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STABILIZATION OF SWITCHED SYSTEMS WITH UNCONTROLLABLE SUBSYSTEMS†

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ABSTRACT. In this paper, we study the stabilization problem of switched systems with both controllable and uncontrollable subsystems. By using an average dwell time approach, we first establish a sufficient condition such that the switched system is exponentially stabilizable under appropriate switching signals. We also extend this result to the switched system with nonlinear impulse effects and disturbances. Numerical examples are given to illustrate the theoretical results.

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1. Introduction

For a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time (discrete-time) subsystems and a rule orchestrating the switching between the subsystems. In recent years, the switched system has attracted considerable efforts, see [1]-[11]. This is because switched systems have strong engineering backgrounds such as computer disk system [12], robotics [13], power systems [14], air traffic management [15], and because that the methods of intelligent control design are based on the idea of switching between different controllers [4] and [16].

For systems that switch among a finite set of controllable linear systems, the stabilization problem of switched systems with arbitrary switching frequency has been studied in [17] and [18] by developing an improved estimation on transition matrices. Very recently, the results in [17] and [18] were further extended to switched systems with nonlinear impulses and perturbations [19].

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In this paper, motivated by the above work, we study continuous-time switched systems with both controllable and uncontrollable subsystems. Switching signals are proposed which ensure that the entire switched system is exponentially stabilizable. By using an average dwell time approach and the multiple Lyapunov functional method, we first show that the switched system is also stabilizable under arbitrary switching signal with any given average dwell time even when there exist both controllable and uncontrollable subsystems. Then, we extend this result to the switched system with nonlinear impulse effects and disturbances.

This paper is organized as follows. In Section 2, the stabilization of switched linear systems with both controllable and uncontrollable subsystems is investigated. We study the stabilization of switched systems with nonlinear impulse effects and disturbances in Section 3. In Section 4, we work out two examples to illustrate the theoretical results. A conclusion is given in Section 5.

2. Stabilization of switched systems with uncontrollable subsystems

We consider the following switched linear system of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, x_0 is the initial state. $\sigma(t) : [t_0, \infty) \to \Lambda = \{1, 2, \cdots N\}$ is a piecewise constant function, called a switching signal. When $\sigma(t) = i$ for $i \in \Lambda$, it means that the ith subsystem $\dot{x}(t) = A_i x(t) + B_i u(t)$ is activated. The switching moments $0 \le t_0 < t_1 < t_2 < \cdots$ of $\sigma(t)$ are defined recursively by $t_{k+1} = \inf\{t > t_k : \sigma(t) \ne \sigma(t_k)\}$ for $k = 0, 1, 2, \cdots$. Throughout this paper, we denote $\sigma(t) = i_k \in \Lambda$ for $t_k \le t < t_{k+1}$.

Unlike some literatures [17], [18] and [20], we here do not assume that all the pairs (A_i, B_i) for $i \in \Lambda$ are controllable. Without loss of generality, we assume that

(H1): (A_1, B_1),$\cdots$,(A_p, B_p) are uncontrollable, and (A_{p+1}, B_{p+1}), $\cdots$, (A_N, B_N) are controllable, where $1 \le p < N$.

For any given switching signal $\sigma(t)$, the switched system (1) is said to be exponentially stabilizable if there exist positive numbers L and λ such that every solution of the system satisfies

$$\|x(t)\| \le Le^{-\lambda(t-t_0)}\|x_0\|, \ t \ge t_0.$$

Lemma 2.1 ([18])**.** *Let $A \in R^{n\times n}$ and $B \in R^{n\times m}$ be constant matrices such that the pair (A, B) is controllable. Then for any $\lambda > 1$, there exists a matrix $K \in R^{m\times n}$ such that*

$$\|e^{(A+BK)t}\| \le M\lambda^{n-1}e^{-\lambda(t-t_0)}, \quad t \ge t_0,$$

where $M > 0$ is a constant, which is independent of λ and can be estimated in terms of A, B and n.

For any $t \ge t_0$, assume that $t_k \le t < t_{k+1}$ for some $k \in \{0, 1, 2, \cdots\}$. When $\sigma(t) = i_k \in \{p + 1, p + 2, \cdots , N\}$, by Lemma 2.1, we see that for any $\lambda > 1$, there exists a gain matrix K_{i_k} such that the state of the closed-loop system $\dot{x} = (A_{i_k} + B_{i_k}K_{i_k})x$ satisfies

$$\|x(t)\| = \|e^{(A_{i_k}+B_{i_k}K_{i_k})(t-t_k)}x(t_k)\| \leq M\lambda^{n-1}e^{-\lambda(t-t_k)}\|x(t_k)\|, \tag{2}$$

where $M = \max\limits_{p<i_k\leq N} M_{i_k}$, and $M_{i_k} > 0$ is determined by $\{A_{i_k}, B_{i_k}\}_{p<i_k\leq N}$ and n.

In order to estimate the norm of the state for those uncontrollable subsystems, we consider the following optimization problem.

Maximize $\lambda_{i_k} > 0$ subject to

$$X_{i_k}A_{i_k}^T + A_{i_k}X_{i_k} + Y_{i_k}^T B_{i_k}^T + B_{i_k}Y_{i_k} < \lambda_{i_k}X_{i_k}, \quad 1 \leq i_k \leq p, \tag{3}$$

where X_{i_k} is a positive-definite matrix, and Y_{i_k} is a free weighting matrix for $i_k = 1, 2\cdots, p$. The above optimization problem can be solved by using the LMI Toolbox of Matlab in [21].

Let $P_{i_k} = X_{i_k}^{-1}$ and $K_{i_k} = Y_{i_k}P_{i_k}$. Multiplying P_{i_k} on both sides of (3), we have that (3) is equivalent to

$$(A_{i_k} + B_{i_k}K_{i_k})^T P_{i_k} + P_{i_k}(A_{i_k} + B_{i_k}K_{i_k}) < \lambda_{i_k}P_{i_k}, \quad 1 \leq i_k \leq p. \tag{4}$$

Define the following piecewise Lyapunov function

$$V_{i_k}(x(t)) = x^T(t)P_{i_k}x(t), \quad t_k \leq t < t_{k+1}. \tag{5}$$

It is easy to see that

$$\lambda_{\min}(P_{i_k})\|x\|^2 \leq V_{i_k}(x) \leq \lambda_{\max}(P_{i_k})\|x\|^2, \tag{6}$$

where $\lambda_{\min}(P_{i_k})$ and $\lambda_{\max}(P_{i_k})$ denote the smallest and the largest eigenvalue of the positive definite symmetric matrix P_{i_k}.

Lemma 2.2. *For any $t \geq t_0$, assume that $t_k \leq t < t_{k+1}$ for some $k \in \{0, 1, 2, \cdots\}$. If $\sigma(t) = i_k \in \{1, 2, \cdots, p\}$, there exists a gain matrix K_{i_k} and a constant $\lambda_{i_k} > 0$ such that the state of the closed-loop system $\dot{x} = (A_{i_k}+B_{i_k}K_{i_k})x$ satisfies*

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P_{i_k})}{\lambda_{\min}(P_{i_k})}}e^{\frac{\lambda_{i_k}}{2}(t-t_k)}\|x(t_k)\|, \quad t_k \leq t < t_{k+1}.$$

Proof. Let the piecewise Lyapunov function be defined by (5). Along the solution of system (1) on $[t_k, t]$, we have

$$\dot{V}_{i_k}(x(t)) = x^T(t)[(A_{i_k} + B_{i_k}K_{i_k})^T P_{i_k} + P_{i_k}(A_{i_k} + B_{i_k}K_{i_k})]x(t). \tag{7}$$

By (4) and (7) we obtain

$$\begin{aligned} &\dot{V}_{i_k}(x(t)) - \lambda_{i_k}V_{i_k}(x(t)) \\ &= x^T(t)[(A_{i_k} + B_{i_k}K_{i_k})^T P_{i_k} + P_{i_k}(A_{i_k} + B_{i_k}K_{i_k}) - \lambda_{i_k}P_{i_k}]x(t) \qquad (8) \\ &< 0. \end{aligned}$$

Multiplying $e^{-\lambda_{i_k}(t-t_k)}$ on both sides of (8) and integrating it from t_k to t, we get

$$V_{i_k}(x(t)) \leq e^{\lambda_{i_k}(t-t_k)} V_{i_k}(x(t_k), \quad t_k \leq t < t_{k+1}. \tag{9}$$

By (6) and (9), we have

$$\lambda_{\min}(P_{i_k})\|x(t)\|^2 \leq e^{\lambda_{i_k}(t-t_k)} \lambda_{\max}(P_{i_k})\|x(t_k)\|^2, \quad t_k \leq t < t_{k+1}.$$

Thus

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P_{i_k})}{\lambda_{\min}(P_{i_k})}} e^{\frac{\lambda_{i_k}}{2}(t-t_k)} \|x(t_k)\|, \ t_k \leq t < t_{k+1}, 1 \leq i_k \leq p.$$

This completes the proof of Lemma 2.2. □

Now, we apply Lemma 2.1 and lemma 2.2 to the stabilization problem of the switched system. We first introduce the concept of average dwell time. For any switching signal $\sigma(t)$ and any $t \geq \tau$, let $N_\sigma(\tau, t)$ denote the number of switchings of $\sigma(t)$ over the interval $[\tau, t)$ [22]. Let $S_a[\tau_a, N_0]$ denote the set of all switching signals satisfying

$$N_\sigma(\tau, t) \leq N_0 + \frac{t-\tau}{\tau_a},$$

where the constant τ_a is called the average dwell time and N_0 denotes the chatter bound. If we ignore the first N_0 switchings, then the average time interval between consecutive switchings is at least τ_a. Without loss of generality, we assume that $N_0 = 0$.

We denote j_1 the number of activated uncontrollable subsystems over $[t_0, t)$, and denote j_2 the number of activated controllable subsystems over $[t_0, t)$. Thus, we have that $j_1 + j_2 = N_\sigma(t_0, t)$. For any switching signal $\sigma(t)$, we denote $T_1(t_0, t)$ the total activation time of uncontrollable subsystems on $[t_0, t)$, and denote $T_2(t_0, t)$ the total activation time of controllable subsystems on $[t_0, t)$. It is easy to see that $T_1(t_0, t) + T_2(t_0, t) = t - t_0$.

Define

$$\lambda_+ = \max_{1 \leq q \leq p} \lambda_q.$$

Determine the switching signal $\sigma(t)$ so that

$$\frac{T_2(t_0, t)}{T_1(t_0, t)} \geq \gamma \tag{10}$$

holds for some $\gamma > 0$ and any $t > t_0 \geq 0$. By (10) and the fact that $T_1(t_0, t) + T_2(t_0, t) = t - t_0$, we have that

$$T_1(t_0, t) \leq \frac{1}{1+\gamma}(t - t_0), \quad T_2(t_0, t) \geq \frac{\gamma}{1+\gamma}(t - t_0). \tag{11}$$

Theorem 2.3. *Assume that (H1) and* (4) *hold. For any given* $\tau_a^* > 0$ *and* $\gamma > 0$*, there exist a set of feedback matrices* $\{K_i\}_{i \in \Lambda}$ *such that the resulting closed-loop system of* (1) *is exponentially stabilizable for any switching signal satisfying* (10) *and* $\tau_a > \tau_a^*$.

Proof. Set

$$L = \max_{1 \le i \le p} \left\{ \sqrt{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}}, M \right\}. \tag{12}$$

It is easy to see that $L > 1$. For any $\tau_a^* > 0$ and $\gamma > 0$, choose $\lambda > 1$ sufficiently large such that

$$c_1 := \frac{2\gamma\lambda - \lambda_+}{2(1+\gamma)} - \frac{\ln L + (n-1)\ln\lambda}{\tau_a^*} > 0. \tag{13}$$

For any $t \ge t_0$, assume that $t_k \le t < t_{k+1}$ and $\sigma(t) = i_k$. If $i_k \in \{p+1, p+2, \cdots, N\}$, by Lemma 2.1, there exists a feedback matrix K_{i_k} such that (2) holds. By (12), we have that

$$\|x(t)\| \le L\lambda^{n-1} e^{-\lambda(t-t_k)} \|x(t_k)\|. \tag{14}$$

If $i_k \in \{1, 2, \cdots, p\}$, by Lemma 2.2, there exists a feedback matrix K_{i_k} such that

$$\begin{aligned} \|x(t)\| &\le \sqrt{\frac{\lambda_{\max}(P_{i_k})}{\lambda_{\min}(P_{i_k})}} e^{\frac{\lambda_{i_k}}{2}(t-t_k)} \|x(t_k)\| \\ &\le L e^{\frac{\lambda_+}{2}(t-t_k)} \|x(t_k)\|. \end{aligned} \tag{15}$$

Similarly, there exist a series of feedback matrices $\{K_{i_{j-1}}\}$ for $j \in \{1, 2, \cdots, k\}$ such that

$$\|x(t_j)\| \le \begin{cases} L\lambda^{n-1} e^{-\lambda(t_j - t_{j-1})} \|x(t_{j-1})\|, \ \sigma(t_{j-1}) \in \{p+1, \cdots, N\}; \\ L e^{\frac{\lambda_+}{2}(t_j - t_{j-1})} \|x(t_{j-1})\|, \ \sigma(t_{j-1}) \in \{1, 2, \cdots, p\}. \end{cases} \tag{16}$$

Therefore, under the feedback law $u(t) = K_{i_j} x(t)$ for $j = 0, 1, \cdots, k$, we get from (14)-(16) by induction that

$$\begin{aligned} \|x(t)\| &\le (L\lambda^{n-1})^{j_2} e^{-\lambda T_2(t_0,t)} L^{j_1} e^{\frac{\lambda_+}{2} T_1(t_0,t)} \|x_0\| \\ &\le (L\lambda^{n-1})^{j_1+j_2} e^{\frac{\lambda_+}{2} T_1(t_0,t) - \lambda T_2(t_0,t)} \|x_0\| \\ &= e^{N_\sigma(t_0,t)[\ln L + (n-1)\ln\lambda] + \frac{\lambda_+}{2} T_1(t_0,t) - \lambda T_2(t_0,t)} \|x_0\|. \end{aligned} \tag{17}$$

Noting that

$$N_\sigma(t_0, t) \le \frac{t - t_0}{\tau_a^*}. \tag{18}$$

By (10), (11), (17) and (18), we get

$$\begin{aligned} \|x(t)\| &\le e^{\frac{\ln L + (n-1)\ln\lambda}{\tau_a^*}(t-t_0) + \frac{\lambda_+}{2(1+\gamma)}(t-t_0) - \frac{\gamma\lambda}{1+\gamma}}(t-t_0)\|x_0\| \\ &= e^{-\left(\frac{2\gamma\lambda-\lambda_+}{2(1+\gamma)} - \frac{\ln L+(n-1)\ln\lambda}{\tau_a^*}\right)(t-t_0)} \|x_0\| \\ &= e^{-c_1(t-t_0)} \|x_0\|, \end{aligned}$$

where $c_1 > 0$ is defined by (13). This completes the proof of Theorem 2.3. □

3. Stabilization of Switched Systems with Nonlinear Impulse Effects and Disturbances

In this section, we consider the following switched system with nonlinear impulse effects and disturbances

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + f_{\sigma(t)}(x(t)), \quad t \neq t_k, \\ x(t_k^+) = g_k(x(t_k)), \quad x(t_0) = x_0, \end{cases} \tag{19}$$

where $x(t)$, $u(t)$, $\sigma(t)$ and t_k $(k = 0, 1, \cdots)$ are the same as above, $x(t_k^{\pm}) = \lim_{v\to 0+} x(t_k \pm v)$, $x(t_k^-) = x(t_k)$, $g_k : \mathbf{R}^n \to \mathbf{R}^n$ is the incremental change of the state variable at the time t_k, and $f_i : \mathbf{R}^n \to \mathbf{R}^n$ is the piecewise continuous disturbance for $i \in \Lambda$, A_i and B_i $(i \in \Lambda)$ are constant matrices of appropriate dimensions.

Throughout this section, we need the following assumption

(H2): $\|f_i(x)\| \le \varepsilon\|x\|$, and $\|g_k(x)\| \le \delta\|x\|$ for $i \in \Lambda$ and $k = 1, 2, \cdots$, where ε and δ are positive constants.

We also need the following inequality of Gronwall type ([23], Corollary 3, p.3).

Lemma 3.1. *Let y and h be real continuous functions defined on $[a, b]$ with $b > a \ge 0$, $h(t) \ge 0$ for $t \in [a, b]$, $c \ge 0$ is a constant. Then,*

$$y(t) \le c + \int_a^t h(s)y(s)ds, \quad t \in [a, b]$$

implies that

$$y(t) \le ce^{\int_a^t h(s)ds}, \quad t \in [a, b].$$

Lemma 3.2. *Assume that (H2) holds and $t_k \le t < t_{k+1}$ for some $k \in \{0, 1, 2, \cdots\}$. If $\sigma(t) = i_k \in \{p+1, p+2, \cdots N\}$, there exists a gain matrix K_{i_k} for any $\lambda > 1$ such that the state of the subsystem*

$$\begin{cases} \dot{x}(t) = (A_{i_k} + B_{i_k}K_{i_k})x(t) + f_{i_k}(x(t)), \quad t \in (t_k, t_{k+1}), \\ x(t_k^+) = g_k(x(t_k)) \end{cases} \tag{20}$$

satisfies

$$\|x(t)\| \le \delta M\lambda^{n-1}e^{(\varepsilon M\lambda^{n-2}-1)\lambda(t-t_k)}\|x(t_k)\|.$$

Proof. By Lemma 2.1, there exist a gain matrix K_{i_k} and a constant $M > 0$ such that

$$\|e^{(A_{i_k}+B_{i_k}K_{i_k})t}\| \le M\lambda^{n-1}e^{-\lambda t}, \quad t_k \le t < t_{k+1}. \tag{21}$$

For the system (20), we have

$$\begin{aligned} x(t) \;\; &= \;\; e^{(A_{i_k}+B_{i_k}K_{i_k})(t-t_k)}x(t_k^+) \\ &\quad + \int_{t_k}^t e^{(A_{i_k}+B_{i_k}K_{i_k})(t-s)}f_{i_k}(x(s))ds, \; t \in (t_k, t_{k+1}). \end{aligned} \tag{22}$$

From (21) and (22), we get

$$\begin{aligned}\|x(t)\| &\leq M\lambda^{n-1}e^{-\lambda(t-t_k)}\|x(t_k^+)\| \\ &\quad +\varepsilon M\lambda^{n-1}\int_{t_k}^{t} e^{-\lambda(t-s)}\|x(s)\|ds.\end{aligned}$$

Set $z(t) = e^{\lambda(t-t_k)}x(t)$. We have

$$\|z(t)\| \leq M\lambda^{n-1}\|x(t_k^+)\| + \varepsilon M\lambda^{n-1}\int_{t_k}^{t}\|z(s)\|ds,\ t \in (t_k, t_{k+1}). \quad (23)$$

By (23) and Lemma 3.1, we have

$$\|z(t)\| \leq M\lambda^{n-1}e^{\varepsilon M\lambda^{n-1}(t-t_k)}\|x(t_k^+)\|,\ t \in (t_k, t_{k+1}).$$

Therefore,

$$\begin{aligned}\|x(t)\| &\leq M\lambda^{n-1}e^{(\varepsilon M\lambda^{n-2}-1)\lambda(t-t_k)}\|x(t_k^+)\| \\ &\leq \delta M\lambda^{n-1}e^{(\varepsilon M\lambda^{n-2}-1)\lambda(t-t_k)}\|x(t_k)\|.\end{aligned}$$

This completes the proof of Lemma 3.2. □

For uncontrollable subsystems, from (H2) we obtain

$$f_{i_k}^T(x(t))f_{i_k}(x(t)) \leq \varepsilon^2 x^T(t)x(t), \quad 1 \leq i_k \leq p. \quad (24)$$

On the other hand, for any given feedback matrices $\{K_{i_k}\}_{1\leq i_k\leq p}$, there always exist a set of positive scalars $\{\mu_{i_k}\}_{1\leq i_k\leq p}$ such that

$$(A_{i_k} + B_{i_k}K_{i_k})^T + (A_{i_k} + B_{i_k}K_{i_k}) + (\varepsilon^2 + 1)I_n < \mu_{i_k}I_n. \quad (25)$$

Note that the above inequality can be easily solved by using the LMI Toolbox [21].

Lemma 3.3. *Assume that (H2) and* (25) *hold, and* $t_k \leq t < t_{k+1}$ *for some* $k \in \{0, 1, 2, \cdots\}$. *If* $\sigma(t) = i_k \in \{1, 2, \cdots p\}$, *there exists a gain matrix* K_{i_k} *and constant* $\mu_{i_k} > 0$ *such that the state of the subsystem* (20) *satisfies*

$$\|x(t)\| \leq \delta e^{\frac{\mu_{i_k}}{2}(t-t_k)}\|x(t_k)\|.$$

Proof. Choose a common Lyapunov function $V(x(t)) = x^T(t)x(t)$ for all uncontrollable subsystems of the switched system (19). Along the solution of system (20) on (t_k, t_{k+1}), we have

$$\dot{V}(x(t)) = x^T(t)[(A_{i_k} + B_{i_k}K_{i_k})^T + (A_{i_k} + B_{i_k}K_{i_k})]x(t) + 2x^T(t)f_{i_k}(x(t)). \quad (26)$$

Note that

$$2x^T(t)f_{i_k}(x(t)) \leq x^T(t)x(t) + f_{i_k}^T(x(t))f_{i_k}(x(t)). \quad (27)$$

By (24), (26) and (27), we obtain

$$\dot{V}(x(t)) \leq x^T(t)[(A_{i_k} + B_{i_k}K_{i_k})^T + (A_{i_k} + B_{i_k}K_{i_k}) + (\varepsilon^2 + 1)]x(t).$$

We obtain from (25) that

$$\dot{V}(x(t)) < \mu_{i_k}V(x(t)),\ t \in (t_k, t_{k+1}). \quad (28)$$

Multiplying $e^{-\mu_{i_k}(t-t_k)}$ on both sides of (28) and integrating it from t_k to t, we get

$$V(x(t)) \leq e^{\mu_{i_k}(t-t_k)} V(x(t_k^+)), \quad t_k < t < t_{k+1}.$$

Therefore,

$$\|x(t)\|^2 \leq e^{\mu_{i_k}(t-t_k)} \|x(t_k^+)\|^2.$$

It implies that

$$\begin{aligned} \|x(t)\| &\leq e^{\frac{\mu_{i_k}}{2}(t-t_k)} \|x(t_k^+)\| \\ &\leq \delta e^{\frac{\mu_{i_k}}{2}(t-t_k)} \|x(t_k)\|. \end{aligned}$$

This completes the proof of Lemma 3.3. □

In the following, define

$$\mu^+ = \max_{1 \leq q \leq p} \mu_q. \tag{29}$$

Theorem 3.4. *Assume that (H1), (H2) and* (25) *hold. For any given* $\gamma > 0$, $\tau_a^* > 0$ *and* $\delta > 0$, *there exist a set of gain matrices* $\{K_i\}_{i\in\Lambda}$ *and a sufficiently small constant* $\varepsilon > 0$, *such that the close-loop system of* (19) *is globally exponentially stabilizable for any switching signal satisfying* (10) *and* $\tau_a > \tau_a^*$.

Proof. For any given $\tau_a^* > 0$ and $\gamma > 0$, choose $\lambda > 1$ such that

$$c_2 := \frac{\gamma\lambda - \mu^+}{2(1+\gamma)} - \frac{\ln(\delta M \lambda^{n-1})}{\tau_a^*} > 0. \tag{30}$$

For any $t \geq t_0$, assume that $t_k \leq t < t_{k+1}$ and $\sigma(t) = i_k$. If $i_k \in \{p+1, p+2, \cdots, N\}$, by Lemma 3.2, there exists a feedback matrix K_{i_k} such that

$$\|x(t)\| \leq \delta M \lambda^{n-1} e^{(\varepsilon M \lambda^{n-2} - 1)\lambda(t-t_k)} \|x(t_k)\|. \tag{31}$$

If $i_k \in \{1, 2, \cdots, p\}$, by Lemma 3.3, there exists a feedback matrix K_{i_k} such that

$$\|x(t)\| \leq \delta e^{\frac{\mu_{i_k}}{2}(t-t_k)} \|x(t_k)\| \leq \delta e^{\frac{\mu^+}{2}(t-t_k)} \|x(t_k)\|. \tag{32}$$

Similar to the analysis in Theorem 2.3, there exist a series of feedback matrices $\{K_{i_{j-1}}\}$ for $j \in \{1, 2, \cdots, k\}$ such that

$$\|x(t_j)\| \leq \begin{cases} \delta M \lambda^{n-1} e^{(\varepsilon M \lambda^{n-2} - 1)\lambda(t_j - t_{j-1})} \|x(t_{j-1})\|, \\ \qquad \sigma(t_{j-1}) \in \{p+1, p+2, \cdots, N\}; \\ \delta e^{\frac{\mu^+}{2}(t_j - t_{j-1})} \|x(t_{j-1})\|, \\ \qquad \sigma(t_{j-1}) \in \{1, 2, \cdots, p\}. \end{cases} \tag{33}$$

Therefore, under the feedback law $u(t) = K_{i_j} x(t)$ for $j = 0, 1, \cdots, k$, we get from (10), (31)-(33) by induction that

$$\begin{aligned} \|x(t)\| &\leq \delta^{j_1} e^{\frac{\mu^+}{2} T_1(t_0,t)} (\delta M \lambda^{n-1})^{j_2} e^{(\varepsilon M \lambda^{n-2} - 1)\lambda T_2(t_0,t)} \|x_0\| \\ &\leq e^{N_\sigma(t_0,t) \ln(\delta M \lambda^{n-1}) + \frac{\mu^+ + 2(\varepsilon M \lambda^{n-2} - 1)\gamma\lambda}{2(1+\gamma)}(t-t_0)} \|x_0\|. \end{aligned} \tag{34}$$

Since

$$N_{\sigma(t_0,t)} \le \frac{t-t_0}{\tau_a^*}, \tag{35}$$

we get from (34) and (35) that

$$\|x(t)\| \le e^{-\left(\frac{2(1-\varepsilon M\lambda^{n-2})\gamma\lambda-\mu^+}{2(1+\gamma)} - \frac{\ln(\delta M\lambda^{n-1})}{\tau_a^*}\right)(t-t_0)}\|x_0\|,\ t\in(t_k,t_{k+1}). \tag{36}$$

Let $\varepsilon \le 1/(2M\lambda^{n-2})$. By (30) and (36), we have that

$$\begin{aligned}\|x(t)\| &\le e^{-(\frac{\gamma\lambda-\mu^+}{2(1+\gamma)} - \frac{\ln(\delta M\lambda^{n-1})}{\tau_a^*})(t-t_0)}\|x_0\| \\ &= e^{-c_2(t-t_0)}\|x_0\|,\ t\in(t_k,t_{k+1}),\end{aligned}$$

where $c_2 > 0$ is defined by (30). This completes the proof of Theorem 3.4. □

4. Numerical Examples

In order to illustrate the theoretical results, we present two numerical examples as follows.

Example 4.1. Consider the switched system (1) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It is not difficult to verify that (A_1, B_1), (A_2, B_2) are controllable, and (A_3, B_3) is uncontrollable.

By Lemma 2.1, we get $M = 211$. Choosing $\lambda = 30$, we have that $K_1 = \begin{bmatrix} 90 & -1799 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -80.2 & 255.4 \end{bmatrix}$. Let $K_3 = \begin{bmatrix} -2 & 1 \end{bmatrix}$. Under the linear feedback law $u(t) = K_\sigma x(t)$, system (1) reduces to the following closed-loop system

$$\dot{x}(t) = (A_\sigma + B_\sigma K_\sigma)x(t). \tag{37}$$

Let $\gamma = 1$ and $\tau_a^* = 2$. By (4), we can choose $\lambda_3 = 5$ and a positive definite symmetric matrix $P_3 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. By (12), we have $L = M = 211$. For $\lambda_+ = \lambda_3 = 5$, we get

$$\frac{2\gamma\lambda - \lambda_+}{2(1+\gamma)} - \frac{\ln L + (n-1)\ln\lambda}{\tau_a^*} = 9.3735 > 0.$$

By Theorem 2.3, the switched system (37) is exponentially stable for any switching signal satisfying $\frac{T_2(0,t)}{T_1(0,t)} \ge 1$ and $\tau_a > 2$.

Example 4.2. Consider the switched system (19) with the nonlinear disturbance

$$f(x) = \varepsilon \begin{bmatrix} \sin(x_1) & \sin(x_2) & \sin(x_3) \end{bmatrix}^T, \quad \varepsilon > 0,$$

and

$$A_1 = \begin{bmatrix} -1 & -4 & -2 \\ 0 & 6 & -1 \\ 1 & 7 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

It is not difficult to verify that (A_1, B_1), (A_2, B_2) are controllable, and (A_3, B_3) is uncontrollable.

By Lemma 3.2, we get $M = 226290$. By choosing $\lambda = 10$, we have

$$K_1 = \begin{bmatrix} 45.600 & -1567.8 & -27.300 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 11.2125 & -268.80 & 298.375 \end{bmatrix}.$$

Let $\varepsilon = 2 \times 10^{-7} < \frac{1}{2M\lambda^{n-2}}$, $\gamma = 2$, $\delta = 1$ and $\tau_a^* = 16$. For given $K_3 = \begin{bmatrix} -2 & -5 & -1 \end{bmatrix}$, by (25), we can choose $\mu_3 = 5$. For $\mu^+ = \mu_3 = 5$, we get

$$c_2 = \frac{\gamma\lambda - \mu^+}{2(1+\gamma)} - \frac{\ln(\delta M \lambda^{n-1})}{\tau_a^*} = 1.4416 > 0.$$

By Theorem 3.4, the close-loop system of (19) is exponentially stable for any switching signal satisfying $\frac{T_2(0,t)}{T_1(0,t)} \geq 2$ and $\tau_a > 16$.

5. Conclusion

This paper studies the stabilization problem of the switched system with both controllable and uncontrollable subsystems. By estimating solutions for controllable subsystems and uncontrollable subsystems, respectively, and by using an average dwell time approach, we first establish a sufficient condition such that the switched system is exponentially stabilizable under appropriate switching signals. Then, we extend this result to the switched system with nonlinear impulse effects and disturbances. Finally, two numerical examples are worked out to illustrate the theoretical results.

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