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ON ENTIRE SOLUTIONS OF NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the non-existence of finite order entire solutions of nonlinear differential-difference of the form

$$f^n + Q(z,f) = h,$$

where $n \geq 2$ is an integer, $Q(z,f)$ is a differential-difference polynomial in f with polynomial coefficients, and h is a meromorphic function of order ≤ 1.

1. Introduction and main results

In what follows, a meromorphic function $f(z)$ is always meromorphic in the whole complex plane. We use the standard notations of value distribution theory, such as $T(r,f)$, $m(r,f)$ and $N(r,f)$, and we assume that the reader is familiar with the lemma on the logarithmic derivatives, the first and second fundamental theorems and so on (see [4]). Nevanlinna value distribution theory of meromorphic functions has been extensively applied to study the properties of linear and nonlinear differential equations (see e.g. [5, 6, 10]).

Let $a(z)$ be a meromorphic function. If $T(r,a) = S(r,f)$, then $a(z)$ is called a small function of $f(z)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside of a set of r with finite logarithmic measure.

If $P(z,f)$ is a polynomial in f and its derivatives, with small functions of f as the coefficients, then $P(z,f)$ is said to be a differential polynomial in f.

Given a constant c, $f(z+c)$ is called a shift of f. And a difference monomial of type $\prod_{i=1}^{m} f^{n_i}(z+c_i)$ is called a difference product, where $c_1, \ldots, c_m \in \mathbb{C}$, and $n_1, \ldots, n_m \in \mathbb{N}$.

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A differential-difference polynomial (resp. a difference polynomial) in f is a finite sum of products of f, derivatives of f and their shifts (resp. of difference products of f and its shifts), with all the coefficients of these monomials being small functions of f.

Yang and Laine have studied the existence of meromorphic solutions of nonlinear differential equations (see, e.g. [8]), and recently, the existence of entire solutions of differential-difference equations (see [9]). The following two theorems were included in the paper [9].

Theorem A. *Let $L(z,f)$ be a linear differential-difference polynomial of f with polynomial coefficients, and $p(z)$ be a polynomial. Then the equation*

$$f^2(z) + L(z,f) = p(z)$$

has no transcendental entire solutions of finite order.

Theorem B. *A nonlinear difference equation*

$$f^3(z) + q(z)f(z+1) = c\sin bz,$$

where $q(z)$ is a nonconstant polynomial and $b, c \in \mathbb{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a nonzero constant, then the equation possesses three distinct entire solutions of finite order, provided $b = 3n\pi$ and $q^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a nonzero integer n.

To generalize Theorems A and B, we prove the following Theorems 1 and 2.

Theorem 1. *Suppose that a nonlinear differential-difference equation is*

$$f^n(z) + \sum_{i=1}^{m} H_i(z,f) = p(z), \tag{1}$$

where $n, m \in \mathbb{Z}^+$, $p(z)$ is a polynomial, and the terms $H_i(z,f)$ are differential-difference monomials with polynomial coefficients. If

$$n > (m+1)\max_{1\le i\le m}\deg(H_i) - \sum_{i=1}^{m}\deg(H_i), \tag{2}$$

then the equation has no transcendental entire solutions of finite order.

From Theorem 1, it is easy to see that if $\deg(H_1) = \cdots = \deg(H_m) = k$, then for the differential-difference equation

$$f^{k+1}(z) + \sum_{i=1}^{m} H_i(z,f) = p(z),$$

a similar conclusion holds. If $k = 1$, the result is just Theorem A.

Theorem 2. *For two integers $n \ge 3$, $k \ge 0$ and a nonlinear differential-difference equation*

$$f^n(z) + q(z)f^{(k)}(z+t) = ae^{ibz} + de^{-ibz}, \tag{3}$$

where $q(z)$ is a polynomial and t, a, b, d are complex numbers such that $|a|+|d| \neq 0, bt \neq 0$,

(i) *Let $n = 3$. If $q(z)$ is nonconstant, then the equation* (3) *does not admit entire solutions of finite order. If $q := q(z)$ is constant, then equation* (3) *admits three distinct transcendental entire solutions of finite order, provided that*

$$bt = 3m\pi \ (m \neq 0, \ \text{if } q \neq 0), \ q^3 = (-1)^{m+1}(\frac{3i}{b})^{3k}27ad, \tag{4}$$

when k is even, or

$$bt = \frac{3\pi}{2} + 3m\pi \ (\text{if } q \neq 0), \ q^3 = i(-1)^m(\frac{3i}{b})^{3k}27ad, \tag{5}$$

when k is odd, for an integer m.

(ii) *Let $n > 3$. If $ad \neq 0$, then the equation* (3) *does not admit entire solutions of finite order. If $ad = 0$, then equation* (3) *admits n distinct transcendental entire solutions of finite order, provided that $q := q(z) \equiv 0$.*

2. Lemmas

The following lemma (see [1, 3]) on quotients of shifts can be seen as the difference counterpart of the lemma on the logarithmic derivatives, but it fails for meromorphic functions of infinite order, such as $f(z) = \exp(e^z)$ (see [9]).

Lemma 1. *Let $f(z)$ be a transcendental meromorphic function of finite order ρ. Then for any given complex numbers c_1, c_2, and for each $\varepsilon > 0$,*

$$m(r, \frac{f(z+c_1)}{f(z+c_2)}) = O(r^{\rho-1+\varepsilon}).$$

In 1962, Clunie [2] obtained Lemma 2 which has been extensively applied in studying the value distribution.

Lemma 2. *Let $f(z)$ be a transcendental meromorphic function, and $P(z, f)$, $Q(z, f)$ be two differential polynomials of f. If*

$$f^n(z)P(z, f) = Q(z, f)$$

holds and if the total degree of $Q(z, f)$ in f and its derivatives is $\leq n$, then

$$m(r, P(z, f)) = S(r, f).$$

In 2007, Lemma 2 was generalized for the differential polynomial case in [11], and Laine and Yang [7] got the Clunie theorem for difference polynomial. Recently they have pointed out that the Clunie theorem for difference polynomial is also true for a type of differential-difference polynomial (see [9]).

Lemma 3. *Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of a differential-difference equation of the form*

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z,f), Q(z,f)$ are differential-difference polynomials in f, and the total degree of $Q(z,f)$ in f, its derivatives and their shifts is $\leq n$. Then for any $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

The following Lemma 4 (see [10]), plays a key role in the proof of Theorem 2.

Lemma 4. *Suppose c is a nonzero constant and α is a nonconstant meromorphic function. Then the differential equation*

$$f^2 + (cf^{(n)})^2 = \alpha$$

has no transcendental meromorphic solutions satisfying $T(r,\alpha) = S(r,f)$.

3. Proofs of the theorems

Proof of Theorem 1. Let $f(z)$ be a transcendental entire solution of finite order ρ of the equation (1). Set

$$k = \max_i \deg(H_i), \ k_i = \deg(H_i) \ (1 \leq i \leq m).$$

Without loss of generality, we may assume that there exists

$$H_i(z,f) = q_i(z) f^{k_{i1}}(z)[f^{(l)}(z)]^{k_{i2}} f^{k_{i3}}(z+c_{i1})[f^{(s)}(z+c_{i2})]^{k_{i4}},$$

where $\sum_{j=1}^4 k_{ij} = k_i (k_{ij} \in \mathbb{N})$, $l, s \in \mathbb{Z}^+$, $q_i(z) \not\equiv 0$ is a polynomial, and $c_{i1}, c_{i2} \in \mathbb{C}$ are nonzero constants. Then

$$\frac{H_i(z,f)}{f^k(z)}$$
$$= q_i(z)\Big(\frac{f^{(l)}(z)}{f(z)}\Big)^{k_{i2}}\Big(\frac{f(z+c_{i1})}{f(z)}\Big)^{k_{i3}}\Big(\frac{f^{(s)}(z+c_{i2})}{f^{(s)}(z)}\Big)^{k_{i4}}\Big(\frac{f^{(s)}(z)}{f(z)}\Big)^{k_{i4}}\Big(f(z)\Big)^{k_i-k}.$$

Hence by Lemma 1 and the logarithmic derivatives lemma, we conclude that

$$m(r, \frac{H_i(z,f)}{f^k}) \leq O(r^{\rho-1+\varepsilon}) + S(r,f) + (k-k_i)T(r,f)$$

for all r sufficiently large, outside of an exceptional set of finite logarithmic measure. Combining this with (1), we obtain that

$$\begin{aligned} nT(r,f) &= m(r, p(z) - \sum_{i=1}^m H_i(z,f)) \\ &\leq m(r, \frac{\sum_{i=1}^m H_i(z,f)}{f^k(z)}) + kT(r,f) + O(\log r) \\ &\leq \sum_{i=1}^m m(r, \frac{H_i(z,f)}{f^k}) + kT(r,f) + O(\log r) \end{aligned}$$

$$\leq (k + \sum_{i=1}^{m}(k - k_i))T(r, f) + O(r^{\rho-1+\varepsilon}) + S(r, f).$$

Therefore,

$$[n - ((m+1)k - \sum_{i=1}^{m} k_i)]T(r, f) \leq O(r^{\rho-1+\varepsilon}) + S(r, f).$$

Now it follows from (2) that $\rho(f) < \rho$, a contradiction. Theorem 1 is thus proved. □

Proof of Theorem 2. Suppose that $f(z)$ is a transcendental entire solution of finite order to the equation (3). Differentiating (3), we have

$$nf^{n-1}(z)f'(z) + q'(z)f^{(k)}(z+t) + q(z)f^{(k+1)}(z+t) = ib(ae^{ibz} - de^{-ibz}).$$

Combining this with (3), we obtain

$$b^2\Big(f^n(z) + q(z)f^{(k)}(z+t)\Big)^2 + \Big(nf^{n-1}(z)f'(z) + q'(z)f^{(k)}(z+t) + q(z)f^{(k+1)}(z+t)\Big)^2 = 4adb^2.$$

This implies that

$$f^{2n-2}(z)(b^2f^2(z) + n^2f'^2(z)) = Q(z, f),$$

where $Q(z, f)$ is a differential-difference polynomial of f with the total degree at most $n+1$.

If $Q(z, f) \equiv 0$, then it can be deduced from $b^2f^2(z) + n^2f'^2(z) = 0$ that

$$f'' + \frac{b^2}{n^2}f = 0. \tag{6}$$

If $Q(z, f) \not\equiv 0$, then by Lemma 3, we have

$$T(r, b^2f^2(z) + n^2f'^2(z)) = S(r, f).$$

Thus $\alpha := b^2f^2(z) + n^2f'^2(z)(\not\equiv 0)$ is a small function of f. By Lemma 4, α must be a constant. Differentiating $\alpha = b^2f^2(z) + n^2f'^2(z)$, we get (6) again.

From (6), one can see that the form of the solution f must be

$$f(z) = c_1e^{\frac{ibz}{n}} + c_2e^{-\frac{ibz}{n}}. \tag{7}$$

Substituting (7) into (3) and denoting $\omega(z) := e^{\frac{ibz}{n}}$, we get

$$\begin{aligned} &c_1^n\omega^{2n} + C_n^1c_1^{n-1}c_2\omega^{2n-2} + C_n^2c_1^{n-2}c_2^2\omega^{2n-4} + \cdots + C_n^{n-2}c_1^2c_2^{n-2}\omega^4 \\ &\quad + C_n^{n-1}c_1c_2^{n-1}\omega^2 + c_2^n + c_1e^{\frac{ibt}{n}}(\frac{ib}{n})^kq(z)\omega^{n+1} \\ &\quad + c_2e^{-\frac{ibt}{n}}(-\frac{ib}{n})^kq(z)\omega^{n-1} = a\omega^{2n} + d, \end{aligned} \tag{8}$$

where $C_n^i = \frac{n!}{i!(n-i)!}$ $(1 \leq i \leq n-1)$.

Case (i): $n = 3$. We have

$$a_6\omega^6 + a_4\omega^4 + a_2\omega^2 + a_0 = 0,$$

where

$$\left\{\begin{array}{l} a_6 = c_1^3 - a, \\ a_4 = 3c_1^2c_2 + c_1e^{\frac{ibt}{3}}(\frac{ib}{3})^k q(z), \\ a_2 = 3c_1c_2^2 + c_2e^{-\frac{ibt}{3}}(-\frac{ib}{3})^k q(z), \\ a_0 = c_2^3 - d. \end{array}\right. \tag{9}$$

Since $\omega(z)$ is a transcendental function, we have

$$a_6 = a_4 = a_2 = a_0 = 0.$$

Subcase (i)-1: $a \neq 0,\ d \neq 0$. Then $c_1 \neq 0, c_2 \neq 0,\ q := q(z)$ is a nonzero constant and

$$e^{\frac{ibt}{3}}(\frac{ib}{3})^k = e^{-\frac{ibt}{3}}(-\frac{ib}{3})^k.$$

We obtain from this that

$$\left\{\begin{array}{ll} bt = 3m\pi(m \neq 0),\ 3c_1c_2 + (-1)^m(\frac{ib}{3})^k q = 0 & \text{if } k \text{ is even}, \\ bt = \frac{3\pi}{2} + 3m\pi,\ 3c_1c_2 + i(-1)^m(\frac{ib}{3})^k q = 0 & \text{if } k \text{ is odd}, \end{array}\right. \tag{10}$$

where $m \in \mathbb{Z}$. Therefore, (4) and (5) hold.

Subcase (i)-2: $a \neq 0$ and $d = 0$. Then $c_1 \neq 0, c_2 = 0$. From $a_4 = 0$, we have $q \equiv 0$. Therefore, (4) and (5) hold.

Subcase (i)-3: $a = 0$ and $d \neq 0$. Then $c_1 = 0, c_2 \neq 0$. From $a_2 = 0$, we have $q \equiv 0$. Thus (4) and (5) still hold.

Case (ii): $n > 3$. We get from (8) that

$$a_{2n}\omega^{2n} + a_{2n-2}\omega^{2n-2} + \cdots + a_2\omega^2 + a_0 = 0, \tag{11}$$

where

$$a_{2n} = c_1^n - a,\ a_0 = c_2^n - d.$$

Since $2n - 2 > n + 1$ and $2 < n - 1$, then

$$a_{2n-2} = nc_1^{n-1}c_2,\ a_2 = nc_1c_2^{n-1}.$$

Since $\omega(z)$ is a transcendental function, we get

$$a_{2n} = a_{2n-2} = \cdots = a_2 = a_0 = 0.$$

Subcase (ii)-1: $a \neq 0,\ d \neq 0$. Then it can be deduced from $a_{2n} = 0$ and $a_0 = 0$ that

$$c_1 \neq 0, c_2 \neq 0.$$

This is a contradiction to $a_{2n-2} = 0$ and $a_2 = 0$. Hence the equation (3) does not admit entire solutions of finite order.

Subcase (ii)-2: $a \neq 0$ and $d = 0$. Then $c_1 \neq 0, c_2 = 0$.

If n is even, then $n + 1$ is odd. Therefore, the coefficient of ω^{n+1} in (11) is

$$a_{n+1} = c_1e^{\frac{ibt}{n}}(\frac{ib}{n})^k q(z).$$

Since $a_{n+1} = 0$ and $c_1 \neq 0$, we have $q := q(z) \equiv 0$.

If n is odd, then $n+1$ and $n-1$ are both even. Therefore, the coefficient of ω^{n+1} in (11) is

$$a_{n+1} = C_n^{\frac{n-1}{2}} c_1^{\frac{n+1}{2}} c_2^{\frac{n-1}{2}} + c_1 e^{\frac{ibt}{n}} (\frac{ib}{n})^k q(z).$$

Since $a_{n+1} = 0$, $c_1 \neq 0$ and $c_2 = 0$, we get $q := q(z) \equiv 0$.

Subcase (ii)-3: $a = 0$ and $d \neq 0$. With a similar reasoning as the Subcase (ii)-2, we can prove that $q := q(z) \equiv 0$.

The proof is thus completed. □

4. Examples and remarks

Examples. In the equation

$$f^3 = ie^{-z},$$

since $q = 0$, $a = i$, $b = i$, $d = 0$, then by (7) and (9), three solutions of the equation are

$$f_1(z) = (\frac{\sqrt{3}}{2} + \frac{1}{2}i)e^{-\frac{z}{3}}, f_2(z) = (\frac{\sqrt{3}}{2} + \frac{1}{2}i)\varepsilon e^{-\frac{z}{3}}, f_3(z) = (\frac{\sqrt{3}}{2} + \frac{1}{2}i)\varepsilon^2 e^{-\frac{z}{3}},$$

where $\varepsilon := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a cubic root of unity.

The differential-difference equation

$$f^3(z) + 2f'(z+1) = \pi^3 e^{\frac{i3\pi}{2}} + \frac{1}{27}e^{-\frac{i3\pi}{2}}$$

satisfies the condition (5). Therefore, combing (7) with (10), its three finite order entire solutions are

$$f_1(z) = \pi e^{\frac{i\pi z}{2}} + \frac{1}{3}e^{-\frac{i\pi z}{2}} = 2\pi i \sin\frac{\pi z}{2} + (\pi + \frac{1}{3})e^{-\frac{i\pi z}{2}},$$

$$f_2(z) = \pi\varepsilon e^{\frac{i\pi z}{2}} + \frac{1}{3}\varepsilon^2 e^{-\frac{i\pi z}{2}} = 2\pi\varepsilon i \sin\frac{\pi z}{2} + (\pi\varepsilon + \frac{1}{3}\varepsilon^2)e^{-\frac{i\pi z}{2}},$$

$$f_3(z) = \pi\varepsilon^2 e^{\frac{i\pi z}{2}} + \frac{1}{3}\varepsilon e^{-\frac{i\pi z}{2}} = 2\pi\varepsilon^2 i \sin\frac{\pi z}{2} + (\pi\varepsilon^2 + \frac{1}{3}\varepsilon)e^{-\frac{i\pi z}{2}}.$$

Next we give some remarks.

Remark 1. Let $t = 0$ and $b \neq 0$ in (3). Then by Lemma 2, Lemma 4 and the similar proof of Theorem 2, we can find that neither

$$f^n(z) + q(z)f^{(k)}(z) = ae^{ibz} + de^{-ibz} \ (n > 3)$$

nor

$$f^3(z) + q(z)f^{(2k+1)}(z) = ae^{ibz} + de^{-ibz}$$

does admit transcendental entire solutions if $ad \neq 0$.

In addition, the equation

$$f^3(z) + q(z)f^{(2k)}(z) = ae^{ibz} + de^{-ibz}$$

admits three distinct transcendental entire solutions, provided that

$$q^3 = (-1)^{k+1}(\frac{9}{b^2})^{3k}27ad.$$

If $q(z)$ is a nonconstant polynomial, then the equation above does not admit entire solutions of finite order (For the case $k = 1$, see [9]).

Remark 2. Now, we can conclude that the equation

$$f^3(z) = ae^{ibz} + de^{-ibz} \tag{12}$$

has no transcendental entire solutions in the complex plane if $abd \neq 0$, where a, b, d are constants.

In fact, (12) determines a 3-valued algebroid function, and the function is transcendental entire in the Riemann surface.

Remark 3. We may ask the following question: for the difference-differential equation of the form

$$f^n(z) + L(z, f) = ae^{ibz} + de^{-ibz} \qquad (n \geq 3),$$

where $L(z, f)$ is some linear difference-differential polynomial of f with polynomial coefficients, what can we say considering Theorem 2?

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