

SOME PROPERTIES OF EVOLUTION ALGEBRAS

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ABSTRACT. The paper is devoted to the study of finite dimensional complex evolution algebras. The class of evolution algebras isomorphic to evolution algebras with Jordan form matrices is described. For finite dimensional complex evolution algebras the criterium of nilpotency is established in terms of the properties of corresponding matrices. Moreover, it is proved that for nilpotent n -dimensional complex evolution algebras the possible maximal nilpotency index is $1 + 2^{n-1}$.

1. Introduction

In 20s and 30s of the last century a new object was introduced to mathematics, which was the product of interactions between Mendelian genetics and mathematics. Mendel established the basic laws for inheritance, which are summarized as Mendel's Law of Segregation and Mendel's Law of Independent Assortment. These laws were mathematically formulated by Serebrowsky [10], who was also the first to give an algebraic interpretation of the “ \times ” sign, which indicated sexual reproduction. Later Glivenkov [6] used the notion of Mendelian algebras in his work. Also Kostitzin [7] independently introduced a “symbolic multiplication” to express Mendel's laws. In his several papers Etherington [3]–[5] introduced the formal language of abstract algebra to the study of genetics. These algebras, in general, are non-associative.

However, in the beginning of the XX century in genetics there were discovered several examples of inheritances, where traits do not segregate in accordance with Mendel's laws. In the present day, non-Mendelian genetics is a basic language of molecular genetics. Non-Mendelian inheritance plays an important role in several disease processes. Naturally, the question arises: What non-Mendelian genetics offers to mathematics? The evolution algebras, introduced in [12] serve as the answer to this question.

The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebras; dynamically, they represent discrete dynamical systems. Evolution algebras

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have many connections with various branches of mathematics, such as graph theory, group theory, stochastic processes, mathematical physics etc. Since evolution algebras are not defined by identities, they can not belong to any well-known classes of non-associative algebras, as Lie, alternative and Jordan algebras. In recent years several authors have tried to investigate the evolution algebras (see [1], [2], [12]).

The foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed, with pointers to some further research topics in book [11].

In genetic, the action of genes is manifested statistically in sufficiently large communities of matching individuals (belonging to the same species). These communities are called *populations* [8]. The population exists not only in space but also in time, i.e., it has its own life cycle. The basis for this phenomenon is reproduction by mating. Mating in a population can be free or subject to certain restrictions.

The evolution of a population comprises a determined change of state in the next generations as a result of reproduction and selection. This evolution can be studied by using the properties of the evolution algebras. It is possible to characterize the disappearance of a population by means of the nilpotency and solvability of the algebra. More precisely, we identify the multiplication of algebra as the reproduction of the species. Hence the solvability is the reproduction of the same generations and the nilpotency is the reproduction of the species but not necessarily of the same generation. Both concepts allow us to study the disappearance of a population.

In this paper, we study some properties of finite dimensional complex evolution algebras. Since any evolution algebra in a natural basis is defined by a quadratic matrix, we study the connection between the algebraic structure of evolution algebras and matrices. More precise results are obtained for evolution algebras with non-singular matrices. For example, the only automorphisms for such algebras are the composition of basis permutation and the multiplication of basic vectors by scalars. Since in the matrix theory the Jordan form of the matrix is essential topic, we investigate a class of evolution algebras isomorphic to evolution algebras with Jordan form matrices. Thus we can distinguish the class of evolution algebras with a matrix in which the eigenvalues are known. Therefore, corresponding algebras can be investigated by the eigenvalues from an algebraical point of view. Namely, the problem of reconstruction of Markov chains on trees [9] which depends on the second eigenvalue can be studied by above evolution algebras.

In [5] it was pointed out for general genetic algebras that the nilpotent property is essential to these algebras and the definition as train algebras and baric algebras were formulated. By this means, we define nil, solvable, right-nilpotent and nilpotent evolution algebras as in [1] and study some properties of n -dimensional nilpotent evolution algebras. The notions as right nilpotency and nility for finite dimensional evolution algebras are equivalent [1]. In this work,

we prove that any n -dimensional right-nilpotent evolution algebra is nilpotent. Moreover, for evolution algebras of dimension n we describe some possible values for indexes of nilpotency and prove that $1+2^{n-1}$ is a maximal nilpotency index.

2. Preliminaries

Now we define the main object of the paper.

Definition 2.1 ([11]). Let E be a vector space over a field K with defined multiplication \cdot and a basis $\{e_1, e_2, \dots\}$ such that

$$e_i \cdot e_j = 0, \quad i \neq j,$$

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \quad i \geq 1,$$

then E is called an evolution algebra and basis $\{e_1, e_2, \dots\}$ is said to be natural basis.

From the above definition it follows that evolution algebras are commutative.

Let E be a finite dimensional evolution algebra with natural basis $\{e_1, \dots, e_n\}$, then

$$e_i \cdot e_i = \sum_{j=1}^n a_{ij} e_j, \quad 1 \leq i \leq n,$$

where remaining products are equal to zero.

The matrix $A = (a_{ij})$ with $1 \leq i, j \leq n$ is called matrix of the algebra E in natural basis $\{e_1, \dots, e_n\}$.

In [11] conditions for basis transformations that preserve naturalness of the basis are given. Also, the relation between the matrices in a new and old natural basis is established in terms of new defined operation on matrices. Since this approach is not practical for our further purposes, below we give the following brief version in terms of its matrix elements.

Now let us consider non-singular linear transformation of the given natural basis $\{e_1, \dots, e_n\}$ by matrix $T = (t_{ij})$ with $1 \leq i, j \leq n$:

$$f_i = \sum_{j=1}^n t_{ij} e_j, \quad 1 \leq i \leq n.$$

This transformation is an isomorphism if and only if $f_i \cdot f_j = 0$ for all $i \neq j$. Thus,

$$f_i \cdot f_j = \sum_{p=1}^n t_{ip} t_{jp} (e_p \cdot e_p) = \sum_{k=1}^n \left(\sum_{p=1}^n t_{ip} t_{jp} a_{pk} \right) e_k = 0.$$

Hence, if T is an isomorphism, then for $i \neq j$ and $1 \leq k \leq n$ we have

$$(2.1) \quad \sum_{p=1}^n t_{ip} t_{jp} a_{pk} = 0.$$

Observe that

$$f_i \cdot f_i = \sum_{p=1}^n t_{ip}^2 (e_p \cdot e_p) = \sum_{p=1}^n t_{ip}^2 \sum_{k=1}^n a_{pk} e_k = \sum_{k=1}^n \left(\sum_{p=1}^n t_{ip}^2 a_{pk} \right) e_k.$$

Now let T_{ij} be the elements of matrix T^{-1} . Then $e_k = \sum_{s=1}^n T_{ks} f_s$ and

$$f_i \cdot f_i = \sum_{k=1}^n \left(\sum_{p=1}^n t_{ip}^2 a_{pk} \right) \sum_{s=1}^n T_{ks} f_s = \sum_{s=1}^n \left(\sum_{k=1}^n \sum_{p=1}^n t_{ip}^2 a_{pk} T_{ks} \right) f_s.$$

Hence, for the elements of the matrix $B = (b_{is})$ with $1 \leq i, s \leq n$ of the evolution algebra E in natural basis $\{f_1, \dots, f_n\}$ we have

$$(2.2) \quad b_{is} = \sum_{k=1}^n \sum_{p=1}^n t_{ip}^2 a_{pk} T_{ks}.$$

Definition 2.2. An element a of the evolution algebra E is called nil if there exists $n(a) \in \mathbb{N}$ such that $\underbrace{(\dots((a \cdot a) \cdot a) \cdot \dots)}_{n(a) \text{ times}} \cdot a = 0$. The evolution algebra

E is called nil if any element of the algebra is nil.

We introduce the following sequences:

$$E^{(1)} = E, \quad E^{(k+1)} = E^{(k)} E^{(k)}, \quad k \geq 1,$$

$$E^{\langle 1 \rangle} = E, \quad E^{\langle k+1 \rangle} = E^{\langle k \rangle} E, \quad k \geq 1,$$

$$E^1 = E, \quad E^k = \sum_{i=1}^{k-1} E^i E^{k-i}, \quad k \geq 1.$$

Note that is not difficult to prove the following inclusions for $k \geq 1$:

$$E^{\langle k \rangle} \subseteq E^k, \quad E^{(k+1)} \subseteq E^{2^k}.$$

Also, note that since E is a commutative algebra we obtain

$$E^k = \sum_{1 \leq i \leq \frac{k}{2}} E^i E^{k-i}.$$

Definition 2.3. An evolution algebra E is called

- (i) solvable if there exists $n \in \mathbb{N}$ such that $E^{(n)} = 0$;
- (ii) right nilpotent if there exists $n \in \mathbb{N}$ such that $E^{\langle n \rangle} = 0$;
- (iii) nilpotent if there exists $n \in \mathbb{N}$ such that $E^n = 0$.

The minimal numbers s, r, n such that $E^{(s)}, E^{\langle r \rangle}, E^n$ are equal to zero are called indexes of solvability, right nilpotency and nilpotency, respectively.

Observe that if an evolution algebra is nilpotent, then it is right nilpotent and solvable. The following example shows that a solvable evolution algebra is not necessarily a right nilpotent algebra.

Proposition 2.4. *Let E be an n -dimensional complex evolution algebra such that $\dim E^{(2)} = 1$. Then $E^{(3)} = 0$ if and only if E is isomorphic to an evolution algebra with natural basis $\{e_1, \dots, e_n\}$ and $k \in \{1, \dots, n\}$, with the following multiplication:*

$$e_i e_i = \lambda_i (e_1 + \dots + e_k), \quad 1 \leq i \leq n,$$

where $\lambda_i \in \mathbb{C}$, $\sum_{j=1}^k \lambda_j = 0$, $\sum_{j=1}^n |\lambda_j| \neq 0$.

Proof. Since $\dim E^{(2)} = 1$ and $E^{(2)}$ is spanned by $e_i e_i$, $1 \leq i \leq n$ we obtain that they are collinear to a non-zero vector $x = a_1 e_1 + \dots + a_n e_n$. Let us suppose that there exist k elements, $\{a_{i_1}, \dots, a_{i_k}\}$, with $1 \leq k \leq n$ such that all of them are non-zeros. With the suitable natural basis change, one can assume that $x = e_1 + \dots + e_k$ for some $1 \leq k \leq n$.

Let $e_i e_i = \lambda_i x$, $1 \leq i \leq n$ and $\sum_{j=1}^n |\lambda_j| \neq 0$. Then $E^{(3)}$ is spanned by

$$xx = (e_1 + \dots + e_k)^2 = \sum_{j=1}^k \lambda_j x.$$

Hence, $E^{(3)} = 0$ if and only if $\sum_{j=1}^k \lambda_j = 0$. □

The following example is a special case of the above proposition:

Example 2.5. Let E be an evolution algebra with natural basis $\{e_1, \dots, e_n\}$ and the following multiplication:

$$\begin{aligned} e_i e_i &= e_1 + \dots + e_n, \quad 1 \leq i \leq n-1, \\ e_n e_n &= (1-n)(e_1 + \dots + e_n). \end{aligned}$$

Then $E^{(3)} = 0$, but $E^k = \langle e_1 + \dots + e_n \rangle$ for $k \geq 2$.

Remark 2.6. Actually, the multiplication obtained in Proposition 2.4 can be divided into two disjoint classes. First one, when $\lambda_i = 0$ for all $1 \leq i \leq k$, then this evolution algebra is nilpotent. The second one is when $\lambda_i \neq 0$ for some $1 \leq i \leq k$. Then by natural basis transformation one can assume that $e_1 e_1 = e_1 + \dots + e_k$ and hence, this evolution algebra is not nilpotent.

In [1] the equivalence of right nilpotency and nility for finite dimensional evolution algebras is proved.

Theorem 2.7 ([1]). *The following statements are equivalent:*

a) *The matrix of an evolution algebra E can be transformed by natural basis permutation to*

$$(2.3) \quad A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix};$$

- b) Evolution algebra E is right nilpotent algebra;
- c) Evolution algebra E is nil algebra.

3. Isomorphisms

In case of evolution algebras with non-singular evolution matrices the problem of finding isomorphic algebras to the given one can be solved more precisely.

Let E be an evolution algebra with matrix A such that $\det A \neq 0$.

Proposition 3.1. *If we define $T_\pi = (t_{ij})$, $1 \leq i, j \leq n$, such that $t_{ij} \neq 0$ if and only if $j = \pi(i)$ with $\pi \in S_n$, then $\text{Aut}(E) = \{T : T \text{ is of the form } T_\pi \text{ for some } \pi \in S_n\}$. Moreover, if T_π is an automorphism of evolution algebra E and $B = (b_{ij})$ with $1 \leq i, j \leq n$ is the matrix of E in basis $T_\pi(e_1), \dots, T_\pi(e_n)$, then*

$$(3.1) \quad b_{ij} = \frac{t_{i,\pi(i)}^2}{t_{j,\pi(j)}} \cdot a_{\pi(i)\pi(j)}.$$

Proof. Let T be an automorphism and $(t_{ij})_{1 \leq i, j \leq n}$ the matrix. Consider (2.1) as a linear homogeneous system of equations in terms of unknowns $t_{i1}t_{j1}, \dots, t_{in}t_{jn}$. If A is a non-singular matrix, then from (2.1) we obtain

$$\begin{cases} t_{i1}t_{j1} = 0 \\ t_{i2}t_{j2} = 0 \\ \dots \\ t_{in}t_{jn} = 0 \end{cases}$$

where $i \neq j$.

Since matrix $T = (t_{ij})$ with $1 \leq i, j \leq n$ is non-singular, in every row there is at least one non-zero element. But for any non-zero element t_{ip} (in the i -th row) we have $t_{ip}t_{jp} = 0$ for all $j \neq i$. Therefore, $t_{jp} = 0$ for $j \neq i$. Now if for some $m \neq p$ we have $t_{im} \neq 0$, then similarly, we obtain $t_{jm} = 0$ for all $j \neq m$. But this contradicts non-singularity of matrix T . Therefore, in every row and every column we have exactly one non-zero element, i.e., the matrix T has the form described in the statement of the proposition.

Note that $\det T = (-1)^{\sigma(\pi)} t_{1\pi(1)} t_{2\pi(2)} \cdots t_{n\pi(n)}$, where $\sigma(\pi)$ is the sign of π .

We obtain that the group of automorphisms of E is $\{T_\pi \mid \pi \in S_n\}$ and $T_\pi \circ T_\tau = T_{\tau \circ \pi}$.

Let us fix one $\pi \in S_n$. Then $T(e_i) = t_{i,\pi(i)} e_{\pi(i)}$ for all $1 \leq i \leq n$. Now

$$\begin{aligned} T(e_i) \cdot T(e_i) &= t_{i,\pi(i)}^2 (e_{\pi(i)} \cdot e_{\pi(i)}) = t_{i,\pi(i)}^2 \sum_{k=1}^n a_{\pi(i)k} e_k \\ &= t_{i,\pi(i)}^2 \sum_{k=1}^n a_{\pi(i)\pi(k)} e_{\pi(k)} = \sum_{k=1}^n \frac{t_{i,\pi(i)}^2}{t_{k,\pi(k)}} a_{\pi(i)\pi(k)} T(e_k). \end{aligned}$$

Hence, the elements of evolution matrix $B = (b_{ij})_{1 \leq i, j \leq n}$ of an algebra isomorphic to E satisfy (3.1). \square

For a $\pi \in S_n$ denote by $s_\pi : \{1, 2, \dots, n\} \setminus \{\pi^{-1}(n)\} \rightarrow \{1, 2, \dots, n\}$ a one-to-one mapping defined by $s_\pi(i) = \pi^{-1}(1 + \pi(i))$.

Proposition 3.2. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix of an evolution algebra isomorphic to an evolution algebra with Jordan cell matrix with non-zero eigenvalue λ . Then the only non-zero elements of A are the diagonal elements and $a_{i, s_\pi(i)}$ for all $i \neq \pi^{-1}(n)$ and $\lambda = \frac{a_{ii}^2}{a_{i, s_\pi(i)} a_{s_\pi(i) s_\pi(i)}}$ for all $i \neq \pi^{-1}(n)$.*

Proof. First consider the isomorphism of evolution algebra with Jordan cell matrix with non-zero eigenvalue λ . Since the matrix is non-singular, by the proof of Proposition 3.1 we obtain that it is in the form T_π .

For fixed $\pi \in S_n$ we put $T_\pi(e_i) = f_i$ and derive

$$f_i \cdot f_i = \begin{cases} t_{i, \pi(i)} \lambda f_i + \frac{t_{i, \pi(i)}^2}{t_{s_\pi(i), \pi(s_\pi(i))}} f_{s_\pi(i)} & \text{for } i \neq \pi^{-1}(n) \\ t_{i, \pi(i)} \lambda f_i & \text{for } i = \pi^{-1}(n). \end{cases}$$

Hence the matrix of the new evolution algebra is a sum of non-singular diagonal matrix and a matrix that has exactly one non-zero element on each row except the $\pi^{-1}(n)$ -th, which is a zero row and at most one non-zero element in each column.

Now let us fix a permutation $\pi \in S_n$ and consider matrix $A = (a_{ij})_{i, j=1}^n$ with zero elements except the diagonal elements and $a_{i, s_\pi(i)}$ for all $i \neq \pi^{-1}(n)$ and $s_\pi(i) = \pi^{-1}(1 + \pi(i))$. If this evolution algebra is isomorphic to an evolution algebra with Jordan cell matrix with eigenvalue λ , then $a_{ii} = \lambda t_{i, \pi(i)}$ for all $1 \leq i \leq n$ and $a_{i, s_\pi(i)} = \frac{t_{i, \pi(i)}^2}{t_{s_\pi(i), \pi(s_\pi(i))}}$.

Since $t_{i, \pi(i)} = \frac{1}{\lambda} a_{ii}$ and $t_{s_\pi(i), \pi(s_\pi(i))} = \frac{1}{\lambda} a_{s_\pi(i) s_\pi(i)}$ we obtain

$$a_{i, s_\pi(i)} = \frac{a_{ii}^2}{\lambda^2} \cdot \frac{\lambda}{a_{s_\pi(i) s_\pi(i)}} = \frac{1}{\lambda} \cdot \frac{a_{ii}^2}{a_{s_\pi(i) s_\pi(i)}} \text{ and hence } \lambda = \frac{a_{ii}^2}{a_{i, s_\pi(i)} a_{s_\pi(i) s_\pi(i)}}.$$

Hence, if matrix A satisfies $\lambda = \frac{a_{ii}^2}{a_{i, s_\pi(i)} a_{s_\pi(i) s_\pi(i)}}$ for all $i \neq \pi^{-1}(n)$, then evolution algebra with matrix A is isomorphic to evolution algebra with Jordan cell matrix with eigenvalue λ . This isomorphism has the matrix which is the inverse to $T = (t_{ij})_{i, j=1}^n$, where $t_{i\pi(i)} = \frac{1}{\lambda} a_{ii}$ and zero otherwise. \square

The above result can be generalized to the case of Jordan form matrices. Let $J = J_1 \oplus J_2 \oplus \dots \oplus J_r$, where J_i are Jordan cells of dimension n_i with non-zero eigenvalue λ_i .

Now let us denote $\mu_k = \lambda_i$ for $n_1 + \dots + n_{i-1} + 1 \leq k \leq n_1 + \dots + n_i$, $1 \leq i \leq r$.

Take $\pi \in S_n$ and denote $s'_\pi : \{1, \dots, n\} \setminus \{\pi^{-1}(n_1), \dots, \pi^{-1}(n_r)\} \rightarrow \{1, \dots, n\}$ a one-to-one mapping defined by $s'_\pi(i) = \pi^{-1}(1 + \pi(i))$.

Corollary 3.3. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix of an evolution algebra isomorphic to an evolution algebra with Jordan form matrix J . Then the only*

non-zero elements of A are the diagonal elements and $a_{i,s'_\pi(i)}$ such that

$$\frac{a_{ii}^2}{a_{i,s'_\pi(i)}a_{s'_\pi(i)s'_\pi(i)}} = \frac{\mu_i^2}{\mu_{s'_\pi(i)}}$$

for $i \notin \{\pi^{-1}(n_1), \dots, \pi^{-1}(n_r)\}$.

4. Nilpotency of evolution algebras

Let us now consider an evolution algebra E with Jordan cell with eigenvalue λ .

Proposition 4.1. *If $\lambda \neq 0$, then E is neither solvable nor right nilpotent and therefore is not nilpotent.*

Proof. Since $\lambda \neq 0$ then evolution matrix is non-degenerated. Therefore, $E^2 = E^{(2)} = E^{(2)} = E$. By simple induction we obtain $E^k = E^{(k)} = E^{(k)} = E$ and the statement of the proposition is verified. \square

Proposition 4.2. *For an evolution algebra with Jordan cell matrix and eigenvalue $\lambda = 0$ the following statements hold:*

- (i) E is one generated;
- (ii) E is solvable with index of solvability $n + 1$;
- (iii) E is right nilpotent with index of right nilpotency $n + 1$;
- (iv) E is nilpotent with index of nilpotency $2^{n-1} + 1$.

Proof. (i) From $\lambda = 0$ it follows that for basis elements e_i of E we have $e_i \cdot e_i = e_{i+1}$ for all $1 \leq i \leq n - 1$ and $e_n \cdot e_n = 0$.

Therefore, E is one-generated: $E = \text{id}\langle e_1 \rangle$.

(ii) It follows from $E^{(k)} = \langle e_k, e_{k+1}, \dots, e_n \rangle$, $1 \leq k \leq n$ and $E^{(n+1)} = 0$.

(iii) It is similar to (ii).

(iv) We claim that

$$E^{2^k+1} = E^{2^k+2} = \dots = E^{2^{k+1}} = \langle e_{k+2}, \dots, e_n \rangle$$

for all $0 \leq k \leq n - 2$.

Indeed, for $k = 0$ we have $E^2 = EE = \langle e_2, \dots, e_n \rangle$.

For $k = 1$ we have

$$E^{2+1} = E^3 = EE^2 = \langle e_3, \dots, e_n \rangle,$$

$$E^{2^2} = E^4 = EE^3 + E^2E^2 = \langle e_3, \dots, e_n \rangle.$$

Assume that

$$E^{2^{k-1}+1} = E^{2^{k-1}+2} = \dots = E^{2^k} = \langle e_{k+1}, \dots, e_n \rangle.$$

Using this assumption we obtain

$$\begin{aligned} E^{2^k+1} &= EE^{2^k} + E^2E^{2^k-1} + \dots + E^{2^{k-1}}E^{2^{k-1}+1} \\ &= EE^{2^k} + E^2E^{2^k} + \dots + E^{2^{k-1}}E^{2^k} \end{aligned}$$

$$= (E + E^2 + E^3 + \cdots + E^{2^{k-1}})E^{2^k} = EE^{2^k} = \langle e_{k+2}, \dots, e_n \rangle.$$

Also

$$\begin{aligned} E^{2^{k+1}} &= EE^{2^{k+1}-1} + E^2 E^{2^{k+1}-2} + \cdots + E^{2^k} E^{2^k} \\ &\supseteq E^{2^k} E^{2^k} = \langle e_{k+2}, \dots, e_n \rangle. \end{aligned}$$

So we obtain

$$\langle e_{k+2}, \dots, e_n \rangle = E^{2^k+1} \supseteq E^{2^k+2} \supseteq \cdots \supseteq E^{2^{k+1}} \supseteq \langle e_{k+2}, \dots, e_n \rangle.$$

Hence,

$$E^{2^k+1} = E^{2^k+2} = \cdots = E^{2^{k+1}} = \langle e_{k+2}, \dots, e_n \rangle.$$

Therefore, $E^{2^{n-1}} = \langle e_n \rangle$ and $E^{2^{n-1}+1} = 0$.

Hence, E is nilpotent with nilpotency index equal to $1 + 2^{n-1}$. \square

Remark 4.3. We should note that the statements (ii)-(iv) of Proposition 4.2 are equivalent, since one can show that each of them is equivalent to $\lambda = 0$. However, statement (i) is not equivalent to $\lambda = 0$ since for $\lambda = 1$ one can prove that E is generated by the element $e_1 + e_2$.

Observe that any evolution subalgebra of an evolution algebra is an ideal [11]. Therefore if we consider an evolution algebra E_J with matrix J in Jordan form $J = J_1 \oplus J_2 \oplus \cdots \oplus J_r$, where J_i are Jordan cells of dimension n_i with eigenvalues λ_i , then

$$E_J = E_1 \oplus E_2 \oplus \cdots \oplus E_r,$$

where $E_i = \langle e_{n_{i-1}+1}, \dots, e_{n_i} \rangle$.

Now we have $E_J^k = E_1^k \oplus E_2^k \oplus \cdots \oplus E_r^k$ and therefore E_J is nilpotent if and only if every E_i is nilpotent. Since we have obtained the criterium of nilpotency of Jordan blocks, we obtain:

Corollary 4.4. *E_J is nilpotent (with index of nilpotency equal to $\max_{1 \leq i \leq r} \{1 + 2^{n_i-1}\}$) if and only if J has only zero eigenvalues. The same assertion holds for right nilpotency and solvability with corresponding indexes equal to $1 + \max_{1 \leq i \leq r} \{n_i\}$.*

Note that from Corollary 4.4 it follows that for every $1 \leq k \leq n$ we obtain an example of nilpotent evolution algebra with index of nilpotency equal to $1 + 2^{k-1}$.

The following theorem represents the criterium of nilpotency of a finite dimensional evolution algebra.

Theorem 4.5. *Let E be an n -dimensional evolution algebra. Then E is nilpotent if and only if the matrix of evolution algebra A can be transformed by the natural basis permutation to form (2.3). Moreover, the index of nilpotency of the evolution algebra E is not greater than $2^{n-1} + 1$.*

Proof. Let E be a nilpotent. Then it is right nilpotent and therefore, by Theorem 2.7 the matrix of this evolution algebra can be transformed by the natural basis permutation to form (2.3).

Now let the matrix A of E can be transformed by the natural basis permutation to form (2.3).

Assume that $a_{12}a_{23} \cdots a_{n-1n} \neq 0$. Similar to the proof of (iv) in Proposition 4.2 one can verify

$$E^{2^k+1} = E^{2^k+2} = \cdots = E^{2^{k+1}} = \langle e_{k+2}, \dots, e_n \rangle$$

for all $0 \leq k \leq n-2$.

Therefore, $E^{2^{n-1}} = \langle e_n \rangle$ and $E^{2^{n-1}+1} = 0$.

Hence, E is nilpotent with nilpotency index equal to $1 + 2^{n-1}$.

Now assume that $a_{12}a_{23} \cdots a_{n-1n} = 0$. In this case we claim that

$$\langle e_{k+2}, \dots, e_n \rangle \supseteq E^{2^k+1}$$

for all $0 \leq k \leq n-2$.

Indeed, for $k=0$ we have $E^2 = EE \subseteq \langle e_2, \dots, e_n \rangle$.

For $k=1$ we have

$$E^{2+1} = E^3 = EE^2 \subseteq \langle e_3, \dots, e_n \rangle.$$

Assume that

$$\langle e_{k+1}, \dots, e_n \rangle \supseteq E^{2^{k-1}+1}.$$

Using this assumption we obtain

$$\begin{aligned} E^{2^k+1} &= EE^{2^k} + E^2E^{2^k-1} + \cdots + E^{2^{k-1}}E^{2^{k-1}+1} \\ &\subseteq EE^{2^{k-1}+1} + E^2E^{2^{k-1}+1} + \cdots + E^{2^{k-1}}E^{2^{k-1}+1} \\ &= (E + E^2 + E^3 + \cdots + E^{2^{k-1}})E^{2^{k-1}+1} \subseteq EE^{2^{k-1}+1} \subseteq \langle e_{k+2}, \dots, e_n \rangle. \end{aligned}$$

So we obtain

$$\langle e_{k+2}, \dots, e_n \rangle \supseteq E^{2^k+1}.$$

Therefore, $\langle e_n \rangle \supseteq E^{2^{n-2}+1}$.

Hence,

$$\begin{aligned} E^{2^{n-1}+1} &= EE^{2^{n-1}} + E^2E^{2^{n-1}-1} + \cdots + E^{2^{n-2}}E^{2^{n-2}+1} \\ &\subseteq (E + E^2 + \cdots + E^{2^{n-2}})\langle e_n \rangle \subseteq E\langle e_n \rangle = 0. \end{aligned}$$

Thus, E is nilpotent with nilpotency index not greater than $1 + 2^{n-1}$. \square

Corollary 4.6. *For finite dimensional complex evolution algebra notions as nil, nilpotent and right nilpotent algebras are equivalent. However, the indexes of nility, right nilpotency and nilpotency do not coincide in general.*

The following proposition excludes significantly many possible values that a nilpotency indexes of n -dimensional evolution algebras can take.

Proposition 4.7. *Let E be a nilpotent evolution algebra with index of nilpotency not equal to $2^{n-1} + 1$. Then it is not greater than $2^{n-2} + 1$.*

Proof. Since E is nilpotent, we assume that the matrix A of E in the natural basis $\{e_1, \dots, e_n\}$ is in the form (2.3).

From the proof of Proposition 4.5 it follows that $a_{12}a_{23} \dots a_{n-1n} = 0$ and

$$\langle e_{k+2}, \dots, e_n \rangle \supseteq E^{2^k+1}$$

for all $0 \leq k \leq n-2$.

Assume that E is nilpotent with index of nilpotency greater than $2^{n-2} + 1$ and not equal to $2^{n-1} + 1$.

Then $\langle e_n \rangle \supseteq E^{2^{n-2}+1}$ and since $E^{2^{n-2}+1} \neq 0$ we obtain $E^{2^{n-2}+1} = \langle e_n \rangle$.

Therefore, $\langle e_{n-1}, e_n \rangle \supseteq E^{2^{n-3}+1} \supseteq E^{2^{n-3}+2} \supseteq \dots \supseteq E^{2^{n-2}} \supseteq \langle e_n \rangle$.

Now if $E^{2^{n-3}+1} = E^{2^{n-3}+2} = \dots = E^{2^{n-2}} = \langle e_n \rangle$, then

$$\begin{aligned} E^{2^{n-2}+1} &= EE^{2^{n-2}} + E^2E^{2^{n-2}-1} + \dots + E^{2^{n-3}}E^{2^{n-3}+1} \\ &\subseteq (E + E^2 + \dots + E^{2^{n-3}})\langle e_n \rangle = E\langle e_n \rangle = 0 \end{aligned}$$

which is a contradiction. Hence, $\langle e_{n-1}, e_n \rangle = E^{2^{n-3}+1}$.

Now assume that $\langle e_{n-k}, \dots, e_n \rangle = E^{2^{n-k-2}+1}$. Then

$$\begin{aligned} \langle e_{n-k-1}, e_{n-k}, \dots, e_n \rangle &\supseteq E^{2^{n-k-3}+1} \supseteq E^{2^{n-k-3}+2} \\ &\supseteq \dots \supseteq E^{2^{n-k-2}} \supseteq \langle e_{n-k}, \dots, e_n \rangle. \end{aligned}$$

If $E^{2^{n-k-3}+1} \neq \langle e_{n-k-1}, e_{n-k}, \dots, e_n \rangle$, then $E^{2^{n-k-3}+1} = E^{2^{n-k-3}+2} = \dots = E^{2^{n-k-2}} = \langle e_{n-k}, \dots, e_n \rangle$ and

$$\begin{aligned} E^{2^{n-k-2}+1} &= EE^{2^{n-k-2}} + \dots + E^{2^{n-k-3}}E^{2^{n-k-3}+1} \\ &= E\langle e_{n-k}, e_{n-k}, \dots, e_n \rangle \subseteq \langle e_{n-k+1}, e_{n-k}, \dots, e_n \rangle \end{aligned}$$

which contradicts $\langle e_{n-k}, \dots, e_n \rangle = E^{2^{n-k-2}+1}$.

Hence this assumption is true and therefore $E^2 = \langle e_2, \dots, e_n \rangle$ which is also a contradiction $a_{12} \dots a_{n-1n} = 0$. \square

The following example shows that there exist evolution algebras with index of nilpotency greater than $1 + 2^{k-3}$ and less than $1 + 2^{k-2}$ for all $4 \leq k \leq n$.

Example 4.8. Let n and k be positive integers such that $4 \leq k \leq n$. Consider an evolution algebra E_k with basis $\{e_1, \dots, e_n\}$ and the following multiplication table:

$$\begin{aligned} e_1 \cdot e_1 &= e_2 + e_3 + \dots + e_k, \\ e_2 \cdot e_2 &= -e_4, \\ e_i \cdot e_i &= e_{i+1}, & 3 \leq i \leq k-1. \end{aligned}$$

Then one can show that $E_k^{3 \cdot 2^i} = \langle e_{4+i}, \dots, e_k \rangle$ for $0 \leq i \leq k-4$ and index of nilpotency of this algebra is $1 + 3 \cdot 2^{k-4}$.

Now we will consider a nilpotent evolution algebra with matrix (2.3) and a condition $\dim E^2 = n - 2$. Then $\text{rank} A = n - 2$. This implies that there are $1 \leq i \leq n - 1$ and $2 \leq j \leq n$ such that i -th row is linear dependent to other rows and j -th column is linear dependent to other columns.

The following proposition presents a relation between the dimension of E^2 and E^3 . In genetics, this property gives information on behavior of population of species from second and third generations.

Proposition 4.9. *Let $\dim E^2 = n - 2$ and i -th row ($1 \leq i \leq n - 1$) is linear dependent to other rows and j -th column ($2 \leq j \leq n$) is linear dependent to other columns. Then*

$$\dim E^3 = \begin{cases} n - 3 & \text{if } i = 1 \text{ or} \\ & j = n \text{ or} \\ & j \neq n \text{ and } j\text{-th column is non-zero or} \\ & i \neq 1, j \neq n, j = n - 1 \text{ or } j = n - 2 \text{ and} \\ & j\text{-th column is zero} \\ & i \neq 1, j \neq n, j \leq n - 3, i = j + 1, \text{ and} \\ & j\text{-th column is zero} \\ n - 4 & \text{if } i \neq 1, j \neq n, i \neq j + 1 \text{ and } j\text{-th column is zero.} \end{cases}$$

Moreover, $i = 1$ implies $j = 2$ and $j = n$ implies $i = n - 1$.

Proof. Consider

$$\begin{aligned} E^3 = E \cdot E^2 &= \left\langle \begin{array}{cccc} e_2(e_1e_1), & e_3(e_1e_1), & \dots & e_{n-1}(e_1e_1), \\ & e_3(e_2e_2), & \dots & e_{n-1}(e_2e_2), \\ & & \ddots & \vdots \\ & & & e_{n-1}(e_{n-2}e_{n-2}) \end{array} \right\rangle \\ (4.1) \quad &= \left\langle \begin{array}{cccc} a_{12}(e_2e_2), & a_{13}(e_3e_3), & \dots & a_{1\ n-1}(e_{n-1}e_{n-1}), \\ & a_{23}(e_3e_3), & \dots & a_{2\ n-1}(e_{n-1}e_{n-1}), \\ & & \ddots & \vdots \\ & & & a_{n-2\ n-1}(e_{n-1}e_{n-1}) \end{array} \right\rangle. \end{aligned}$$

Denote by $L := \langle e_2e_2, e_3e_3, \dots, e_{n-1}e_{n-1} \rangle$. Obviously, $E^3 \subseteq L$.

If $i = 1$, then we obtain $a_{12} = 0$ and $a_{23} \dots a_{n-1n} \neq 0$. Hence, $E^2 = \langle e_3, \dots, e_n \rangle$ and $E^3 = \langle e_4, \dots, e_n \rangle$. Moreover, applying the same arguments as in the proof of Proposition 4.2(i) we obtain

$$E^{2^{k-1}+1} = E^{2^k+2} = \dots = E^{2^k} = \langle e_{k+2}, \dots, e_n \rangle$$

for all $1 \leq k \leq n - 2$ and index of nilpotency for this algebra in this case is $1 + 2^{n-2}$.

Now let $2 \leq i \leq n - 1$. Then $\dim L = n - 3$.

If $j = n$, then $a_{n-1n} = 0$ and therefore, $i = n - 1$. Hence, from (4.1) one obtains $E^3 = \langle e_2e_2, e_3e_3, \dots, e_{n-2}e_{n-2} \rangle$. Thus, $\dim E^3 = n - 3$.

Now if $j \neq n$ and j -th column is non-zero column, then one can easily see from (4.1) that again $E^3 = L$. Hence, $\dim E^3 = n - 3$.

If $j \neq n$ and j -th column is zero column, then

$$E^3 = \langle (e_2e_2), \dots, (e_je_j), (e_{j+2}e_{j+2}), \dots, (e_{n-1}e_{n-1}) \rangle.$$

If $j = n - 1$ or $j = n - 2$, then $\dim E^3 = n - 3$.

Let $j \leq n - 3$. If $i = j + 1$, then $E^3 = L$. In this case $\dim E^3 = \dim L = n - 3$.

If $i \neq j + 1$, then $\dim E^3 = \dim L - 1$.

Hence, the statement of the proposition is verified. \square

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