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MORPHIC PROPERTY OF A QUOTIENT RING OVER POLYNOMIAL RING

Kai Long, Qichuan Wang, and Lianggui Feng

Abstract. A ring R is called *left morphic* if $R/Ra \cong l(a)$ for every $a \in R$. Equivalently, for every $a \in R$ there exists $b \in R$ such that $Ra = l(b)$ and $l(a) = Rb$. A ring R is called *left quasi-morphic* if there exist b and c in R such that $Ra = l(b)$ and $l(a) = Rc$ for every $a \in R$. A result of T.-K. Lee and Y. Zhou says that R is unit regular if and only if $R[x]/(x^2) \cong R \propto R$ is morphic. Motivated by this result, we investigate the morphic property of the ring $S_n \overset{\text{def}}{=} R[x_1, x_2, \ldots, x_n]/(\{x_i x_j\})$, where $i, j \in \{1, 2, \ldots, n\}$. The morphic elements of S_n are completely determined when R is strongly regular.

1. Introduction

Morphic rings were first introduced by W. K. Nicholson and E. Sánchez Campos in [6]. A ring R is called *left morphic* if $R/Ra \cong l(a)$ for every $a \in R$. Equivalently, for every $a \in R$, there exists $b \in R$ such that $Ra = l(b)$ and $l(a) = Rb$ [6, Lemma 1]. Right morphic rings are defined analogously. A left and right morphic ring is simply called a *morphic ring*. If there exist $b, c \in R$ such that $Ra = l(b)$ and $l(a) = Rc$, the element a is called *left quasi-morphic* [1]. Morphic and quasi-morphic rings were discussed in great detail in [1], [6] and [7]. The morphic property of the trivial extension $R \propto M$ of a ring R with a bimodule M over R is discussed in [2]. In particular, R is unit regular if and only if $R[x]/(x^2)$ is morphic [5].

Motivated by these results, we investigate the morphic property of the ring $S_n \overset{\text{def}}{=} R[x_1, x_2, \ldots, x_n]/(\{x_i x_j\})$, where $i, j \in \{1, 2, \ldots, n\}$. By converting to the case of the elements of the type $\alpha = e + fx_1 + a_2 x_2 + \cdots + a_n x_n$, where e, f are idempotents in R, and $a_i \in (1-e)R(1-e)$, $i \geq 2$, We completely determine the morphic elements of S_n [Theorem 6]. Further, from the proof of this theorem, we know that the result is also right to *left quasi-morphic*

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elements in S_n (Remark 1). The case of $n = 2$ has a very close relationship with trivial extension (Remark 2).

All rings here are associative with identity. The set of units of a ring R is denoted by $U(R)$. We simply write $l(a)$ as its left annihilators, and $r(a)$ as its right annihilators. The $n \times n$ matrix ring over R is denoted by $M_n(R)$. We write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for integers module n, respectively. Regular rings here mean von Neumann regular rings.

2. The unit elements in $R[x_1, x_2, \ldots, x_n]/(\{x_i x_j\})$

Denote $S_n = R[x_1, x_2, \ldots, x_n]/(\{x_i x_j\}),\ n \geq 2$, and $\alpha = a_0 + \sum_{i=1}^n a_i x_i$ in S_n, where $a_i \in R,\ x_i x_j = 0$, for all $i, j \in \{1, 2, \ldots, n\}$ and x_i commute with R. *Throughout this article, we adopt this notation, and $n \geq 2$ is indispensable.*

Lemma 1. *Let S_n and R be as above, denote $U(S_n)$ as the set of units of ring S_n. Then $U(S_n) = \{u + \sum_{i=1}^n r_i x_i \mid u \in U(R),\ r_i \in R\}$.*

Proof. First of all, the identity of S_n is the same with R, we denote $1_{S_n} = 1_R = 1$. Assume $\alpha = a_0 + \sum_{i=1}^n a_i x_i$ in S_n is a unit, then there exists an element $\beta = a_0{}' + \sum_{i=1}^n a_i{}' x_i$ such that

$$\begin{aligned}\alpha\beta &= (a_0 + \sum_{i=1}^n a_i x_i)(a_0{}' + \sum_{i=1}^n a_i{}' x_i) \\ &= a_0 a_0{}' + \sum_{i=1}^n (a_0 a_i{}' + a_0{}' a_i) x_i = 1.\end{aligned}$$

Hence we have $a_0 a_0{}' = 1$. Similarly, we can get $a_0{}' a_0 = 1$ by considering $\beta\alpha = 1$. Thus $U(S_n) \subset \{u + \sum_{i=1}^n r_i x_i \mid u \in U(R),\ r_i \in R\}$.

Conversely, assume $uv = vu = 1$, then

$$(u + \sum_{i=1}^n r_i x_i)(v - \sum_{i=1}^n v r_i v x_i) = (v - \sum_{i=1}^n v r_i v x_i)(u + \sum_{i=1}^n r_i x_i) = 1.$$

Thus $U(S_n) \supset \{u + \sum_{i=1}^n r_i x_i \mid u \in U(R),\ r_i \in R\}$. Hence the proof is completed. □

If R is unit regular, we have the following result, which can help us to convert $\alpha \in S_n$ into a simpler form.

Claim 1. Let R be a unit regular ring. Then for any $\alpha = a_0 + \sum_{i=1}^n a_i x_i$ in S_n, we have $\alpha = u_{s_n}(e + f x_1 + a_2 x_2 + \cdots + a_n x_n) v_{s_n}$, where $u_{s_n}, v_{s_n} \in U(S_n)$, e, f are idempotents in R, and $f, a_i \in (1-e)R(1-e),\ i \geq 2$.

Proof. Since R is unit regular, every element of R is the product of a unit and an idempotent. By multiplying α with a suitable unit of R we can assume $\alpha = e_0 + a_1 x_1 + \cdots + a_n x_n$. Then we have

$$\prod_{i=0}^{n-1} \big(1 - (1 - e_0) a_{n-i} x_{n-i}\big) \cdot \alpha \cdot \prod_{i=1}^{n} \big(1 - a_i x_i\big)$$

$$= e_0 + (1-e_0)a_1(1-e_0)x_1 + \cdots + (1-e_0)a_n(1-e_0)x_n.$$

We also have $(1-e_0)a_1(1-e_0) = u_0 f$, $u_0 \in U\big((1-e_0)R(1-e_0)\big)$, $f \in (1-e_0)R(1-e_0)$ is idempotent, since $(1-e_0)R(1-e_0)$ is unit regular by [3]. Further there exists an element $v_0 \in U\big((1-e_0)R(1-e_0)\big)$, such that $u_0v_0 = 1-e_0$. Hence $(e_0+u_0)(e_0+v_0) = e_0 + u_0v_0 = 1$. That is to say $e_0 + v_0 \in U(R)$, then

$$(e_0+v_0)\alpha = e_0 + fx_1 + a_2{}'x_2 + \cdots + a_n{}'x_n.$$

All the factors are units of S_n by Lemma 1, thus the claim is proved. □

Claim 2. Let α' be an element of S_n satisfying $\alpha' = e + fx_1 + a_2x_2 + \cdots + a_nx_n$, where e, f are idempotents in R, and $f, a_i \in (1-e)R(1-e)$, $i \geq 2$. Then

$$l(\alpha') = l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n) + l(e)x_1 + l(e)x_2 + \cdots + l(e)x_n.$$

Proof. Assume $r_0 + \sum_{i=1}^n r_ix_i \in l(\alpha')$, then

$$\begin{aligned}(r_0 + \sum_{i=1}^n r_ix_i)(e + fx_1 + \sum_{i=2}^n a_ix_i) &= r_0e + (r_0f + r_1e)x_1 + \sum_{i=2}^n (r_0a_i + r_ie)x_i \\ &= 0.\end{aligned}$$

So we have

$$(*) \qquad r_0e = 0,\ r_0f + r_1e = 0,\ r_0a_i + r_ie = 0,\ i \geq 2.$$

Noticing that $ef = fe = a_ie = ea_i = 0$, $i \geq 2$ and $e^2 = e$, by multiplying e on the respective two sides of equations $(*)$, we get

$$r_0e = r_0f = r_0a_i = 0,\ r_1e = 0,\ r_ie = 0,\ i \geq 2.$$

Hence $l(\alpha') \subset l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n) + l(e)x_1 + l(e)x_2 + \cdots + l(e)x_n$, and it is a routine way to verify $l(\alpha') \supset l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n) + l(e)x_1 + l(e)x_2 + \cdots + l(e)x_n$. Thus the result is established. □

3. The morphic elements in S_n

The following lemma comes from the paper [1] of V. Camillo, W. K. Nicholson and Z. Wang.

Lemma 2 ([1])**.** *Let R be a left quasi-morphic ring. Then the intersection of finite principal left ideals of R is again principal.*

A ring is called *strongly regular* if $a \in a^2R$ for every $a \in R$ [4]. Strongly regular rings are unit regular, hence are morphic and quasi-morphic [1, 6]. It is well known that R is strongly regular if and only if R is regular and every idempotent in R is center.

Lemma 3. *Let R be a strongly regular ring. Then for any $a, b \in R$, $Rab \subset Ra$, particularly if $b \in U$, $Rab = Ra$.*

Proof. Since R is strongly regular, for any $a \in R$, there exist $u \in U(R)$ and an idempotent element e such that $a = ue$ and $Rab = Rueb = Reb = Rbe \subset Re = Ra$. Of course, if $b \in U$, then "$\subset$" can be replaced by "$=$". □

Lemma 4. *Let R be a strongly regular ring. Then for any $\alpha \in S_n$, we have $l(\alpha) = l(e + fx_1 + a_2x_2 + \cdots + a_nx_n)$, where e, f are idempotents in R, and $f, a_i \in (1-e)R(1-e)$, $i \geq 2$.*

Proof. By Claim 1, we have $\alpha = u_{s_n}(e + fx_1 + a_2x_2 + \cdots + a_nx_n)v_{s_n}$, where $u_{s_n}, v_{s_n} \in U(S_n)$, and e, f are idempotents in R, and $f, a_i \in (1-e)R(1-e)$, $i \geq 2$. Since $u_{s_n}^{-1} \in U(S_n)$, by Lemma 1, we can assume $u_{s_n}^{-1} = u + r_1x_1 + \cdots + r_nx_n$, where $u \in U(R)$, $r_i \in R$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} l(\alpha) &= l(e + fx_1 + a_2x_2 + \cdots + a_nx_n)u_{s_n}^{-1} \\ &= \{(t_0 + t_1x_1 + \cdots + t_nx_n)(u + r_1x_1 + \cdots + r_nx_n) \mid \\ &\qquad t_0 \in l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n),\ t_i \in l(e), i \geq 1\} \\ &= \{t_0u + (t_1u + t_0r_1)x_1 + \cdots + (t_iu + t_0r_i)x_i + \cdots \\ &\qquad + (t_nu + t_0r_n)x_n \mid t_0 \in l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n), \\ &\qquad t_i \in l(e), i \geq 1\}. \end{aligned}$$

By Lemma 2 and Lemma 3, we know that $t_0u, t_0r_i \in l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n)$, $t_iu \in l(e)$. Noticing that $l(e) \cap l(f) \cap l(a_2) \cap \cdots \cap l(a_n) \subset l(e)$, hence $t_iu + t_0r_i$ is in $l(e)$. Thus

$$(**) \qquad \begin{aligned} l(\alpha) &= l(e + fx_1 + a_2x_2 + \cdots + a_nx_n)u_{s_n}^{-1} \\ &\subset l(e + fx_1 + a_2x_2 + \cdots + a_nx_n). \end{aligned}$$

In fact, by the proof, we know that the equation $(**)$ is also right for an arbitrary $u_{s_n}^{-1} \in U(S_n)$. Then

$$\begin{aligned} l(e + fx_1 + a_2x_2 + \cdots + a_nx_n) &= l(e + fx_1 + a_2x_2 + \cdots + a_nx_n)u_su_s^{-1} \\ &\subset l(e + fx_1 + a_2x_2 + \cdots + a_nx_n)u_s^{-1} \\ &= l(\alpha). \end{aligned}$$

Combing these together, we get $l(\alpha) = l(e + fx_1 + a_2x_2 + \cdots + a_nx_n)$, where e, f are idempotents in R, and $f, a_i \in (1-e)R(1-e)$, $i \geq 2$. □

Above lemma tells us that if $\alpha = e + fx_1 + a_2x_2 + \cdots + a_nx_n$ is left morphic, and there is an element β such that $S_n\alpha = l(\beta), l(\alpha) = S_n\beta$, then we can further assume that β has the form of $e' + f'x_1 + a_2'x_2 + \cdots + a_n'x_n$, where e', f' are idempotents in R, and $f', a_i' \in (1-e')R(1-e')$, $i \geq 2$.

Lemma 5. *Let R be a strongly regular ring, $\alpha = e + fx_1 + a_2x_2 + \cdots + a_nx_n$ in S_n, where e, f are idempotents in R, and $f, a_i \in (1-e)R(1-e)$, $i \geq 2$. Then*

$$\alpha \text{ is left quasi-morphic} \Rightarrow Rf = Ra_2 = \cdots = Ra_n.$$

Proof. If $e = 1$, then $f, a_i \in (1-e)R(1-e) = 0$, and $Rf = Ra_2 = \cdots = Ra_n = 0$. So we assume α is quasi-morphic and $e \neq 1$. By Lemma 4 and Claim 2, we have

$$S_n\alpha = \{t_0e + (t_1e + t_0f)x_1 + \cdots + (t_ie + t_0a_i)x_i + \cdots$$

$$+ (t_n e + t_0 a_n)x_n \mid t_i \in R\}$$
$$= l(\beta)$$
$$= l(e') \cap l(f') \cap l({a_2}') \cap \cdots \cap l({a_n}') + l(e')x_1$$
$$+ l(e')x_2 + \cdots + l(e')x_n.$$

Then there must be $Re + Rf = Re + Ra_i = l(e')$. Multiplying both sides by $1 - e$, we get $Rf = Ra_i,\ i = 2, 3, \ldots, n$. □

Theorem 6. *Let R be a strongly regular ring, denote by T the set of morphic elements of ring S_n. Then $T = \{u_{s_n} e \mid u_{s_n} \in U(S_n), e^2 = e \in R\}$.*

Proof. Since multiplying units does not change the morphic property of an element [6], by Claim 1, we can assume $\alpha = e + fx_1 + a_2x_2 + \cdots + a_nx_n$ in S_n, where e, f are idempotents in R, and $f, a_i \in (1 - e)R(1 - e),\ i \geq 2$.

By the proof of Lemma 5, we get $Re = l(e') \cap l(f') \cap l({a_2}') \cap \cdots \cap l({a_n}')$, $Re+Rf = Re+Ra_i = l(e'),\ i = 2, 3, \ldots, n$. Thus $l(\beta) = Re+\sum_{i=1}^{n}(Re+Rf)x_i$. Considering the element of the type $\alpha_0 = fx_1 + ex_2$, since

$$\alpha_0 \in Re + (Re + Rf)x_1 + \cdots + (Re + Rf)x_n$$
$$= l(\beta) = S_n\alpha$$
$$= \{t_0e + (t_1e + t_0f)x_1 + \cdots + (t_ie + t_0a_i)x_i + \cdots + (t_ne + t_0a_n)x_n \mid t_i \in R\}$$

so we have $f = t_1e + t_0f,\ e = t_2e + t_0a_2$.

Multiplying both sides by e, we get $t_1e = 0,\ e = t_2e$, and $f = t_0f,\ t_0a_2 = 0$. By Lemma 5 and noticing that R is strongly regular, we have $a_2 = rf = fr$, then $R(1 - f)a_2 = R(1 - f)fr = 0$. That is to say $l(f) \subset l(a_2)$. Assume $1 - t_0 = r_0,\ r_0 \in l(f)$, then $t_0a_2 = (1 - r_0)a_2 = a_2 = 0$. Thus $Rf = Ra_2 = \cdots = Ra_n = 0$, so $f = a_2 = \cdots = a_n = 0$.

Hence $T \subset \{u_{s_n}e \mid u_{s_n} \in U(S_n), e^2 = e \in R\}$. Since $T \supset \{u_{s_n}e \mid u_{s_n} \in U(S_n), e^2 = e \in R\}$ is trivial, we complete the proof. □

Remark 1. In fact, our proof just uses the property that R is strongly regular and $S_n\alpha = l(\beta)$, hence the theorem is also right for *left quasi-morphic* elements in S_n.

Remark 2. The case of $n = 2$ has a very close relationship with trivial extension, since

$$R \propto (R \propto R) \cong \left\{ \left(\begin{smallmatrix} a & 0 & b & c \\ 0 & a & 0 & b \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{smallmatrix} \right) \Big| \ a, b, c \in R \right\} \cong R[x, y]/(x^2, y^2, xy).$$

Corollary 7. *The ring S_n could never be a morphic ring.*

Proof. We proof the corollary by showing that the element of the type $x_1 + x_2$ could never be a morphic element.

Assume $x_1 + x_2$ is morphic. There must be an element $\beta = {a_0}' + \sum_{i=1}^{n} {a_i}'x_i$ such that

$$l(x_1 + x_2) = Rx_1 + Rx_2 + \cdots + Rx_n = S_n\beta$$

$$= \{(a_0a_0' + \sum_{i=1}^{n}(a_0a_i' + a_ia_0')x_i \mid a_0, a_i, , a_0', a_i' \in R\}.$$

Then $a_0' = 0,\ a_i' \in U(R)$, and $l(\beta) = Rx_1 + Rx_2 + \cdots + Rx_n$. But we have

$$S_n(x_1 + x_2) = \{(a_0 + \sum_{i=1}^{n} a_ix_i)(x_1 + x_2) \mid a_0, a_i \in R\}$$
$$= \{a_0x_1 + a_0x_2 \mid a_0 \in R\} \neq l(\beta).$$

This is a contradiction.

Hence the element of the type $x_1 + x_2$ could never be a morphic element, and the ring S_n could never be a morphic ring. □

Further, we know that every idempotent $e_{s_n} \in S_n$ is morphic. By Theorem 6, we assume $e_{s_n} = u_{s_n}e$, then

$${e_{s_n}}^2 = u_{s_n}eu_{s_n}e = {u_{s_n}}^2e = u_{s_n}e, \text{ so } u_{s_n}e = e, \text{ that is } e_{s_n} = e.$$

Corollary 8. *Let R be a strongly regular ring. Then the idempotents in S_n are just the idempotents in R.*

Proof. Here we give another normal way to prove the corollary, and take $n = 2$ for example.

Suppose $\alpha = a + bx_1 + cx_2 \in S_2$ is idempotent, we get

$$(a + bx_1 + cx_2)^2 = a^2 + (ab + ba)x_1 + (ac + ca)x_2 = a + bx_1 + cx_2$$

then

$$a^2 = a, ab + ba = b, ac + ca = c$$

thus

$$ab + aba = ab,\ aba = 0.$$

Since R is strongly regular, every idempotent is in center. Then $ab = ba = aba = 0$ thus $b = 0$. With the same method we can get $c = 0$. □

Finally, we give an example that R is unit regular, and an element $\alpha \in S_n$ is morphic but not of the form $u_{s_n}e$.

Example 9. Let $R = M_2(\mathbb{Z}_2)$, considering the element $\alpha = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)x \in S_2$. We show that α is a morphic element but could not be the form of $u_{s_2}e$.

Proof. First verify the morphic property of α, denote

$$A = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\ B = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$$

and we have

$$A^2 = A,\ AB + BA = B$$
$$\Rightarrow \alpha^2 = (A + Bx)^2 = A^2 + (AB + BA)x = A + Bx = \alpha.$$

Thus α is an idempotent in S_2, so it is a morphic element. But if

$$\alpha = u_{s_2}e = (u + ax + by)e = ue + aex$$

$$\Rightarrow A = ue,\ B = ae,$$

then

$$B = au^{-1}A = \left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & 0 \\ * & 0 \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

This is a contradiction. □

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Kai Long
Department of Mathematics
Aviation University of Air Force
130022, Changchun, Jilin, P. R. China
E-mail address: `lkkkkkkkk@hotmail.com`

Qichuan Wang
No. 8, Anningqu Road, 830001, Urumqi, Xinjiang, P. R. China
E-mail address: `wangqichuan1026@163.com`

Lianggui Feng
College of science
National University of Defense Technology
410073, Changsha, Hunan, P. R. China
E-mail address: `fenglg2002@sina.com`