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THE FUNCTION ANALYTIC IN THE EXTERIOR OF A DISC AND ITS APPLICATION TO PERIODIC COMPLEX OSCILLATION

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Abstract. We consider the value distribution of a class of the functions analytic in the exterior of a disc and their applications to complex oscillation theory of differential equations with periodic coefficients in the complex plane.

1. Introduction and the function analytic in the exterior of a disc

We use standard notations from the value distribution theory (see [13, 14]). We denote the order of growth of a meromorphic function $f(z)$ by $\sigma(f)$, and denote respectively the exponents of convergence of zeros and distinct zeros for $f(z)$ by $\lambda(f)$ and $\overline{\lambda}(f)$. We define respectively

$$\sigma_e(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{r},$$
$$\lambda_e(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ N(r,1/f)}{r}, \tag{1.1}$$
$$\overline{\lambda}_e(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ \overline{N}(r,1/f)}{r}$$

to be the e-type order, the e-type exponents of convergence of zeros and distinct zeros for $f(z)$ ([7]).

Assume $A(z)$ to be a periodic entire function with period $2\pi i$. Thus, we can write $A(z) = B(e^z)$, where $B(\zeta)$ is clearly analytic in $0 < |\zeta| < \infty$ (see [1, p. 7]). For $B(\zeta)$, it is easy to see that it has the representation (from its Laurent expansion)

$$B(\zeta) = g_1\left(\frac{1}{\zeta}\right) + g_2(\zeta), \tag{1.2}$$

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where both $g_1(t)$ and $g_2(t)$ are entire functions. It is shown in [7] that

$$\sigma_e(A) = \max\{\sigma(g_1), \sigma(g_2)\}. \tag{1.3}$$

If $f(z)$ is an analytic function in $R_0 < |z| < \infty$, then it has the representation (see [16, p. 15])

$$f(z) = z^m \psi(z) F(z), \tag{1.4}$$

where m is an integer, $\psi(z)$ is analytic and non-vanishing on $|z| > R_0$ (including $z = \infty$), $F(z)$ is an entire function with

$$F(z) = u(z)e^{h(z)}, \tag{1.5}$$

where $u(z)$ is a Weierstrass product formed by zeros of $f(z)$ in $R_0 < |z| < \infty$, $h(z)$ is an entire function. Thus, since $B(\zeta)$ is analytic in $R_0 < |z|$, by (1.4), we have the representation

$$B(\zeta) = \zeta^n \phi(\zeta) b(\zeta), \tag{1.6}$$

where n is an integer, $\phi(\zeta)$ is analytic and non-vanishing on $|\zeta| > R_0$ (including $\zeta = \infty$), $b(\zeta)$ is entire. It is shown in [7] that

$$\sigma(g_2) = \sigma(b). \tag{1.7}$$

Setting $t = 1/\zeta$, $B(\zeta)$ analytic in $0 < |\zeta| < R_0$ is changed to $B^*(t) = B(1/t)$ analytic in $\frac{1}{R_0} < |t| < \infty$. Thus, it has a similar representation as (1.6) with $b(\zeta)$ replaced by another entire function, denoted by $b^*(t)$. We have similarly as (1.7)

$$\sigma(g_1) = \sigma(b^*). \tag{1.8}$$

Thus, (1.7) and (1.8) together with (1.3) yield

$$\sigma_e(A) = \max\{\sigma(b), \sigma(b^*)\}. \tag{1.9}$$

It is shown in [7] (it is proved only in the case $R_0 = 1$ there, but we will explain that it is still valid for any $R_0 > 0$ later)

$$\lambda_e(A) = \max\{\lambda(b), \lambda(b^*)\}. \tag{1.10}$$

By the same reasoning, we also have

$$\overline{\lambda}_e(A) = \max\{\overline{\lambda}(b), \overline{\lambda}(b^*)\}. \tag{1.11}$$

We also need to make use of the functions meromorphic in $R_0 < |z| < \infty$. If $w(z)$ is such a function, then by a similar argument as for (1.4), $w(z)$ has the representation

$$w(z) = z^{m_0} \psi_0(z) h(z), \tag{1.12}$$

where m_0 is an integer, $\psi_0(z)$ is analytic and non-vanishing on $|z| > R_0$ (including $z = \infty$), $h(z)$ is meromorphic in the plane. In fact, we may write

$$h(z) = \frac{u(z)}{v(z)} e^{g(z)}, \tag{1.13}$$

where $u(z), v(z)$ are Weierstrass products formed, respectively, from the zeros and poles of $w(z)$ in $R_0 < |z| < \infty$, and $g(z)$ is an entire function. The properties of $w(z)$ can be described by a Nevanlinna type theory in $R_0 < |z| < \infty$ (see [3, pp. 97–99]). Denote the characteristic function of $w(z)$ by $T_1(r, w)$ ([1]) which is defined by

$$T_1(r, w) = m_1(r, w) + N_1(r, w),$$

where

$$m_1(r, w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\theta})| d\theta, \tag{1.14}$$

$N_1(r, w)$ is the counting function for the poles of $w(z)$ in $R_0 < |z| \leq r$. It is easy to deduce from (1.12) that

$$m_1(r, w) = m(r, h) + O(\log r). \tag{1.15}$$

Evidently,

$$N_1(r, w) = N(r, h). \tag{1.16}$$

Therefore,

$$T_1(r, w) = T(r, h) + O(\log r). \tag{1.17}$$

We get from $T(r, h)$

$$T_1\left(r, \frac{1}{w}\right) = T_1(r, w) + O(\log r). \tag{1.18}$$

We define that $w(z)$ meromorphic in $R_0 < |z| < \infty$ is rational, if $w(z)$ is analytic or has a pole at $z = \infty$. Contrary, we say that it is transcendental. It is easy to see that $w(z)$ is rational if and only if $T_1(r, w) = O(\log r)$.

If $w(z)$ is transcendental, (1.17) may be easy written to

$$T_1(r, w) = (1 + o(1))T(r, h); \tag{1.17a}$$

and if $w(z)$ is rational, then

$$T_1(r, w) = O(\log r), \quad T(r, h) = O(\log r). \tag{1.17b}$$

We denote the order of growth of $w(z)$ by $\sigma_1(w)$, and denote respectively exponents of convergence of zeros and distinct zeros of $w(z)$ by $\lambda_1(w)$ and $\overline{\lambda}_1(w)$. They are defined by

$$\begin{aligned} \sigma_1(w) &= \overline{\lim_{r\to\infty}} \frac{\log^+ T_1(r, w)}{\log r}, \\ \lambda_1(w) &= \overline{\lim_{r\to\infty}} \frac{\log^+ N_1(r, \frac{1}{w})}{\log r}, \\ \overline{\lambda}_1(w) &= \overline{\lim_{r\to\infty}} \frac{\log^+ \overline{N_1}(r, \frac{1}{w})}{\log r}. \end{aligned} \tag{1.19}$$

From (1.17a), (1.17b) and an analogue of (1.16), we get

$$\sigma_1(w) = \sigma(h),\ \lambda_1(w) = \lambda(h),\ \overline{\lambda}_1(w) = \overline{\lambda}(h). \tag{1.20}$$

Since $B(\zeta)$ is analytic in $0 < |\zeta| < \infty$ and (1.20), by (1.6), $B^*(t) = B(\frac{1}{t})$ and denotation of $b^*(t)$, we obtain

$$\begin{aligned} \sigma_1(B) &= \sigma(b),\ \sigma_1(B^*) = \sigma(b^*), \\ \lambda_1(B) &= \lambda(b),\ \lambda_1(B^*) = \lambda(b^*), \\ \overline{\lambda}_1(B) &= \overline{\lambda}(b),\ \overline{\lambda}_1(B^*) = \overline{\lambda}(b^*). \end{aligned} \tag{1.21}$$

Thus, (1.21) together with (1.9), (1.10) and (1.11) gives

$$\begin{aligned} \sigma_e(A) &= \max\{\sigma_1(B), \sigma_1(B^*)\}, \\ \lambda_e(A) &= \max\{\lambda_1(B), \lambda_1(B^*)\}, \\ \overline{\lambda}_e(A) &= \max\{\overline{\lambda}_1(B), \overline{\lambda}_1(B^*)\}. \end{aligned} \tag{1.22}$$

Remark 1.1. We need to show that, for a function $B(\zeta)$ analytic in $0 < |\zeta| < \infty$, the quantities $\sigma_1(B)$, $\sigma_1(B^*)$, $\lambda_1(B)$, $\lambda_1(B^*)$, $\overline{\lambda}_1(B)$, $\overline{\lambda}_1(B^*)$ have not any relations to $R_0(> 0)$, i.e., they are only the quantities which portray the properties of $B(\zeta)$ in the neighborhood of $\zeta = \infty$ and $\zeta = 0$.

2. Main results

Many authors (see [1, 2, 4–12, 15, 17]) consider complex oscillation theory of differential equations. Bank and Langley proved in [2]:

Theorem A. *Suppose that $k \geq 2$ and that $D_0, \ldots, D_{k-2}$ are entire functions with period $2\pi i$ such that D_0 is transcendental in e^z with*

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, D_0)}{r} = c < \frac{1}{2}. \tag{2.1}$$

Suppose further that if $k \geq 3$, then for each j with $1 \leq j \leq k-2$, the coefficient D_j is either rational in e^z or satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, D_j)}{r} < c. \tag{2.2}$$

Then the equation

$$f^{(k)} + D_{k-2}f^{(k-2)} + \cdots + D_0 f = 0 \tag{2.3}$$

cannot have linearly independent solutions f, g with

$$\log^+ N(r, 1/fg) = O(r). \tag{2.4}$$

The main purpose of this article is to improve Theorem A, and prove the following theorems.

Theorem 2.1. *Let $k \geq 3$, and $A(z), A_0(z), \ldots, A_{k-2}(z)$ be periodic entire functions with period $2\pi i$, i.e., $A(z) = B(e^z), A_j(z) = B_j(e^z)$ with $B(\zeta)$ and $B_j(\zeta)$ analytic in $0 < |\zeta| < \infty$, satisfying for $j = 0, \ldots, k-2$,*

(i) $\overline{\lambda}_1(B) < \sigma_1(B)$;

(ii) $\sigma_1(B_j) < \sigma_1(B)$;

(iii) $\sigma_1(B_j^*) < \max\{\sigma_1(B), \sigma_1(B^*)\}$, *where* $B^*(t) = B(1/t)$ *and* $B_j^*(t) = B_j(1/t)$.

If the equation

$$w^{(k)} + A_{k-2}w^{(k-2)} + \cdots + (A_0 + A)w = 0 \tag{2.5}$$

has a solution $f(z) \not\equiv 0$ satisfying

(iv) $\overline{\lambda}_e(f) < \sigma_1(B)$,

then $B(\zeta)$ has no zeros in $0 < |\zeta| < \infty$ (and thus $A(z)$ has no zeros there), and has the form

$$B(\zeta) = \zeta^m e^{g(\zeta)}, \tag{2.6}$$

where m is an integer, $g(\zeta)$ is analytic in $0 < |\zeta| < \infty$.

Remark 2.1. (1) The condition (i) in Theorem 2.1 only makes demand on the property of the dominant coefficient $B(\zeta)$ in the neighborhood of $\zeta = \infty$, and $B(\zeta)$ can be arbitrary in the neighborhood of $\zeta = 0$, i.e., $B^*(t)$ can be arbitrary in the neighborhood of $t = \infty$, we call the property of $B(\zeta)$ the "single-side property".

(2) If we replace $\overline{\lambda_1}(B) < \sigma_1(B)$ by $\lambda_1(B) < \sigma_1(B)$ in the condition (i), then $\sigma_1(B)$ must be a positive integer or infinity from (1.21). Thus, the "single-side" order of dominant coefficient $B(\zeta)$ in the condition (i) is more general than either positive integer order or infinity order. The same explanations apply to Theorem 2.2 below.

Theorem 2.2. *Suppose the assumption of Theorem 2.1 for k, $A(z), A_0(z), \ldots, A_{k-2}(z)$ and the hypotheses* (i)-(iii) *holds. If $A(z)$ has at least one zero, then every solution $f(z) \not\equiv 0$ of equation* (2.5) *satisfies*

$$\lambda_e(f) \geq \overline{\lambda}_e(f) \geq \sigma_1(B),$$

and hence, $\lambda(f) = \overline{\lambda}(f) = \infty$.

Remark 2.2. Though Theorems 2.1 and 2.2 does not contain Theorem A completely, the conditions of Theorems 2.1 and 2.2 are more general than one of Theorem A. This can be seen from the following.

(i) In Theorem A, set $D_0(z) = d(\zeta)$ ($\zeta = e^z$), so that, by (1.22), we see that

$$\sigma_e(D_0) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r,\ D_0)}{r} = c < \frac{1}{2}$$

if and only if

$$\sigma_e(D_0) = \max\{\sigma_1(d),\ \sigma_1(d^*)\} = c < \frac{1}{2}.$$

Thus, in Theorem 2.1, we omitted the condition "$c < \frac{1}{2}$" (this is a very slashing condition) of Theorem A.

(ii) Though Theorem A does not involve zeros of coefficients obviously, the condition $\max\{\sigma_1(d),\ \sigma_1(d^*)\} = c < \frac{1}{2}$ (that is (2.1)) implies

$$\lambda_e(D_0) = \max\{\lambda_1(d),\ \lambda_1(d^*)\} = \max\{\sigma_1(d),\ \sigma_1(d^*)\} = \sigma_e(D_0) = c < \frac{1}{2}$$

and $\lambda_e(D_j) = \sigma_e(D_j) < \sigma_e(D_0)$ $(j = 1, \ldots, k-2)$.

In Theorem 2.1, the condition (i) implies $0 \le \overline{\lambda_1}(B) < \lambda_1(B) \le \sigma_1(B)$ or $0 \le \overline{\lambda_1}(B) = \lambda_1(B) < \sigma_1(B)$, and A_j $(j = 0,\ 1, \ldots, k-2)$ do not involve zeros essentially. Theorem 2.2 only demands that "$A(z)$ has at least one zero".

Thus, our Theorems 2.1 and 2.2 greatly generalized Theorem A, also generalized results of [7, 9, 11] (note: in [7, 9, 11], the order of the outer function is supposed that is not equal to positive integer or infinity). And the our proofs of following Theorems 2.1 and 2.2 are also totally different from proofs of [2, 7, 9, 11].

3. Lemmas for proof of main results

Lemma 3.1. *If $w(z)$ is meromorphic in $R_0 < |z| < \infty$, then*

$$m_1\left(r, \frac{w^{(k)}}{w}\right) = S_1(r, w), \tag{3.1}$$

where $S_1(r, w)$ denotes any non-negative quantity satisfying

$$S_1(r, w) = o\{T_1(r, w)\} \ n.e.$$

("n.e." denotes: as r $\to \infty$ outside possibly a set of r with finite linear measure).

Proof. Since (1.12) and $m_1\left(r, \frac{\psi_0^{(j)}}{\psi_0}\right) = O(1)$, $j = 1, \ldots, k$, we easy get that

$$m_1\left(r, \frac{w^{(k)}}{w}\right) \le \sum_{j=1}^{k} c_j m\left(r, \frac{h^{(j)}}{h}\right) + O(1) = S(r, h) = o\{T(r, h)\} \text{ n.e.},$$

where c_j $(j = 1, \ldots, k)$ are positive integers. By (1.17a) and (1.17b), we see (3.1) holds. ☐

The following Lemma 3.2 is a generalized Clunie type lemma. Apart from some obvious modification, its proof is identical to that for Lemma 2.1 in [12] and for Lemma 3.3 in [13]. So we omit its proof.

Lemma 3.2. *Let $w(z)$ be a function meromorphic in $R_0 < |z| < \infty$ and satisfying $w^n P(w) = Q(w)$, where $P(w)$ and $Q(w)$ are differential polynomials in $w(z)$ with coefficients $b_j(z)$ meromorphic in $R_0 < |z| < \infty$, and the degree* *of $Q(w)$ is at most n. Then for $r > R_0$,*

$$m_1(r, P(w)) = O\left\{\sum_j m_1(r, b_j) + S_1(r, w)\right\}.$$

Remark 3.1. Let $R_0 > 0$ be a fixed constant. Two conditions have been used frequently in order to characterize the growth of a nonnegative and increasing function $R(r)$ in $(R_0, +\infty)$. That is, either $R(r) = S_1(r, w) = o\{T_1(r, w)\}$ n.e. as $r \to \infty$, or $\sigma_R < \sigma_1(w)$, where

$$\sigma_R = \overline{\lim_{r\to\infty}} \frac{\log R(r)}{\log r}.$$

However, they are not equivalent in general. To unify these practices, we apply the following "combined dominant condition". There exists a constant $d < \sigma_1(w)$ such that $R(r) = S_1(r, w) + o(r^d)$ as $r \to \infty$. For using this dominance, we require the following fact, which is easily checked. If at least one of $R(r) = S_1(r, w)$ and $R(r) = o(r^d)$ as $r \to \infty$ holds, then $R(r) = S_1(r, w) + o(r^d)$ as $r \to \infty$ must hold; equivalently, if $R(r) \neq S_1(r, w) + o(r^d)$ as $r \to \infty$, then $R(r) \neq S_1(r, w)$ and $R(r) \neq o(r^d)$ as $r \to \infty$ must simultaneously hold. Hence, the "combined dominant condition" is more general that either one of the two conditions $R(r) = S_1(r, w)$ and $\sigma_R < \sigma_1(w)$.

The following Lemma 3.3 is a little revised result of Tumure-Clunie type, that is a generalization of Theorem 3.9 in [13]. We use the "combined dominant condition" in Lemma 3.3. Its proof is similar to that of Theorem 3.9 in [13].

Lemma 3.3. *Let $w(z)$ be a function meromorphic and transcendental in $R_0 < |z| < \infty$, and*

$$g(z) = w(z)^n + P_{n-1}(w),$$

providing that there exists a positive constant $d < \sigma_1(w)$ (if $\sigma_1(w) = 0$, set $d = 0$) such that

$$N_1(r, w) + \overline{N}_1(r, 1/g) = S_1(r, w) + o(r^d),$$

where $P_{n-1}(w)$ is a differential polynomial in $w(z)$ with degree at most $n-1$, its coefficients $b_j(z)$ are meromorphic in $R_0 < |z| < \infty$ and satisfies

$$T_1(r, b_j) = S_1(r, w) + o(r^d).$$

Then $g(z) = h(z)^n$, where $h(z) = w(z) + a(z), a(z)$ is a function meromorphic in $R_0 < |z| < \infty$ and satisfies

$$T_1(r, a) = S_1(r, w) + o(r^d).$$

The function $na(z)h(z)^{n-1}$ can be obtained by the substituting h for w, h' for w', etc in terms of degree $n-1$ in $P_{n-1}(w)$.

The following Lemma 3.4 is Lemma 3.5 in [13].

Lemma 3.4. *Suppose that $F(z)$ is meromorphic in a domain D and set*

$$F'(z)/F(z) = w(z).$$

Then we have for $n \geq 1$

$$\frac{F^{(n)}(z)}{F(z)} = w^n + \frac{n(n-1)}{2}w^{n-2}w' + \alpha_n w^{n-3}w'' + \beta_n w^{n-4}w'^2 + P_{n-3}(w),$$

where $\alpha_n = \frac{1}{6}n(n-1)(n-2), \beta_n = \frac{1}{8}n(n-1)(n-2)(n-3)$, and $P_{n-3}(w)$ is a differential polynomial in $w(z)$ with constant coefficients, which vanishes identically for $n \leq 3$ and has the term of degree $n-3$ when $n > 3$.

Lemma 3.5. *Let $F(z)$ be meromorphic in $R_0 < |z| < \infty$ with F'/F transcendental. And let $k \geq 2$, and $D_0(z), D_1(z), \dots, D_{k-1}(z)$ be analytic in $R_0 < |z| < \infty$. Suppose again that there exists a positive constant $d < \sigma_1(F'/F)$ (if $\sigma_1(F'/F) = 0$, set $d = 0$) such that $T_1(r, D_j) = S_1(r, F'/F) + o(r^d)$, $j = 0, 1, \dots, k-1$, and*

$$\overline{N}_1(r, F) + \overline{N}_1(r, \frac{1}{F}) + \overline{N}_1(r, \frac{1}{L_k(F)}) = S_1(r, \frac{F'}{F}) + o(r^d), \tag{3.2}$$

where

$$L_k(F) = F^{(k)} + D_{k-1}F^{(k-1)} + \cdots + D_0F. \tag{3.3}$$

Then $L_k(F)/F = e^{k\varphi(z)+c}$, where c is a constant, both $\varphi'(z)$ and $e^{k\varphi(z)+c}$ are analytic in $R_0 < |z| < \infty$ with $e^{k\varphi(z)+c}$ no zeros in $R_0 < |z| < \infty$, and we have

$$\frac{k(k-1)(k+1)}{24}(\varphi'^2 - 2\varphi'') - \frac{k-1}{2k}D_{k-1}^2 - \frac{k-1}{2}D_{k-1}' + D_{k-2} \equiv 0. \tag{3.4}$$

Proof. Set $w(z) = F'(z)/F(z)$. We get from Lemma 3.4

(3.5)

$$\begin{aligned}\frac{L_k(F)}{F} &= w^k + \frac{k(k-1)}{2}w^{k-2}w' + D_{k-1}w^{k-1} + \alpha_k w^{k-3}w'' + \beta_k w^{k-4}w'^2 \\ &\quad + D_{k-1}\frac{(k-1)(k-2)}{2}w^{k-3}w' + D_{k-2}w^{k-2} + \widetilde{P}_{k-3}(w) = g(z),\end{aligned}$$

where $\widetilde{P}_{k-3}(w)$ is a differential polynomial in w with degree at most $k-3$, its coefficients are polynomials in $D_0, D_1, \dots, D_{k-1}$. Recall that $w(z)$ is transcendental, it follows from (3.2) that

$$\overline{N}_1(r, \frac{1}{g}) = \overline{N}_1(r, \frac{F}{L_k(F)}) \leq \overline{N}_1(r, F) + \overline{N}_1(\frac{1}{L_k(F)}) = S_1(r, w) + o(r^d),$$

$$N_1(r, w) = N_1(r, \frac{F'}{F}) = \overline{N}_1(r, F) + \overline{N}_1(r, \frac{1}{F}) = S_1(r, w) + o(r^d).$$

If we now take into the consideration of the hypotheses on $D_0, D_1, \dots, D_{k-1}$, then the conditions of Lemma 3.3 are satisfied. Therefore, (3.5) becomes $g(z) = h(z)^k$, where $h(z) = w(z) + a(z)$ and $a(z)$ is meromorphic in $R_0 < |z| < \infty$.

$$ka(z)h(z)^{k-1} = \frac{k(k-1)}{2}h(z)^{k-2}h'(z) + D_{k-1}h(z)^{k-1},$$

i.e.,

$$a(z) = \frac{k-1}{2}\frac{h'}{h} + \frac{D_{k-1}}{k}.$$

By Lemma 3.1 and the hypotheses of the lemma, we see that

$$T_1(r,a) = S_1(r,w) + o(r^d).$$

Setting $\varphi' = \frac{2}{k-1}a - \frac{2}{k(k-1)}D_{k-1}$, then

$$a = \frac{k-1}{2}\varphi' + \frac{D_{k-1}}{k},$$

and

$$h' = \varphi' h.$$

From $h' = \varphi' h$ we get $h = e^{\varphi + c_1}$, where c_1 is a constant. Thus, $g(z) = L_k(F)/F = e^{k\varphi + c}$, where $c = kc_1$.

Now we need only to prove that both φ' and $e^{k\varphi+c}$ are analytic in $R_0 < |z| < \infty$, (3.4) holds and that $e^{k\varphi+c}$ has no zeros in $R_0 < |z| < \infty$.

From $h' = \varphi' h$ we get $h'' = (\varphi'' + \varphi'^2)h, \ldots$. Substituting

$$w = h - a, w' = \varphi' h - a', w'' = (\varphi'' + \varphi'^2)h - a'', \ldots$$

into (3.5) gives

$$(h-a)^k + \frac{k(k-1)}{2}(h-a)^{k-2}(\varphi' h - a') + D_{k-1}(h-a)^{k-1} + \tag{3.6}$$
$$\alpha_k (h-a)^{k-3}\{(\varphi'' + \varphi'^2)h - a''\} + \beta_k (h-a)^{k-4}(\varphi' h - a')^2 +$$
$$D_{k-1}\frac{(k-1)(k-2)}{2}(h-a)^{k-3}(\varphi' h - a') + D_{k-2}(h-a)^{k-2} + \widetilde{P}_{k-3}(h-a) \equiv h^k.$$

Expanding the left-hand side of (3.6) and gathering the terms according to the degree of h, and noting that $a = \frac{k-1}{2}\varphi' + \frac{D_{k-1}}{k}$, we get

$$B_{k-2}h^{k-2} + B_{k-3}h^{k-3} + \cdots + B_0 \equiv 0, \tag{3.7}$$

where $B_j (j = 0, \ldots, k-2)$ are differential polynomials in φ' and D_{k-1}, its coefficients are linear polynomials in $D_0, D_1, \ldots, D_{k-2}$ with constant coefficients. It is easy to see that

$$T_1(r, B_j) = S_1(r,w) + o(r^d).$$

If $k = 2$, then $B_{k-2} = B_0 \equiv 0$ from (3.7). If $k > 2$, applying Lemma 3.2 to (3.7) gives

$$m_1(r, B_{k-2}h) = O\{\sum_{j=0}^{k-3} m_1(r, B_j) + S_1(r,h)\} = S_1(r,w) + o(r^d).$$

Since

$$\begin{aligned} N_1(r, B_{k-2}h) &\le N_1(r, B_{k-2}) + N_1(r,h) \\ &= N_1(r, B_{k-2}) + O\{N_1(r,g)\} \\ &= N_1(r, B_{k-2}) + O\{N_1(r,w)\} \end{aligned}$$

$$= S_1(r,w) + o(r^d),$$

we get

$$T_1(r, B_{k-2}h) = S_1(r,w) + o(r^d).$$

If $B_{k-2} \not\equiv 0$, then since $w(z)$ is transcendental, we get

$$\begin{aligned} T_1(r,h) &= T_1(r, B_{k-2}h/B_{k-2}) \\ &\leq T_1(r, B_{k-2}h) + T_1(r, B_{k-2}) + O(\log r) \\ &= S_1(r,w) + o(r^d) + O(\log r) \\ &= S_1(r,w) + o(r^d). \end{aligned}$$

Thus,

$$T_1(r,w) = T_1(r, h-a) = S_1(r,w) + o(r^d),$$

if $\sigma_1(\frac{F'}{F}) = \sigma_1(w) > d$, but above formula implies $\sigma_1(w) \leq d$, this is a contradiction; if $\sigma_1(w) = 0$, then $d = 0$, and above formula implies $T_1(r,w) = S_1(r,w)$, this is also a contradiction. Hence, we must have $B_{k-2} \equiv 0$. Applying the same reasoning repeatedly, we can successful prove $B_{k-3} \equiv \cdots \equiv B_0 \equiv 0$. Calculating (3.6) directly gives

$$B_{k-2} = \frac{k(k-1)(k+1)}{24}(\varphi'^2 - 2\varphi'') - \frac{k-1}{2k}D_{k-1}^2 - \frac{k-1}{2}D'_{k-1} + D_{k-2}. \tag{3.8}$$

(3.8) together with $B_{k-2} \equiv 0$ yields (3.4). We can check that φ' is analytic in $R_0 < |z| < \infty$ from (3.4). Otherwise, assume z_0 is a pole of φ' in $R_0 < |z| < \infty$. Noting that both D_{k-1} and D_{k-2} are analytic in $R_0 < |z| < \infty$, it can be seen from (3.4) that z_0 must be a simple pole of φ', and the principal part of Laurent expansion of φ' in the neighborhood of z_0 is $-\frac{2}{z-z_0}$. From this and

$$\frac{h'}{h} = \varphi', \quad \frac{F'}{F} = h - \frac{k-1}{2}\varphi' - \frac{D_{k-1}}{k},$$

we can obtain in the neighborhood of z_0 that

$$F = (z-z_0)^m e^{b(z)} \exp\left\{\frac{d_0}{z-z_0}\right\},$$

where m is an integer, $b(z)$ is analytic in this neighborhood, d_0 is a non-zero constant. Thus, $F(z)$ has an essential singularity at z_0. This contradicts that $F(z)$ is meromorphic in $R_0 < |z| < \infty$.

This establishes that φ' is analytic and so it has the expansion $\varphi' = \sum_{-\infty}^{+\infty} a_j z^j$ in $R_0 < |z| < \infty$. Thus,

$$e^{\varphi(z)+c_1} = z^{a_{-1}} \exp\left\{\int (\sum_{j=-\infty, j\neq -1}^{+\infty} a_j z^j) dz\right\}. \tag{3.9}$$

Since h is single valued, a_{-1} must be an integer. This indicates that e^{φ} is analytic and has no zeros in $R_0 < |z| < \infty$. Thus, $e^{k\varphi + c}$ is analytic and has no zeros in $R_0 < |z| < \infty$. □

Lemma 3.6. *Let $k \geq 2$, and $D_0(z), D_1(z), \ldots, D_{k-1}(z), D(z)$ be analytic in $R_0 < |z| < \infty$ with $D(z)$ transcendental. Also assume that there exists a positive constant $d < \sigma_1(D)$ (if $\sigma_1(D) = 0$, set $d = 0$) such that*

$$\overline{N}_1(r, 1/D) = S_1(r, D) + o(r^d),$$

$$T_1(r, D_j) = S_1(r, D) + o(r^d),\ j = 0, 1, \ldots, k-1.$$

If the equation

$$y^{(k)} + D_{k-1}y^{(k-1)} + \cdots + D_1 y' + (D_0 + D)y = 0 \tag{3.10}$$

has an analytic solution $F(z) \not\equiv 0$ in $R_0 < |z| < \infty$, and $F(z)$ satisfies that there exists a positive constant $d_1 < \sigma_1(D)$ (if $\sigma_1(D) = 0$, set $d_1 = 0$) such that

$$\overline{N}_1(r, 1/F) = S_1(r, D) + o(r^{d_1}),$$

then D has no zeros in $R_0 < |z| < \infty$ and it has the form $D = e^{k\varphi + c_0}$, where φ' is analytic in $R_0 < |z| < \infty$, c_0 is a constant, and we have the relation (3.4).

Proof. Clearly, if $\sigma_1(D) > 0$, without loss of generality, we may assume $d_1 < d$. Substituting $F(z)$ for y in (3.10) gives

$$L_k(F) = -DF, \tag{3.11}$$

where $L_k(F)$ is defined in (3.3). Hence

$$\overline{N}_1(r, \frac{1}{L_k(F)}) = \overline{N}_1(r, \frac{1}{DF}) \leq \overline{N}_1(r, \frac{1}{F}) + \overline{N}_1(r, \frac{1}{D}) = S_1(r, D) + o(r^d). \tag{3.12}$$

In addition, from Lemma 3.4, setting $w = F'/F$, we have

$$\frac{L_k(F)}{F} = w^k + P_{k-1}(w), \tag{3.13}$$

where $P_{k-1}(w)$ is a differential polynomial in w with degree at most $k-1$, each coefficient of which is a polynomial in $D_0, D_1, \ldots, D_{k-1}$. Combining (3.11) and (3.13) gives

$$w^k + P_{k-1}(w) = -D. \tag{3.14}$$

It is easy to get from (3.14)

$$\begin{aligned} T_1(r, D) &= O\{T_1(r, w) + \sum_{j=0}^{k-1} T_1(r, D_j)\} \text{ n.e.} \\ &= O\{T_1(r, w) + S_1(r, D) + r^d\} \text{ n.e..} \end{aligned}$$

Thus,

$$T_1(r, D) = O\{T_1(r, w) + r^d\} \text{ n.e..} \tag{3.15}$$

On the other hand, applying Lemma 3.2 to (3.14) gives

$$m_1(r, w) = O\{\sum_{j=0}^{k-1} m_1(r, D_j) + m_1(r, D) + S_1(r, w)\}$$

$$= O\{T_1(r, D) + r^d + S_1(r, w)\} \text{ n.e..}$$

Since (note that $F(z)$ is analytic in $R_0 < |z| < \infty$)

$$N_1(r, w) = N_1(r, \frac{F'}{F}) = \overline{N}_1(r, \frac{1}{F}) = S_1(r, D) + o(r^d),$$

we get

$$T_1(r, w) = O\{T_1(r, D) + r^d\} \text{ n.e..} \tag{3.16}$$

It is easy to see that $\sigma_1(w) = \sigma_1(D)$ from (3.15) and (3.16). Hence, from (3.15), the assumptions of this lemma and (3.12), and, noting that $F(z)$ is analytic in $R_0 < |z| < \infty$, we can deduce

$$T_1(r, D_j) = S_1(r, \frac{F'}{F}) + o(r^d), \ j = 0, 1, \ldots, k-1,$$

$$\overline{N}_1(r, F) + \overline{N}_1(r, \frac{1}{F}) + \overline{N}_1(r, \frac{1}{L_k(F)}) = S_1(r, \frac{F'}{F}) + o(r^d).$$

In addition, (3.15) implies that $w = F'/F$ is transcendental since D is transcendental and $\sigma_1(w) = \sigma_1(D)$. Therefore, the conditions of Lemma 3.5 are satisfied. And thus, $L_k(F)/F = e^{k\varphi + c}$. Combining this and (3.11) gives $D = -e^{k\varphi + c} = e^{k\varphi + c_0}$, where $c_0 = c + \pi i$. The remaining conclusions are the same as stated in Lemma 3.5. □

The following Lemma 3.7 is Theorem 2.1 in [10] which generalizes Theorem 2.1 in [2].

Lemma 3.7. *Let $k \geq 2$, and $A_0(z), \ldots, A_{k-2}(z)$ be periodic entire functions with period $2\pi i$, for $k \geq 3$, suppose that there exists a positive constant $d < \sigma_e(A_0)$ (if $\sigma_e(A_0) = 0$, set $d = 0$) such that*

$$T(r, A_j) = S(r, A_0) + o(e^{dr}), \ j = 1, \ldots, k-2. \tag{3.17}$$

If the equation

$$w^{(k)} + A_{k-2}w^{(k-2)} + \cdots + A_0 w = 0 \tag{3.18}$$

has a solution $f(z) \not\equiv 0$, and $f(z)$ satisfies that there exists a positive constant $d_1 < \sigma_e(A_0)$ (if $\sigma_e(A_0) = 0$, set $d_1 = 0$) such that

$$\log^+ N(r, \frac{1}{f}) = S(r, A_0) + o(e^{d_1 r}), \tag{3.19}$$

then there exists an integer $q, 1 \leq q \leq k$, such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent.

Lemma 3.8. *Let q be a positive integer, $U(z)$ be an entire function with periodic $q2\pi i$ and satisfy*

$$U(z) = G(e^{\frac{z}{q}}) = G(\zeta),$$

where $\zeta = e^{\frac{z}{q}}$, $G(\zeta)$ is analytic in $0 < |\zeta| < \infty$, then

$$\lambda_e(U) = \frac{1}{q} \max\{\lambda_1(G), \lambda_1(G^*)\}; \quad \overline{\lambda_e}(U) = \frac{1}{q} \max\{\overline{\lambda_1}(G), \overline{\lambda_1}(G^*)\},$$

where $G^*(t) = G(\frac{1}{t}),\ t = \frac{1}{\zeta}$.

Proof. Let $n(D, \frac{1}{F})$ be the number of zeros of $F(z)$ in the set D. Set $|\zeta| = \rho$. Since the transformation $\zeta = e^{\frac{z}{q}}$ is an one-one correspondence between the set

$$\{z:\ -q\log\rho \le \text{Re } z \le q\log\rho; -q\pi < \text{Im } z \le q\pi\}$$

and the set

$$\{\zeta:\ \rho^{-1} \le |\zeta| \le \rho\},$$

where $|\zeta| = \rho$, we see that

$$\begin{aligned}
&n\left(\rho^{-1} \le |\zeta| \le \rho,\ \frac{1}{G(\zeta)}\right) \\
&= n\left((-q\log\rho \le \text{Re } z \le q\log\rho; -q\pi < \text{Im } z \le q\pi),\ \frac{1}{U(z)}\right) \\
&\le n\left(|z| \le q(\log\rho + \pi),\ \frac{1}{U(z)}\right) \\
&\le 2\left(\frac{q(\log\rho + \pi) - q\pi}{2\pi} + 1\right) \\
&\quad n\left((-q\log\rho \le \text{Re } z \le q\log\rho; -q\pi < \text{Im } z \le q\pi),\ \frac{1}{U(z)}\right) \\
&= \left(\frac{q\log\rho}{\pi} + 2\right) n\left((\rho e^{\pi})^{-1} \le |\zeta| \le \rho e^{\pi},\ \frac{1}{G(\zeta)}\right).
\end{aligned}$$

Thus, we get

$$\lambda_e(U) = \frac{1}{q}\varlimsup_{\rho\to\infty}\frac{\log n\left(\rho^{-1} \le |\zeta| \le \rho,\ \frac{1}{G(\zeta)}\right)}{\log\rho}.$$

Since

$$\begin{aligned}
n\left(\rho^{-1} \le |\zeta| \le \rho,\ \frac{1}{G(\zeta)}\right) &= n\left(1 \le |\zeta| \le \rho,\ \frac{1}{G(\zeta)}\right) \\
&\quad + n\left(\rho^{-1} \le |\zeta| \le 1,\ \frac{1}{G(\zeta)}\right),
\end{aligned}$$

we obtain

$$\lambda_e(U) = \frac{1}{q}\max\{\lambda_1(G), \lambda_1(G^*)\}.$$

By the same reasoning, we also get

$$\overline{\lambda_e}(U) = \frac{1}{q}\max\{\overline{\lambda_1}(G), \overline{\lambda_1}(G^*)\},$$

□

Apart from the representation (1.2), the function $B(\zeta)$ has another representation to be given in the following lemma.

Lemma 3.9. *The function* $B(\zeta)$ *is analytic in* $0 < |\zeta| < \infty$ *if and only if it can be represented to*

$$B(\zeta) = \zeta^{n_0} h_1\left(\frac{1}{\zeta}\right) h_2(\zeta), \tag{3.20}$$

where n_0 *is an integer, both* $h_1(t)$ *and* $h_2(t)$ *are entire functions with* $h_1(0)h_2(0) \neq 0$.

Proof. It is evident that $B(\zeta)$ is analytic in $0 < |\zeta| < \infty$ if it has representation (3.20). We now assume $B(\zeta)$ is analytic in $0 < |\zeta| < \infty$, and deduce (3.20) holds. We denote the Weierstrass products formed from zeros of $B(\zeta)$ in $R_0 \leq |\zeta| < \infty$ by $H_2(\zeta)$ and zeros of $B^*(t) = B(1/t)$ in $\frac{1}{R_0} \leq |t| < \infty$ by $H_1(t)$. Set

$$H(\zeta) = H_1(1/\zeta)H_2(\zeta).$$

Then $\frac{B'(\zeta)}{B(\zeta)} - \frac{H'(\zeta)}{H(\zeta)}$ is analytic in $0 < |\zeta| < \infty$. Thus, it has the expansion

$$\frac{B'(\zeta)}{B(\zeta)} - \frac{H'(\zeta)}{H(\zeta)} = \sum_{j=-\infty}^{+\infty} b_j \zeta^j.$$

From this we get

$$\frac{B(\zeta)}{H(\zeta)} = \zeta^{b_{-1}} \exp\left\{\int (\sum_{j=-\infty, j\neq -1}^{+\infty} b_j \zeta^j) d\zeta\right\}.$$

Since the left side of the above representation is a single-valued function, so b_{-1} must be an integer. This yields (3.20) with $n_0 = b_{-1}$ and

$$\begin{cases} h_1(\frac{1}{\zeta}) = H_1(\frac{1}{\zeta})e^{\varphi_1(1/\zeta)}, \quad h_2(\zeta) = H_2(\zeta)e^{\varphi_2(\zeta)}, \\ \varphi_1(\frac{1}{\zeta}) = \int (\sum\limits_{j=-\infty}^{-2} b_j\zeta^j)d\zeta, \quad \varphi_2(\zeta) = \int(\sum\limits_{j=0}^{+\infty} b_j\zeta^j)d\zeta, \end{cases} \tag{3.21}$$

where $\varphi_1(t)$, $\varphi_2(t)$, $h_1(t)$ and $h_2(t)$ are evidently entire functions. It is easy to see that $H_1(0)H_2(0) \neq 0$, and thus $h_1(0)h_2(0) \neq 0$. □

Representation (3.20) may change as R_0 changes, so it is not unique. But from (3.20) we may get that as making from (1.6)

$$\begin{cases} \sigma_e(A) = \max\{\sigma(h_1), \sigma(h_2)\}, \\ \lambda_e(A) = \max\{\lambda(h_1), \lambda(h_2)\}, \quad \overline{\lambda}_e(A) = \max\{\overline{\lambda}(h_1), \overline{\lambda}(h_2)\}, \\ \sigma_1(B) = \sigma(h_2), \quad \sigma_1(B^*) = \sigma(h_1), \\ \lambda_1(B) = \lambda(h_2), \quad \lambda_1(B^*) = \lambda(h_1), \\ \overline{\lambda}_1(B) = \overline{\lambda}(h_2), \quad \overline{\lambda}_1(B^*) = \overline{\lambda}(h_1). \end{cases} \tag{3.22}$$

4. Proofs of Theorems 2.1 and 2.2

Theorem 2.2 can be directly obtained from Theorem 2.1, we only prove Theorem 2.1

Proof of Theorem 2.1. It follows from (ii), (iii) and (1.22) that $\sigma_e(A_j) < \sigma_e(A)$ for $j \geq 0$ and $\sigma_e(A_j) < \sigma_e(A_0 + A)$ for $j > 0$. Thus, there exists a positive constant $d < \sigma_e(A_0 + A)$ such that $T(r, A_j) = o(e^{dr})$ for $j > 0$, also this and Remark 3.1 lead to

$$T(r, A_j) = S(r, A_0 + A) + o(e^{dr}).$$

We assert that

$$N(r, \frac{1}{f}) \leq (k-1)\overline{N}(r, \frac{1}{f}).$$

In fact, if z_0 is a zero of f multiplicity $\tau(> k)$, then the Laurent expansion of $\frac{f^{(k)}}{f}$ in the neighborhood of z_0 is

$$\frac{f^{(k)}}{f} = \frac{\tau(\tau-1)\cdots(\tau-k+1)}{(z-z_0)^k} + \cdots .$$

Substituting this expression into (2.5) yields a contradiction. Therefore, $\overline{\lambda}_e(f) = \lambda_e(f)$, thus (ii) and (iv) together with (1.22) give

$$\lambda_e(f) < \sigma_1(B_0 + B) \leq \sigma_e(A_0 + A),$$

and this implies that there exists a positive constant $d_1 < \sigma_e(A_0 + A)$ such that $N(r, 1/f) = o(e^{d_1 r})$. It further results in

$$\log^+ N(r, 1/f) = S(r, A_0 + A) + o(e^{d_1 r}).$$

Hence, the conditions of Lemma 3.7 are satisfied. So there exists an integer $q, 1 \leq q \leq k$, such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent. It follows that (see [1, p. 14]) $f(z)$ can be represented as $f(z) = e^{\beta z}U(z)$, where β is a constant, $U(z)$ is a periodic entire function with period $q2\pi i$, i.e., we may write $U(z) = G(e^{z/q})$ with $G(\zeta)$ analytic in $0 < |\zeta| < \infty$. Evidently, $\overline{\lambda}_e(f) = \overline{\lambda}_e(U)$. By Lemma 3.8, we have

$$\overline{\lambda}_e(U) = \frac{1}{q}\max\{\overline{\lambda}_1(G), \overline{\lambda}_1(G^*)\},$$

where $G^*(t) = G(1/t)$. Thus,

$$\overline{\lambda}_e(f) = \frac{1}{q}\max\{\overline{\lambda}_1(G), \overline{\lambda}_1(G^*)\}. \tag{4.1}$$

Substituting $f(z) = \zeta^{\beta q}G(\zeta)$ into equation (2.5), and noting that $\zeta = e^{z/q}$, gives

$$G^{(k)} + D_{k-1}(\zeta)G^{(k-1)} + \cdots + D_1(\zeta)G' + (D_0(\zeta) + D(\zeta))G = 0, \tag{4.2}$$

where
(4.3)
$$\begin{cases} D(\zeta) = \frac{q^k}{\zeta^k} B(\zeta^q), \\ D_{k-j}(\zeta) = \frac{1}{\zeta^j}[c_{k-j}^{(1)} + c_{k-j}^{(2)} B_{k-2}(\zeta^q) + \cdots + c_{k-j}^{(j)} B_{k-j}(\zeta^q)], \quad 1 \le j \le k, \end{cases}$$

$c_{k-j}^{(i)}$ $(1 \le i \le j)$ are constants, and for $j \ge 2$, $c_{k-j}^{(j)} = q^j$. Clearly, $D(\zeta)$ and $D_{k-j}(\zeta)$ $(1 \le j \le k)$ are analytic in $0 < |\zeta| < \infty$, and $\sigma_1(D) = q\sigma_1(B)$,

$$\sigma_1(D_{k-j}) \le q \max\{\sigma_1(B_{k-2}), \dots, \sigma_1(B_{k-j})\} < q\sigma_1(B) = \sigma_1(D), \ 1 \le j \le k,$$

$$\overline{\lambda}_1(D) = q\overline{\lambda}_1(B) < \sigma_1(D).$$

Then, taking an arbitrary constant $R_0 > 0$, there exists a positive constant $d < \sigma_1(D)$ such that

$$T_1(r, D_{k-j}) = o(r^d) \quad (1 \le j \le k), \quad \overline{N}_1(r, 1/D) = o(r^d)$$

in $R_0 < |\zeta| < \infty$, and these lead to

$$T_1(r, D_{k-j}) = S_1(r, D) + o(r^d) \ (1 \le j \le k), \quad \overline{N}_1(r, 1/D) = S_1(r, D) + o(r^d).$$

For the solution $G(\zeta)$ of the equation (4.2),

$$\overline{\lambda}_1(G) \le q\overline{\lambda}_e(U) = q\overline{\lambda}_e(f) < q\sigma_1(B) = \sigma_1(D).$$

Hence, there exists a positive constant $d_1 < \sigma_1(D)$ such that

$$\overline{N}_1(r, 1/G) = o(r^{d_1})$$

in $R_0 < |\zeta| < \infty$, and hence, from Remark 2.1, $\overline{N}_1(r, 1/G) = S_1(r, D) + o(r^{d_1})$. In addition, $D(\zeta)$ is clearly transcendental in $R_0 < |\zeta| < \infty$. Therefore, the conditions of Lemma 3.6 are satisfied, and so $D(\zeta)$ has no zeros in $R_0 < |\zeta| < \infty$. Since R_0 is arbitrary, we see that $D(\zeta)$ has no zeros in $0 < |\zeta| < \infty$. In view of (4.3), $B(\zeta)$ has no zeros in $0 < |\zeta| < \infty$ yet. From (3.20) and (3.21) of Lemma 3.9, we may get (2.6). The proof is completed. □

5. Example

To show property of the following equations, we need the following lemma.

Lemma 5.1 ([6]). *Let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max\{\sigma(A_1), \dots, \sigma(A_{k-1}), \ \lambda(A_0)\} < \sigma(A_0) = \sigma \quad (0 < \sigma \le \infty),$$

and that A_0 has at least one zero whose multiplicity is not a multiple of k. Then every solution f of the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0$$

satisfies $\overline{\lambda}(f) \ge \sigma$.

Example 1. Consider the equations

$$f^{(k)} + (e^z - 1)\exp\{e^z\} f = 0 \quad (k \geq 3), \tag{5.1}$$

we see the equation (5.1) satisfies all conditions of Theorem 2.2,

By Lemma 5.1, we can see that every solution of (5.1) satisfies $\overline{\lambda}(f) = \infty$.

Moreover, we can know that in the equation

$$A(z) = B(e^z) = B(\zeta) = (\zeta - 1)e^{\zeta},$$

it do not satisfy the form $B(\zeta) = \zeta^m e^{g(\zeta)}$ in Theorem 2.1.

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