
&\quad + (s'_0f_3 + t'_0f_2 + (f_1 + f_2)a_2s'_3)a_3b_1 + (s'_0f_4 + h'_0f_1 + (f_1 + f_2)a_2s'_4)a_4b_1 \\
&\quad + (t'_0f_4 + h'_0f_2 + (f_1 + f_2)a_2t')a_4b_2.
\end{aligned}$$

For ease of notation, we denote the coefficient of a_kb_l in $\gamma\gamma'$ by $\circledast_{a_kb_l}$. That is

$$\begin{aligned}
\circledast_{a_1b_0} &= r'_0f_1 + s'_0g + (f_1 + f_2)a_2r'_1, \\
\circledast_{a_2b_0} &= r'_0f_2 + t'_0g + (f_1 + f_2)a_2r'_2, \\
&\cdots \\
\circledast_{a_4b_2} &= t'_0f_4 + h'_0f_2 + (f_1 + f_2)a_2t'.
\end{aligned}$$

Since $f_1 + f_2 \neq 0$, it follows that $f'_1 = f'_2 = f'_3 = f'_4 = g' = 0$. From the last five lines of $\gamma\gamma'$ we obtain

$$\circledast_{a_ib_0} = \circledast_{a_ib_1} = \circledast_{a_4b_2} = 0, \quad i = 1, \ldots, 4.$$

Assume that $s'_0 = 1$. If $t'_0 = 1$, then Eq. $\circledast_{a_2b_1}$ implies $\deg((f_1 + f_2)a_2) \le \deg(f_1 + f_2)$, which is impossible since $f_1 \neq f_2$. So $t'_0 = 0$. But then adding Eqs. $\circledast_{a_1b_1}$ and $\circledast_{a_2b_1}$ causes the same contradiction.

Thus we have $s'_0 = 0$. If $t'_0 = 1$, then adding Eqs. $\circledast_{a_2b_1}$ and $\circledast_{a_3b_1}$ reaches the same contradiction as above. Therefore, $t'_0 = 0$.

Suppose that $r'_0 = 1$ and $h'_0 = 1$, then adding Eqs. $\circledast_{a_1b_0}$ and $\circledast_{a_2b_0}$, Eqs. $\circledast_{a_4b_1}$ and $\circledast_{a_4b_2}$, respectively. We also obtain the preceding contradiction. So $r'_0 = 0$ and $h'_0 = 0$. Because $f_1 \neq f_2$, we have $r'_i = s'_j = t' = 0$, where $1 \le i, j \le 4$. Hence $\gamma' = 0$, contradicting our previous assumption that $\gamma' \neq 0$.

This shows that $\gamma r\gamma' = 0$ if $\deg(r) = 1$. Repeating the above argument replacing γ by γr, $\gamma r\gamma' = 0$ also holds when r is a monomial of any positive degree. Clearly, if $r = 1$ then $\gamma r\gamma' = 0$. Since any element of R is just a sum of monomials, putting this all together yields $\gamma r\gamma' = 0$ for all $r \in R$. Therefore, R is a semi-commutative ring. $\square$

Due to Claim 1, one may check that $F(y),\ G(y) \neq 0$ in $R[x][y]$.

Claim 3. The polynomial ring $R[x]$ is not right linearly McCoy.

Proof of Claim 3. We conclude that $r_{R[x]}(a_4) = \{\sum_i k_i b_3 x^i : k_i \in \mathbb{F}_2,\ i \in \mathbb{N}\} + \sum_{j=0}^4 a_j R[x]$. Obviously, by the construction of the ideal I, each polynomial of the right side set in the equation above annihilates a_4 on the right. Meanwhile, for any $p(x) \in r_{R[x]}(a_4)$, write $p(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \cdots + \gamma_n x^n$ with $\gamma_i \in R$ written in the form as Claim 1. So we let $\gamma_i = f_0^{(i)} + f_1^{(i)} a_1 + \cdots + h_0^{(i)} b_3$, $0 \le i \le n$. Since $a_4 \gamma_i = 0$, we have $f_0^{(i)} = r_0^{(i)} = s_0^{(i)} = t_0^{(i)} = 0$. Thus $\gamma_i \in \{k b_3 : k \in \mathbb{F}_2\} + \sum_{j=0}^4 a_j R$, i.e., $p(x) \in \{\sum_i k_i b_3 x^i : k_i \in \mathbb{F}_2,\ i \in \mathbb{N}\} + \sum_{j=0}^4 a_j R[x]$.

For any $p(x) \in R[x]$, if $F(y)p(x) = 0$, then $p(x) \in r_{R[x]}(a_4)$ since $a_4 p(x) = 0$. So let $p(x) = (k_0 b_3 + k_0' p_0) + (k_1 b_3 + k_1' p_1)x + (k_2 b_3 + k_2' p_2)x^2 + \cdots + (k_n b_3 + k_n' p_n)x^n$, where $k_i, k_i' \in \mathbb{F}_2$, $p_l \in \sum_{j=0}^4 a_j R$, $0 \le l \le n$, and each p_l is written uniquely as Claim 1. If some p_{l_0} does not occur in the coefficients of $p(x)$ (*i.e.*, $p_{l_0} = 0$), then let $k_{l_0}' = 0$.

From the equation $F(y)p(x) = 0$, we also get $(a_0 + a_1 x + a_2 x^2 + a_3 x^3)p(x) = 0$. This implies the following system of equations:

(0) $a_0(k_0 b_3 + k_0' p_0) = 0$,
(1) $a_0(k_1 b_3 + k_1' p_1) + a_1(k_0 b_3 + k_0' p_0) = 0$,
(2) $a_0(k_2 b_3 + k_2' p_2) + a_1(k_1 b_3 + k_1' p_1) + a_2(k_0 b_3 + k_0' p_0) = 0$,
(3) $a_0(k_3 b_3 + k_3' p_3) + a_1(k_2 b_3 + k_2' p_2) + a_2(k_1 b_3 + k_1' p_1) + a_3(k_0 b_3 + k_0' p_0) = 0$,
$\vdots$
$(n+1)$ $a_1(k_n b_3 + k_n' p_n) + a_2(k_{n-1} b_3 + k_{n-1}' p_{n-1}) + a_3(k_{n-2} b_3 + k_{n-2}' p_{n-2}) = 0$.

Notice that $a_0 p_i = 0$ and $a_k p_i = a_k b_3 = 0$, where $k = 3, 4$; $i = 0, 1, \ldots, n$. So from Eq.(0) we have $k_0 = 0$, and Eq. (1) implies that $k_0' = 0,\ k_1 = 0$. Continuing this process, Eq.$(i+1)$ yields $k_i' = k_{i+1} = 0$ for $1 \le i \le n-1$, and we get $k_n' = 0$ from Eq.$(n+1)$. Thus, $p(x) = 0$, which implies that $R[x]$ is not right linearly McCoy. □

Claim 4. The ring R is left McCoy (so $R[x]$ is left linearly McCoy).

Proof of Claim 4. For completeness of the proof, we adapt the method used in [15, Claim 8]. Let $\alpha(x),\ \beta(x) \in R[x] \backslash \{0\}$ satisfy $\alpha(x)\beta(x) = 0$. Set $\alpha(x) = \sum_{i=0}^m p_i x^i,\ \beta(x) = \sum_{i=0}^n q_i x^i$. If each q_i has zero constant term, then $b_0 q_i = 0$, whence $b_0 \beta(x) = 0$, and we are done. Next we assume that there exists some q_i has a nonzero constant term. Let l_0 be the smallest index such that q_{l_0} satisfies this property.

For each $p_i \ne 0$, let p_i' be the sum of nonzero terms of p_i with smallest degree. And for $p_i = 0$, put $p_i' = 0$. Also, let k_0 be the smallest index such that, among the members of $\{p_0', p_1', \ldots, p_m'\} \backslash \{0\}$, we have p_{k_0}' with minimal degree, since $\alpha(x) \ne 0$, k_0 exists.

Notice that the degree $l_0 + k_0$ part of $\alpha(x)\beta(x) = 0$ we obtain

$$(*) \qquad \sum_{s,t: s+t = l_0 + k_0} p_s q_t = 0.$$

Since I is a homogeneous ideal, each term of any fixed degree in Eq. $(*)$ must add to zero. From our choice of k_0 and l_0, the term of smallest degree in Eq. $(*)$ is $p'_{k_0} \cdot 1 = p'_{k_0} \neq 0$, which comes from $p_{k_0} q_{l_0}$. So this causes a contradiction.

Hence R is a left McCoy ring. In view of [12, Theorem 1], the polynomial ring $R[x]$ is left linearly McCoy. □

This completes the proof of Theorem 2.2. □

By virtue of Theorem 2.2, a well-known result relating to semi-commutativity can be obtained (cf. [4, Example 8.6], [8, Example 2]).

Corollary 2.3. *The polynomial rings over semi-commutative rings need not be semi-commutative.*

Proposition 2.4. *Let R be a ring and Ω be a multiplicatively closed subset of R consisting of central regular elements. Then R is linearly McCoy if and only if $\Omega^{-1}R$ is linearly McCoy.*

Proof. "⇒". Let $f(x) = \alpha + \beta x,\ g(x) = \alpha' + \beta' x$ be nonzero elements of $\Omega^{-1}R[x]$ such that $f(x)g(x) = 0$, where $\alpha,\ \beta,\ \alpha',\ \beta' \in \Omega^{-1}R$. Then there exist $u,\ v \in \Omega$ such that $\alpha = u^{-1}a,\ \beta = u^{-1}b,\ \alpha' = v^{-1}c,\ \beta' = v^{-1}d$. Since Ω is contained in the center of R, we have $f(x)g(x) = u^{-1}(a+bx)v^{-1}(c+dx) = (uv)^{-1}(a+bx)(c+dx) = 0$. Set $f_1(x) = a + bx,\ g_1(x) = c + dx$. Obviously, $f_1(x),\ g_1(x) \in R[x]\backslash\{0\}$ and $f_1(x)g_1(x) = 0$. There exist nonzero $s,\ t \in R$ such that $f_1(x)s = tg_1(x) = 0$ since R is linearly McCoy. Then $f(x)\gamma = \delta g(x) = 0$, where $\gamma = w^{-1}s,\ \delta = w^{-1}t$ and nonzero $w \in \Omega$. Therefore, $\Omega^{-1}R$ is linearly McCoy.

"⇐". Suppose that $f(x) = a_0 + a_1 x$ and $g(x) = b_0 + b_1 x$ are nonzero elements of $R[x]$ with $f(x)g(x) = 0$. Also, $f(x),\ g(x) \in \Omega^{-1}R[x]\backslash\{0\}$. Because $\Omega^{-1}R$ is linearly McCoy, there exist $\alpha = u^{-1}a,\ \beta = v^{-1}b \in \Omega^{-1}R\backslash\{0\}$ such that $f(x)\alpha = \beta g(x) = 0$. It follows that $f(x)a = bg(x) = 0$. Thus R is linearly McCoy. □

The ring of *Laurent polynomials* in x, coefficients in a ring R, consisting of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Corollary 2.5. *For a ring R, $R[x]$ is linearly McCoy if and only if $R[x; x^{-1}]$ is linearly McCoy.*

Proof. Let $\Omega = \{1, x, x^2, \ldots\}$. Then Ω is a multiplicatively closed subset of $R[x]$ consisting entirely of central regular elements. Since $R[x; x^{-1}] = \Omega^{-1}R[x]$, by Proposition 2.4, we are done. □

3. Matrix rings and classical quotient rings

In this section, we study the property "linearly McCoy" of some subring of the upper triangular matrix ring; the trivial extension of a linearly McCoy ring and its classical quotient ring are also investigated.

Let R be a ring. We consider the ring

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} : a, a_{kl} \in R \right\},$$

where n (≥ 1) is a positive integer.

Proposition 3.1. *For any $n \geq 1$, a ring R is linearly McCoy if and only if the ring R_n is linearly McCoy.*

Proof. "$\Rightarrow$". Let $H(x) \in R_n[x]$. Then $H(x)$ can be expressed as the form of a matrix, and the (i, j)-entry of $H(x)$ is denoted by $h_{ij}(x) = [H(x)]_{i,j}$. Clearly, $h_{ij}(x) \in R[x]$.

Suppose that $F(x)$, $G(x)$ are nonzero linear polynomials of $R_n[x]$ with $F(x)G(x) = 0$. We show that there exist A, $B \in R_n\backslash\{0\}$ such that $F(x)A = BG(x) = 0$. Now we proceed with the following cases.

Case 1. If $f_{11}(x) \neq 0$, $g_{11}(x) \neq 0$, then $f_{11}(x)g_{11}(x) = 0$, where $f_{11}(x) = [F(x)]_{1,1}$, $g_{11}(x) = [G(x)]_{1,1}$. Since R is linearly McCoy, there exist s, $t \in R\backslash\{0\}$ such that $f_{11}(x)s = tg_{11}(x) = 0$. Put $A = sE_{1n}$, $B = tE_{1n}$. Then $F(x)A = BG(x) = 0$.

Case 2. If $f_{11}(x) \neq 0$, $g_{11}(x) = 0$, then there exists $g_{kl}(x) \neq 0$ satisfying $g_{(k+u)l}(x) = 0$ for some k, l and $1 \leq u \leq n - k$ since $G(x) \neq 0$. So $f_{11}(x)g_{kl}(x) = 0$. Hence there exists $s \in R\backslash\{0\}$ such that $f_{11}(x)s = 0$. Write $A = sE_{1n}$. Then $F(x)A = AG(x) = 0$.

Case 3. If $f_{11}(x) = 0$, $g_{11}(x) \neq 0$, then there exist A, $B \in R_n\backslash\{0\}$ such that $F(x)A = BG(x) = 0$. The proof is similar to Case 2.

Case 4. If $f_{11}(x) = 0$, $g_{11}(x) = 0$, then for any $s \in R\backslash\{0\}$, $F(x)A = AG(x) = 0$ with $A = sE_{1n}$.

Therefore, R_n is linearly McCoy.

"$\Leftarrow$". Assume that $f(x)g(x) = 0$, where $f(x)$, $g(x)$ are nonzero linearly polynomials of $R[x]$. Let $F(x) = f(x)E_n$, $G(x) = g(x)E_n$ with E_n the $n \times n$ identity matrix. Then $F(x)$, $G(x) \in R_n[x]\backslash\{0\}$ and $F(x)G(x) = 0$. Since R_n is linearly McCoy, there exist A, $B \in R_n\backslash\{0\}$ such that $F(x)A = BG(x) = 0$. Obviously, there exist nonzero a, $b \in R$ such that $f(x)a = bg(x) = 0$. So the proof is complete. □

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices

$$\begin{pmatrix} r & m \\ 0 & r \end{pmatrix},$$

where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 3.2. *A ring R is linearly McCoy if and only if the trivial extension $T(R,R)$ is linearly McCoy.*

However, the trivial extension $T(R,S)$ of a ring R by a ring S being right linearly McCoy does not imply that of S.

Example 3.3. Let K be a commutative ring, $R = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in K\}$ and $S = \{\left(\begin{smallmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{smallmatrix}\right) : a_{ij} \in K\}$. Then the ring R and the trivial extension $H = T(R,S)$ are right linearly McCoy, but S is not.

Proof. Since $R \cong K$, R is a linearly McCoy ring. But S is not right linearly McCoy by [4, Proposition 10.2]. It is easy to check that S is an (R,R)-bimodule. We next show that $H = T(R,S)$ is right linearly McCoy. Let

$$F(x) = \begin{pmatrix} \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} & \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ 0 & f_{22}(x) \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \end{pmatrix} \quad \text{and}$$

$$G(x) = \begin{pmatrix} \begin{pmatrix} g(x) & 0 \\ 0 & g(x) \end{pmatrix} & \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ 0 & g_{22}(x) \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} g(x) & 0 \\ 0 & g(x) \end{pmatrix} \end{pmatrix}$$

be nonzero linear polynomials of $H[x]$ with $F(x)G(x) = 0$, where $f(x) = a + a'x$, $f_{ij}(x) = a_{ij} + a'_{ij}x$, $g(x) = b + b'x$, $g_{ij}(x) = b_{ij} + b'_{ij}x \in K[x]$ and $1 \leq i \leq j \leq 2$. From $F(x)G(x) = 0$, we have $f(x)g(x) = 0$, $f(x)g_{11}(x) + f_{11}(x)g(x) = 0$, $f(x)g_{12}(x) + f_{12}(x)g(x) = 0$ and $f(x)g_{22}(x) + f_{22}(x)g(x) = 0$. For any $s \in K$, write $E_{12}(s) = \left(\begin{smallmatrix} O & \left(\begin{smallmatrix} s & s \\ 0 & s \end{smallmatrix}\right) \\ O & O \end{smallmatrix}\right) \in H$.

Case 1. If $f(x) = 0$, then for any nonzero $r \in K$, let $A = E_{12}(r)$. Then $F(x)E_{12}(r) = 0$.

Case 2. If $f(x) \neq 0$, $g(x) \neq 0$, then there exists $s \in K \backslash \{0\}$ such that $f(x)s = 0$ since $f(x)g(x) = 0$ and K is linearly McCoy. Let $B = E_{12}(s)$. Then $F(x)B = 0$.

Case 3. If $f(x) \neq 0$, $g(x) = 0$, then $f(x)g_{11}(x) = f(x)g_{12}(x) = f(x)g_{22}(x) = 0$. Note that $G(x) \neq 0$. Without loss of generality, we may assume that $g_{11}(x) \neq 0$. So there exists $t \in K \backslash \{0\}$ such that $f(x)t = 0$. Write $C = E_{12}(t)$. Then $F(x)C = 0$.

Hence, H is a right linearly McCoy ring. □

Remark 3.4. Based on Proposition 3.1, one may suspect that the matrix ring and the upper triangular matrix ring over a linearly McCoy ring are linearly McCoy. But it gives a negative answer by [4, Proposition 10.2]. So the linearly McCoy property is badly behaved with regards to Morita invariance.

In view of [5, Example 2.1], the class of linearly McCoy rings is not closed under homomorphic images. Nevertheless, we have the following theorem.

Theorem 3.5. *Let R be a ring and n any positive integer. If R is linearly McCoy, then $R[x]/(x^n)$ is a linearly McCoy ring, where (x^n) is the ideal generated by x^n.*

Proof. Denote $\overline{x}$ in $R[x]/(x^n)$ by u. Then $R[x]/(x^n) \cong R[u] = R + Ru + \cdots + Ru^{n-1}$, where u commutes with elements in R and $u^n = 0$.

Let $F(y) = f_0(u) + f_1(u)y$, $G(y) = g_0(u) + g_1(u)y$ be nonzero elements of $R[u][y]$ such that $F(y)G(y) = 0$, where $f_i(u) = \sum_{p=0}^{n-1} a_i^p u^p$ and $g_j(u) = \sum_{q=0}^{n-1} b_j^q u^q$ with $i, j = 0, 1$. Then

$$\begin{aligned}0 &= F(y)G(y)\\ &= (f_0(u) + f_1(u)y)(g_0(u) + g_1(u)y)\\ &= (\sum_{p=0}^{n-1} a_0^p u^p + \sum_{p=0}^{n-1} a_1^p u^p y)(\sum_{q=0}^{n-1} b_0^q u^q + \sum_{q=0}^{n-1} b_1^q u^q y)\\ &= [\sum_{p=0}^{n-1}(a_0^p + a_1^p y)u^p][\sum_{q=0}^{n-1}(b_0^q + b_1^q y)u^q].\end{aligned}$$

In particular, we have

$$(*) \qquad (a_0^0 + a_1^0 y)(b_0^k + b_1^k y) = 0$$

with minimal k such that $b_0^k + b_1^k y \neq 0$. Such k exists since $G(y) \neq 0$.

Assume that $a_0^0 = a_1^0 = 0$. Let $h(u) = u^{n-1}$. Then $f_0(u)h(u) = f_1(u)h(u) = 0$, whence $F(y)h(u) = 0$ since $u^n = 0$.

Suppose that $a_i^0 \neq 0$ for some i. Since R is linearly McCoy, Eq. $(*)$ implies that there exists $r \in R\backslash\{0\}$ such that $(a_0^0 + a_1^0 y)r = 0$. Let $h(u) = ru^{n-1}$. Then $f_i(u)h(u) = 0$ for $i = 0, 1$, and thus $F(y)h(u) = 0$.

Hence $R[x]/(x^n) \cong R[u]$ is right linearly McCoy. $R[x]/(x^n)$ is left linearly McCoy can be shown in the same manner. □

A *classical right quotient ring* for R is a ring Q which contains R as a subring in such a way that every regular element (i.e., non-zero-divisor) of R is invertible in Q and $Q = \{a\mu^{-1} : a,\ \mu \in R,\ \mu \text{ regular}\}$. The free algebra $L\langle x, y\rangle$ in two indeterminates over a field L is a well-known example of a domain which does not have a classical right quotient ring.

Theorem 3.6. *Suppose that there exists the classical right quotient ring Q of a ring R. Then R is right linearly McCoy if and only if Q is right linearly McCoy.*

Proof. "⇒". Let $f(x) = \alpha_0 + \alpha_1 x$ and $g(x) = \beta_0 + \beta_1 x \in Q[x]\backslash\{0\}$ satisfy $f(x)g(x) = 0$. By [13, Proposition 2.1.16], we may assume that $\alpha_i = a_i u^{-1}$, $\beta_j = b_j v^{-1}$ with a_i, $b_j \in R$ for $i, j = 0, 1$ and regular elements u, $v \in R$. For each j, there exist $c_j \in R$ and a regular element $w \in R$ such that $u^{-1}b_j = c_j w^{-1}$ also by [13, Proposition 2.1.16]. Denote $f_1(x) = a_0 + a_1 x$ and $g_1(x) = c_0 + c_1 x$. Then the equation

$$\begin{aligned} f_1(x)g_1(x)(vw)^{-1} &= \sum_{i=0}^{1}\sum_{j=0}^{1}(a_i c_j)(vw)^{-1}x^{i+j} \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} a_i(u^{-1}b_j)v^{-1}x^{i+j} \\ &= f(x)g(x) = 0 \end{aligned}$$

implies $f_1(x)g_1(x) = 0$. Since R is right linearly McCoy, there exists $s \in R\backslash\{0\}$ such that $f_1(x)s = 0$, i.e., $a_i s = 0$ for $i = 0, 1$. Then $\alpha_i(us) = a_i s = 0$ for every i, which implies that $f(x)(us) = 0$ and us is nonzero in Q. This proves that Q is right linearly McCoy.

"$\Leftarrow$". Let $f(x) = a_0 + a_1 x$, $g(x) = b_0 + b_1 x \in R[x]\backslash\{0\} \subseteq Q[x]\backslash\{0\}$ satisfy $f(x)g(x) = 0$. Then there exists $\alpha \in Q\backslash\{0\}$ such that $f(x)\alpha = 0$ since Q is right linearly McCoy. Because Q is a classical right quotient ring, we can take $\alpha = au^{-1}$ for some $a \in R\backslash\{0\}$ and regular element u. Then $f(x)au^{-1} = f(x)\alpha = 0$, implies that $f(x)a = 0$. Therefore, R is a right linearly McCoy ring. □

Goldie theorem reveals that if R is a semiprime two-sided Goldie ring, then R has the classical left and right quotient rings. Hence there exists a class of rings satisfying the following hypothesis.

Corollary 3.7. *Suppose that there exists the classical left and right quotient ring Q of a ring R. Then R is linearly McCoy if and only if Q is linearly McCoy.*

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