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OPTIMAL POLICY NETWORKS

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ABSTRACT. This paper focuses on the situation of optimizing the total cost with m given messages and n network nodes. Associated with each network node, a fixed cost is incurred to the receiver if at least one message is received. The mean and variance of the total costs are obtained. Normal approximation is used. Empirical results showed that the derived method reduces research work substantially.

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1. Introduction

Suppose that we have the communication links which consists m messages and n possible network nodes, where a message may be selected and transmitted at any one, but not more than one, of the network nodes. Associated with each node, there is a fixed cost if at least one message is transmitted through a node. Then it is worthwhile to challenge the problem on how to select that messages and network nodes so that the sum of the benefits generated by the messages transmitted is maximizes, subject to a constraint on the total network cost.

Consider the sets of all feasible solutions associated with exactly n nodes in the network. Determine the upper and lower bounds for the number of the nodes and the corresponding sets where optimum solutions are likely to be found in certain statistical sense. Then apply exhaustive search, random sampling, or heuristics to obtain the best solutions in those sets. The following is the mathematical formulation of the problem.

Let $i = 1, 2, ..., m$, denote the messages; and $j = 1, 2, ..., n$ the network nodes. Define

$$x_{ij} = \begin{cases} 1, & \text{if ith message is received from jth network node,} \\ 0, & \text{if otherwise; and} \end{cases} \tag{1}$$

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$$y_j \ = \text{Minimum}\ \{\sum_{i=1}^{m}\ x_{ij}, 1\},\ \ i = 1, ..., m, j = 1, 2, ..., n. \tag{2}$$

In other words, $y_j = 1$, if at least one message is received from the jth network node, otherwise $y_j = 0$. Let p_{ij} be the expense of one unit of the ith message received by the jth network node; and c_j, the fixed cost to be receiver, if at least one message is bought from the jth network node; Then the problem of optimum network cost incurred may be written as;

$$\text{Minimize } C = \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}x_{ij} + \sum_{j=1}^{n} c_j y_j, \tag{3}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij}, x_{ij} \le y_j, \tag{4}$$

where x_{ij}, $y_j = 0, 1$, for all $i = 1, ..., m$, and $j = 1, ..., n$. A feasible solution to the problem may be represented by a vector $s = (x_{ij}, y_j, i = 1, ..., m,$ and $j = 1, ..., n)$. However, those for which (2) does not hold need not be considered. Thus we may define the set of all feasible solutions as

$$S = \text{The space of all } s = (x_{ij}, y_j, i = 1, ..., m, \text{and } j = 1, ..., n) \tag{5}$$

which satisfy (1) and (2). The problem is similar to those of optimum location selections [1, 2, 3, 5]. For any $k = 1, ..., n$, and any subset J_k of k elements out of set $\{1, 2, ..., n\}$, define

$$S(k) \ = \ \left\{ All\ s\ \varepsilon \quad S\ : \sum_{n}^{j=1} \quad y_j \quad = \ k \right\} \text{and} \tag{6}$$

$$S(J_k) \ = \ \{All \quad s \quad \varepsilon \quad S\ :\ y_j \ = \ 1\, if\ and\ only\ if\ j\ \varepsilon\ J_k\}\,.$$

In other words, $S(k)$ represents all the possible ways of receiving the messages from exactly k network nodes, and $S(J_k)$, from k given network nodes. The numbers of feasible solutions in S, $S(k)$, and $S(J_k)$ are respectively n^m, N_k and $A(k, m)$, which are given by [4] as

$$A(k, m) = \sum_{i=0}^{k} \quad (-1)^i \quad \binom{k}{i} \quad (k-i)^m, \quad \text{and } N_k \ = \ \binom{n}{k} A(k, m). \tag{7}$$

These numbers are quite large, even for moderately sized m and n. Consequently, to locate the optimum solution is often a difficult problem. However, for the some reason we believe statistical type of search can be fruitful. The following are the results of a type of search procedure which may be of practical interest and worthy of further investigation.

2. The Conditional Mean and The Variance

The following are a Lemma 2.1, the derivation of the conditional mean.

Lemma 2.1.

$$A(k, m-1) + A(k-1, m-1) = (1/k)A(k, m)$$

From (3) and (5), for any solution s in S, denote the corresponding total cost by

$$C(s) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}x_{ij}(s) + \sum_{j=1}^{n} c_j y_j(s)$$

$$\mu_k = \frac{1}{\binom{n}{k}A(k,m)} \left\{ \sum_{J_k} \sum_{s\varepsilon S(J_k)} \left[\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}x_{ij}(s) + \sum_{j=1}^{n} c_j y_j(s) \right] \right\} \tag{8}$$

$$\sum_{J_k} \sum_{s\varepsilon S(J_k)} \sum_{i} \sum_{j} p_{ij}x_{ij}(s) = \sum_{J_k} \sum_{s\varepsilon S(J_k)} \sum_{i} \sum_{j\varepsilon J_k} p_{ij}x_{ij}(s)$$

$$= \sum_{J_k} \sum_{j\varepsilon J_k} \sum_{s\varepsilon S(J_k)} \sum_{i} p_{ij}\, x_{ij}\, (s) \tag{9}$$

For given i_1 and j in a given J_k,

$$S(J_k) = \{s\varepsilon S(J_k) \text{ and } x_{i_1 j} = 0\} \bigcup \{s\varepsilon S(J_k), x_{i_1 j} = 1, x_{ij} = 0,\ i \neq i_1\}$$

$$\bigcup_{i_2 \neq i_1} \{s\varepsilon S(J_k),\ x_{ij} = 1,\ i = i_1, i_2;\ x_{ij} = 0,\ i \neq i_1,\ i_2\}$$

$$\bigcup_{i_2, i_3 \neq i_1} \{s\varepsilon S(J_k), x_{ij} = 1, i = i_1, i_2, i_3; x_{ij} = 0, \text{otherwise}\} \bigcup \cdots$$

Except for the first subset, the total number of elements in the second, third, $\cdots$ subsets are:

$$A\,(k-1, m-1) + \binom{m-1}{1}A(k-1, m-2) + \binom{m-1}{2}A(k-1, m-3) + \cdots =$$

$$A(k-1, m-1) + \sum_{i=1}^{m-k} \binom{m-1}{i} A(k-1, m-1-i) = A(k-1, m-1) + A(k, m-1).$$

Therefore, (9) is equivalent to

$$\sum_{J_k} \sum_{j\varepsilon J_k} \{A(k-1, m-1) + A(k, m-1)\} \sum_{i=1}^{m} p_{ij}$$

$$= \{A(k-1, m-1) + A(k, m-1)\} \sum_{J_k} \sum_{j\varepsilon J_k} \sum_{i=1}^{m} p_{ij}$$

$$= \{A(k-1,m-1) + A(k,m-1)\} \binom{n-1}{k-1} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}. \tag{10}$$

Similarly, one obtains

$$\sum_{J_k} \sum_{s\varepsilon S(J_k)} \sum_{j} c_j\, y_j(s) = \sum_{J_k} \sum s\varepsilon S(J_k) \,(\sum_{j\varepsilon J_k} c_j)$$

$$= \sum_{J_k} \sum_{s\varepsilon S(J_k)} \sum_{j} c_j y_j(s) = \sum s\varepsilon S(J_k)(\sum_{J\varepsilon J_k} c_j)$$

$$= A(k,m) \sum_{J_k} (\sum_{j\varepsilon J_k} c_j) = A(k,m) \binom{n-1}{k-1} \sum_{j=1}^{n} c_j. \tag{11}$$

From (8), (10), and (11), we have

$$\mu_k = \frac{\binom{n-1}{k-1} \{A(k-1,m-1) + A(k,m-1)\}}{\binom{n}{k} A(k,m)} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} + \frac{\binom{n-1}{k-1} A(k,m))}{\binom{n}{k} A(k,m)} \sum_{j=1}^{n} c_j$$

By the Lemma 2.1, we obtain the conditional mean μ_k as

$$\mu_k = m\overline{p} + k\overline{c}\,, \text{ and} \tag{12}$$

On the other hand, we can find variance σ_k^2 as

$$\sigma_k^2 = (1 - r_{km}) \sum_{i=1}^{m} V_i + (r_{km} - 1/n)V_J + 2\,(1 - k/n)\, V_{JF} \mid k\,(1 - k/n)\, V_F \tag{13}$$

where $p_i. = \sum_{n}^{j=1} p_{ij},\ p_{ij} = \sum_{m}^{i=1} p_{ij},\ \overline{p} = \sum_{n}^{j=1} p_{.j}/n,$

$$\overline{p}_{i.}\; p_{i.}\;/n, \overline{p} = \overline{p}_{.}/m\;, \overline{c} = \sum_{n}^{j=1} c_j/n\;,\; r_{km} = A(k,m-1)\;/\;A(k,m)\;,$$

$$V_i = \sum_{n}^{j=1} (p_{ij} - \overline{p}_{i.})^2/(n-1),\; V_J = \sum_{n}^{j=1} (p_{.j} - \overline{p}_{.})^2/(n-1),$$

$$V_{JF} = \sum_{n}^{j=1} (c_j - \overline{c})\,(p_{.j} - \overline{p}_{.})/(n-1),\; V_F = \sum_{n}^{j=1} (c_j - \overline{c})^2/(n-1).$$

3. The Result from Algorithm Analysis

Since $C(s)$ is a linear sum of random variables, the distribution of $C(s)$ over S_k, $k = 1,...,n$, is approximately normal when n is large, and its mean and variance may be approximated by μ_k and σ_k^2 in (12) and (13) respectively. Now define

$$C_k = \min C(s) \text{ for all } s\;\varepsilon\; S_k. \tag{14}$$

If C_k is viewed as the smallest order statistic associated with a random sample of size N_k defined in (7), from the normal distribution said above, then the asymptotic distribution of

$$X_k \;=\; N_k\, F((C_k - \mu_k)/\sigma_k), \tag{15}$$

where $F(x)$ is the standard normal cumulative distribution function which is exponential with mean equal to 1 (see [2]). Now from (15) and [2, 4], although the detailed approximate procedures are omitted in this paper, we have approximated C_k as

$$C_k \;\approx\; \mu_k \;-\sigma_k\; \{(\pi/2)\; log\; (N_k/4X_k)\}^{1/2} \tag{16}$$

where the term $4X_k^2 \;/\; N_k^2$ is omitted for large N_k. When C_k and X_k are replaced by their expected values, we obtain

$$\tau_k \;=\; E(C_k) \approx\; \mu_k - \sigma_k \{(\pi/2)\; log\; (N_k/4)\}^{1/2}. \tag{17}$$

Using (12), (13), and (17), τ_k may be computed for each $k = 1, ..., n.$

4. Conclusions

We have derived (17) from Section 3. Now if we let t be such that $\tau_t =$ min τ_k for all $k = 1, 2, ..., n$, then we can compute t which minimizes τ_k for $k = 1, 2, ..., n$. Computer simulation of 100 randomly generated procurement problems revealed the following: 50% of the time, the optimum k is either t or. $t-1$, 90% of the time, it is either t, $t-1$, or $t-2$, and 100% of the time, it is either t, $t-1$, $t-2$, or $t-3$. In other words, if $n = 10$, the number of S_k which need to be searched may be reduced by about 60%, and if $n = 20$, by 80%. Finally, sampling methods have also been devised to obtain solutions from those screened S_k.

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