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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SINGULAR SYSTEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS†

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ABSTRACT. In this paper, we study the existence and uniqueness of solutions for a singular system of nonlinear fractional differential equations with integral boundary conditions. We obtain existence and uniqueness results of solutions by using the properties of the Green's function, a nonlinear alternative of Leray-Schauder type, Guo-Krasnoselskii's fixed point theorem in a cone. Some examples are included to show the applicability of our results.

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Key words and phrases : Fractional differential equation, Singular system, Fractional Green's function, Fixed point theorem.

1. Introduction

We consider the singular system of nonlinear fractional differential equations with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,v(t))=0, 0<t<1, \\ D_{0+}^{\beta}v(t)+g(t,u(t))=0, 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, u(1)=\mu\displaystyle\int_0^{\eta}u(s)\mathrm{d}s, \\ v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0, v(1)=b\displaystyle\int_0^{\xi}v(s)\mathrm{d}s, \end{cases} \tag{1}$$

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where $n-1<\alpha,\beta\leq n, n\geq 3$, $0<\eta,c\leq 1, 0<\frac{\mu\eta^{\alpha}}{\alpha},\frac{bc^{\beta}}{\beta}\leq 1, f,g:(0,1]\times[0,+\infty)\to[0,+\infty)$ are two given continuous functions and singular at $t=0$ (that is, $\lim_{t\to 0+}f(t,\cdot)=+\infty, \lim_{t\to 0+}g(t,\cdot)=+\infty$), and $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard fractional Riemann-Liouville's derivatives.

The paper [8] considered the existence of positive solutions of singular coupled system

$$\begin{cases} D^{s}u=f(t,v), 0<t<1,\\ D^{p}v=g(t,u), 0<t<1, \end{cases}$$

where $0<s,p<1$, and $f,g:(0,1]\times[0,+\infty)\to[0,+\infty)$ are two given continuous functions, $\lim_{t\to 0+}f(t,\cdot)=+\infty, \lim_{t\to 0+}g(t,\cdot)=+\infty$ and D^{s}, D^{p} are the standard fractional Riemann-Liouville's derivatives. The existence results of positive solution are obtained by a nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii's fixed point theorem in a cone.

The paper [9] considered the existence of positive solutions of singular coupled system

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,v(t))=0, 0<t<1,\\ D_{0+}^{\beta}v(t)+g(t,u(t))=0, 0<t<1,\\ u(0)=u'(0)=u(1)=v(0)=v'(0)=v(1)=0, \end{cases}$$

where $2<\alpha,\beta\leq 3$ and $f,g:(0,1]\times[0,+\infty)\to[0,+\infty)$ are two given continuous functions, and $\lim_{t\to 0+}f(t,\cdot)=+\infty, \lim_{t\to 0+}g(t,\cdot)=+\infty$ and $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard fractional Riemann-Liouville's derivatives. The two sufficient conditions for the existence of solutions are obtained by a nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii's fixed point theorem in a cone.

Inspired by the work of the above papers and many known results, in this paper, we study the existence of positive solutions of BVP (1). The BVP (1) contains the above equations. The existence of solutions are obtained by a nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii's fixed point theorem in a cone.

2. Background materials and Green's function

For the convenience of the reader, we present here the necessary definitions, lemmas and theorems from fractional calculus theory to facilitate analysis of BVP (1). These definitions, lemmas and theorems can be found in the recent literature, see [1-12].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}y(s)\mathrm{d}s$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to R$ is given by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}t})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the statement.

Lemma 2.3 ([3])**.** *Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation*

$$D^{\alpha}_{0+}u(t) = 0,$$

has $u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}, C_i \in R, i = 1, 2, \dots, N,$ as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 2.4 ([3])**.** *Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N},$$

for some $C_i \in R, i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α.

Remark 2.1 ([10])**.** The following properties are useful for our discussion:

$$I^{\alpha}_{0+}I^{\beta}_{0+}f(t) = I^{\alpha+\beta}f(t), D^{\alpha}_{0+}I^{\alpha}_{0+}f(t) = f(t), \alpha, \beta > 0.$$

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 2.5. *Given $y \in C[0, 1]$. The problem*

$$\begin{cases} D^{\alpha}_{0+}u(t) + y(t) = 0, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \mu \int_0^{\eta} u(s)\mathrm{d}s, \end{cases} \tag{2}$$

where $0 < t < 1, n - 1 < \alpha \le n, 0 < \eta \le 1, 0 < \frac{\mu\eta^{\alpha}}{\alpha} < 1$, has a unique solution

$$u(t) = \int_0^1 G_{\alpha}(t, s)y(s)\mathrm{d}s,$$

where $G_\alpha(t,s)$ is Green's function given by

$$G_\alpha(t,s)=\begin{cases}\frac{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}-(1-\frac{\mu}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}, 0\le s\le t\le 1, s\le\eta;\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(1-\frac{\mu}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}, 0\le\eta\le s\le t\le 1;\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}, 0\le t\le s\le\eta\le 1;\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}, 0\le t\le s\le 1, \eta\le s.\end{cases}$$

Here, $G_\alpha(t,s)$ is called the Green's function of BVP (2). Obviously, $G_\alpha(t,s)$ is continuous on $[0,1]\times[0,1]$.

Proof. We may apply Lemma 2.2 to reduce (2) to an equivalent integral equation

$$u(t)=-I_{0+}^\alpha y(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N},$$

for some $C_1,C_2,\cdots,C_N\in R$. Consequently, the general solution of (2) is

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N}.$$

By $u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0$, there are $C_2=C_3=\cdots=C_N=0$. On the other hand, $u(1)=\mu\int_0^\eta u(s)\mathrm{d}s$ combining with

$$u(1)=-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+C_1$$

$$\begin{aligned}\int_0^\eta u(s)\mathrm{d}s &= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\int_0^x(x-s)^{\alpha-1}y(s)\mathrm{d}s\mathrm{d}x+C_1\int_0^\eta s^{\alpha-1}\mathrm{d}s\\ &= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\int_s^\eta(x-s)^{\alpha-1}y(s)\mathrm{d}x\mathrm{d}s+C_1\int_0^\eta s^{\alpha-1}\mathrm{d}s\\ &= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\frac{(\eta-s)^\alpha}{\alpha}y(s)\mathrm{d}s+\frac{C_1\eta^\alpha}{\alpha}\end{aligned}$$

yieds

$$C_1=\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\frac{\mu\eta^\alpha}{\alpha})}y(s)\mathrm{d}s-\mu\int_0^\eta\frac{(\eta-s)^\alpha}{\alpha\Gamma(\alpha)(1-\frac{\mu\eta^\alpha}{\alpha})}y(s)\mathrm{d}s.$$

Therefore, the unique solution of the problem (2) is

$$\begin{aligned}u(t) &= -\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1\frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\\ &\quad -\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^\eta\frac{\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s.\end{aligned}$$

For $t\le\eta$, one has

$$u(t) = -\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s$$

$$
\begin{aligned}
&+\frac{1}{(1-\frac{\mu}{\alpha}\eta^{\alpha})}\Big[\Big(\int_0^t+\int_t^\eta+\int_\eta^1\Big)\frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\Big]\\
&-\frac{\mu}{(1-\frac{\mu}{\alpha}\eta^{\alpha})}\Big[\Big(\int_0^t+\int_t^\eta\Big)\frac{\frac{1}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\Big]\\
=\;&\int_0^t\frac{t^{\alpha-1}(1-s)^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}-(1-\frac{\mu}{\alpha}\eta^{\alpha})(t-s)^{\alpha-1}}{(1-\frac{\mu}{\alpha}\eta^{\alpha})\Gamma(\alpha)}y(s)\mathrm{d}s\\
&+\int_t^\eta\frac{t^{\alpha-1}(1-s)^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{(1-\frac{\mu}{\alpha}\eta^{\alpha})\Gamma(\alpha)}y(s)\mathrm{d}s\\
&+\int_\eta^1\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\frac{\mu}{\alpha}\eta^{\alpha})\Gamma(\alpha)}y(s)\mathrm{d}s\\
=\;&\int_0^1 G_\alpha(t,s)y(s)\mathrm{d}s.
\end{aligned}
$$

For $t\geq\eta$, one has

$$
\begin{aligned}
u(t)\;=\;&-\Big(\int_0^\eta+\int_\eta^t\Big)\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\\
&+\frac{1}{(1-\frac{\mu}{\alpha}\eta^{\alpha})}\Big[\Big(\int_0^\eta+\int_\eta^t+\int_t^1\Big)\frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\Big]\\
&-\frac{\mu}{(1-\frac{\mu}{\alpha}\eta^{\alpha})}\int_0^\eta\frac{\frac{1}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\\
=\;&\int_0^\eta\frac{t^{\alpha-1}(1-s)^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}-(1-\frac{\mu}{\alpha}\eta^{\alpha})(t-s)^{\alpha-1}}{(1-\frac{\mu}{\alpha}\eta^{\alpha})\Gamma(\alpha)}y(s)\mathrm{d}s\\
&+\int_\eta^t\frac{t^{\alpha-1}(1-s)^{\alpha-1}-(1-\frac{\mu}{\alpha}\eta^{\alpha})(t-s)^{\alpha-1}}{(1-\frac{\mu}{\alpha}\eta^{\alpha})\Gamma(\alpha)}y(s)\mathrm{d}s\\
&+\int_t^1\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\frac{\mu}{\alpha}\eta^{\alpha})\Gamma(\alpha)}y(s)\mathrm{d}s\\
=\;&\int_0^1 G_\alpha(t,s)y(s)\mathrm{d}s.
\end{aligned}
$$

The proof is complete. □

Lemma 2.6. *The function $G_\alpha(t,s)$ has the following properties:*
$(a1)$ $G_\alpha(t,s)>0,\forall t,s\in(0,1)$.
$(a2)$ $G_\alpha(t,s)\geq\frac{\frac{\mu\eta^{\alpha}}{\alpha}}{\Gamma(\alpha)(1-\frac{\mu\eta^{\alpha}}{\alpha})}t^{\alpha-1}s(1-s)^{\alpha-1},\forall t,s\in[0,1]$;
$(a3)$ $G_\alpha(t,s)\leq\Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^{\alpha}}{\alpha})\Gamma(\alpha)}\Big]s(1-s)^{\alpha-1},\forall t,s\in[0,1]$;

Proof. For $s\leq t,s\leq\eta$,

$$
G_\alpha(t,s)\;=\;\frac{1}{(1-\frac{\mu\eta^{\alpha}}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}-(1-\frac{\mu\eta^{\alpha}}{\alpha})(t-s)^{\alpha-1}\Big\}
$$

$$\begin{aligned}
&\geq \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}\eta^\alpha(1-s)^\alpha t^{\alpha-1}-(1-\frac{\mu\eta^\alpha}{\alpha})(t-s)^{\alpha-1}\Big\}\\
&= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}[1-\frac{\mu}{\alpha}\eta^\alpha(1-s)]-(1-\frac{\mu\eta^\alpha}{\alpha})(t-s)^{\alpha-1}\Big\}\\
&= \frac{1}{\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}\Big\}+\frac{\frac{\mu\eta^\alpha}{\alpha}s}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}\\
&\geq \frac{1}{\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-2}[t(1-s)-(t-s)]\Big\}+\frac{\frac{\mu\eta^\alpha}{\alpha}s}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}\\
&\geq \frac{\frac{\mu\eta^\alpha}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-1}.
\end{aligned}$$

$$\begin{aligned}
G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}-(1-\frac{\mu\eta^\alpha}{\alpha})(t-s)^{\alpha-1}\Big\}\\
&= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})[(t(1-s))^{\alpha-1}-(t-s)^{\alpha-1}]\\
&\quad+\frac{\mu}{\alpha}\eta^\alpha t^{\alpha-1}[(1-s)^{\alpha-1}-(1-\frac{s}{\eta})^\alpha]\Big\}\\
&\leq \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})(\alpha-1)\int_{t-s}^{t(1-s)}x^{\alpha-2}\mathrm{d}x\\
&\quad+\frac{\mu}{\alpha}\eta^\alpha t^{\alpha-1}[(1-s)^{\alpha-1}-(1-\frac{s}{\eta})^\alpha(1-s)^{\alpha-1}]\Big\}\\
&\leq \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}s(1-t)\\
&\quad+\frac{\mu}{\alpha}\eta^\alpha t^{\alpha-1}(1-s)^{\alpha-1}[1-(1-\frac{s}{\eta})^n]\Big\}\\
&\leq \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})(\alpha-1)(1-s)^{\alpha-1}s+\frac{\mu n}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1}\Big\}\\
&= \Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]s(1-s)^{\alpha-1}.
\end{aligned}$$

For $\eta\leq s\leq t$,

$$\begin{aligned}
G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-(1-\frac{\mu\eta^\alpha}{\alpha})(t-s)^{\alpha-1}\Big\}\\
&\geq \frac{\frac{\mu\eta^\alpha}{\alpha}s}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}-\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\\
&= \frac{1}{\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-2}t(1-s)-(t-s)^{\alpha-2}(t-s)\Big\}\\
&\quad+\frac{\frac{\mu\eta^\alpha}{\alpha}s}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}\\
&\geq \frac{1}{\Gamma(\alpha)}[t(1-s)]^{\alpha-1}s(1-t)+\frac{\frac{\mu\eta^\alpha}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}s[t(1-s)]^{\alpha-1}\\
&\geq \frac{\frac{\mu\eta^\alpha}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-1}.
\end{aligned}$$

$$\begin{aligned}
G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-(1-\frac{\mu\eta^\alpha}{\alpha})(t-s)^{\alpha-1}\Big\}\\
&= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu}{\alpha}\eta^\alpha+\frac{\mu}{\alpha}\eta^\alpha)[t(1-s)]^{\alpha-1}-(1-\frac{\mu\eta^\alpha}{\alpha})(t-s)^{\alpha-1}\Big\}\\
&= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]\\
&\quad+\frac{\mu}{\alpha}\eta^\alpha[t(1-s)]^{\alpha-1}\Big\}\\
&\le \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})(\alpha-1)\int_{t-s}^{t(1-s)}x^{\alpha-2}\mathrm{d}x\\
&\quad+\frac{\mu}{\alpha}\eta^{\alpha-1}st^{\alpha-1}(1-s)^{\alpha-1}\Big\}\\
&\le \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}s(1-t)\\
&\quad+\frac{\mu}{\alpha}\eta^{\alpha-1}st^{\alpha-1}(1-s)^{\alpha-1}\Big\}\\
&\le \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{(1-\frac{\mu\eta^\alpha}{\alpha})(\alpha-1)(1-s)^{\alpha-1}s+\frac{\mu}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1}\Big\}\\
&\le \Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]s(1-s)^{\alpha-1}.
\end{aligned}$$

For $t\le s\le\eta$,

$$\begin{aligned}
G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}\Big\}\\
&= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}\eta^\alpha(1-\frac{s}{\eta})^\alpha t^{\alpha-1}\Big\}\\
&\ge \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}\eta^\alpha(1-s)^\alpha t^{\alpha-1}\Big\}\\
&= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}[1-\frac{\mu}{\alpha}\eta^\alpha(1-s)]\\
&= \frac{1}{\Gamma(\alpha)}[t(1-s)]^{\alpha-1}+\frac{\frac{\mu\eta^\alpha}{\alpha}s}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}\\
&\ge \frac{\frac{\mu\eta^\alpha}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-1}.
\end{aligned}$$

$$\begin{aligned}
G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big\{[t(1-s)]^{\alpha-1}-\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}\Big\}\\
&\le \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1}\\
&\le \Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]s(1-s)^{\alpha-1}.
\end{aligned}$$

For $\eta \le s, t \le s$,

$$\begin{aligned} G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1} \\ &\ge \frac{\frac{\mu\eta^\alpha}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)} t^{\alpha-1}s(1-s)^{\alpha-1}. \end{aligned}$$

$$\begin{aligned} G_\alpha(t,s) &= \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}[t(1-s)]^{\alpha-1} \\ &\le \Big[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big] s(1-s)^{\alpha-1}. \end{aligned}$$

From above, (a2),(a3),(a4) are complete. Clearly, (a1) is true. The proof is complete. □

Similarly, the general solution of

$$\begin{cases} D_{0+}^{\beta} v(t) + y(t) = 0, \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \\ v(1) = b\displaystyle\int_0^c v(s)\mathrm{d}s, \end{cases} \tag{3}$$

where $0 < t < 1, n-1 < \beta \le n, 0 < c \le 1, 0 < \frac{bc^\beta}{\beta} < 1$, is

$$v(t) = \int_0^1 G_\beta(t,s)y(s)\mathrm{d}s,$$

where $G_\beta(t,s)$ can be obtained from $G_\alpha(t,s)$ by correspondingly replacing α, μ, η with β, b, c and satisfy properties (a1)-(a4) with α, μ, η correspondingly replaced by β, b, c in case of $G_\beta(t,s)$. Let (G_α, G_β) denote Green's function for the boundary value problem (1).

Lemma 2.7 ([13]). *Let E be a Banach space and $P \in E$ be a cone. Assume Ω_1 and Ω_2 be two open subsets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A : (\overline{\Omega}_2 \backslash \Omega_1) \cap P \to P$ be completely continuous. Suppose that one of the two conditions*
(1) $\|Au\| \le \|u\|, \forall u \in P \cap \partial\Omega_1; \|Au\| \ge \|u\|, \forall u \in P \cap \partial\Omega_2$
and
(2)$\|Au\| \ge \|u\|, \forall u \in P \cap \partial\Omega_1; \|Au\| \le \|u\|, \forall u \in P \cap \partial\Omega_2$
is satisfied. Then A has a fixed point in $(\Omega_2 \backslash \overline{\Omega}_1) \cap P$.

Lemma 2.8 ([14]). *Let E be a Banach space and $\Omega \in E$ be closed and convex. Assume U is a relatively open subset of E with $\theta \in U$,and Let operator $A : \overline{U} \to \Omega$ be a continuous compact map. Then either*
(1) A has a fixed point in U; or
(2)there exists $u \in \partial U$ and $\varphi \in (0,1)$ with $u = \varphi Au$.

3. Main results

Let the Banach space $E = C[0,1]$ have the maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Thus $(E \times E, \|\cdot\|)$ is a Banach space with the norm defined by $\|(u,v)\| = \max\{\|u\|, \|v\|\}, \forall (u,v) \in E \times E$. We define the cone $P \subset E \times E$ by

$$P = \{(u,v) \in E \times E \mid u(t) \geq 0, v(t) \geq 0, 0 \leq t \leq 1\}.$$

Lemma 3.1. *Let $n-1 < \alpha, \beta \leq n$. Let $F : (0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and satisfy $\lim_{t\to 0+} F(t,\cdot) = +\infty$. Assume that there exists $0 < \sigma < 1$ such that $t^\sigma F(t)$ is continuous on $[0,1]$. Then $u(t) = \int_0^1 G_\alpha(t,s)F(s)\mathrm{d}s$ is continuous on $[0,1]$.*

Proof. From the continuity of $t^\sigma F(t)$ and $u(t) = \int_0^1 G_\alpha(t,s)t^{-\sigma}t^\sigma F(s)\mathrm{d}s$, we know that $u(0) = 0$. If $u(t) \to u(t_0)$ when $t \to t_0$ for $\forall t_0 \in [0,1]$, then the proof is complete. In the following we separate the process into three cases.

Case 1. For $t_0 = 0$ and $\forall t \in (0,1]$. Owing to the continuity of $t^\sigma F(t)$, there exists an $M > 0$ such that $| t^\sigma F(t) | \leq M$, $\forall t \in [0,1]$, then

$$\begin{aligned}
& | u(t) - u(0) | \\
= \; & \Big| -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s)\mathrm{d}s \\
& + \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})} \int_0^1 \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s)\mathrm{d}s \\
& - \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})} \int_0^\eta \frac{\frac{\mu}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s)\mathrm{d}s \Big| \\
\leq \; & \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s)\mathrm{d}s \Big| \\
& + \frac{1+\frac{\mu}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})} \Big| \int_0^1 \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s)\mathrm{d}s \Big| \\
\leq \; & M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}\mathrm{d}s + M \frac{1+\frac{\mu}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})} \int_0^1 \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}\mathrm{d}s \\
= \; & \frac{Mt^{\alpha-\sigma}}{\Gamma(\alpha)} B(1-\sigma,\alpha) + \frac{(1+\frac{\mu}{\alpha})Mt^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)} B(1-\sigma,\alpha) \\
\leq \; & \frac{(1+\frac{\mu}{\alpha})M\Gamma(1-\sigma)}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(1+\alpha-\sigma)} (t^{\alpha-\sigma} + t^{\alpha-1}) \to 0 (t \to 0).
\end{aligned}$$

$B(\cdot)$ mentioned in the above functions shows the Beta function.

Case 2. For $t_0 \in (0,1)$ and $\forall t \in (t_0, 1]$, then

$$\begin{aligned}
& | u(t) - u(t_0) | \\
= \; & \Big| -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&-\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^\eta \frac{\frac{\mu}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&+\int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&-\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}t_0^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&+\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^\eta \frac{\frac{\mu}{\alpha}(\eta-s)^{\alpha}t_0^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\;|\\
= &\;|-\int_0^{t_0} \frac{(t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&+\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}(t^{\alpha-1}-t_0^{\alpha-1})}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&-\frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^\eta \frac{\frac{\mu}{\alpha}(\eta-s)^{\alpha}(t^{\alpha-1}-t_0^{\alpha-1})}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\\
&-\int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\;|\\
\leq &\;|\int_0^{t_0} \frac{(t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\;|\\
&+|\;\frac{1+\frac{\mu}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}(t^{\alpha-1}-t_0^{\alpha-1})}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\;|\\
&+|\int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\mathrm{d}s\;|\\
\leq &\;M\int_0^{t_0} \frac{(t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}\mathrm{d}s\\
&+\frac{M(1+\frac{\mu}{\alpha})(t^{\alpha-1}-t_0^{\alpha-1})}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}\mathrm{d}s\\
&+M\int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}\mathrm{d}s\\
= &\;\frac{M(t^{\alpha-\sigma}-t_0^{\alpha-\sigma})}{\Gamma(\alpha)}B(1-\sigma,\alpha)+\frac{(1+\frac{\mu}{\alpha})M(t^{\alpha-1}-t_0^{\alpha-1})}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}B(1-\sigma,\alpha)\\
\leq &\;\frac{(1+\frac{\mu}{\alpha})M\Gamma(1-\sigma)}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(1+\alpha-\sigma)}(t^{\alpha-\sigma}-t_0^{\alpha-\sigma}+t^{\alpha-1}-t_0^{\alpha-1})\to 0(t\to t_0).
\end{aligned}
$$

Case 3. For $t_0 \in (0,1]$ and $\forall t \in [0,t_0)$. Similarly to the proof of Case 2, so we omit it. The proof is complete. $\square$

From Lemma 2.3, we can write the system of BVPs (1) as an equivalent system of integral equations

$$\begin{cases} u(t) = \int_0^1 G_\alpha(t,s) f(s,v(s))\mathrm{d}s, 0 \le t \le 1, \\ v(t) = \int_0^1 G_\beta(t,s) g(s,u(s))\mathrm{d}s, 0 \le t \le 1, \end{cases} \tag{4}$$

which can be proved in the same way as Lemma 3.3 in [10]. For convenience, the proof is omitted.

We define $A : E \times E \to E \times E$ to be an operator, i.e.,

$$\begin{aligned} & A(u,v)(t) \\ = \ & \Big(\int_0^1 G_\alpha(t,s) s^{-\sigma_1} s^{\sigma_1} f(s,v(s))\mathrm{d}s, \int_0^1 G_\beta(t,s) s^{-\sigma_2} s^{\sigma_2} g(s,u(s))\mathrm{d}s \Big) \\ =: \ & (A_1 v(t), A_2 u(t)). \end{aligned}$$

Lemma 3.2. *Let $n-1 < \alpha, \beta \le n$. Let $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and satisfy $\lim_{t\to 0+} f(t,\cdot) = +\infty, \lim_{t\to 0+} g(t,\cdot) = +\infty$. Assume that there exists $0 < \sigma_1, \sigma_2 < 1$ such that $t^{\sigma_1} f(t,y), t^{\sigma_2} g(t,y)$ are continuous on $[0,1] \times [0,\infty)$. Then the operator $A : P \to P$ is completely continuous .*

Proof. For any $(u,v) \in P$, we have that

$$u, v \in P_1 = \{y \in E \mid y(t) \ge 0, 0 \le t \le 1\}.$$

Since

$$A_1 v(t) = \int_0^1 G_\alpha(t,s) s^{-\sigma_1} s^{\sigma_1} f(s,v(s))\mathrm{d}s,$$

we get that $A_1 : P_1 \to P_1$ by Lemma 3.1 and the nonnegativity of f. Set $v_0 \in P_1$ and $\|v_0\| = c_0$. If $v \in P_1$ and $\|v - v_0\| < 1$, then $\|v\| < 1 + c_0 := c$. By the continuity of $t^{\sigma_1} f(t,y)$, we get that $t^{\sigma_1} f(t,y)$ is uniformly continuous on $[0,1] \times [0,c]$, namely $\forall \varepsilon > 0, \exists \delta > 0 (\delta < 1)$, when $\mid y_1 - y_2 \mid < \delta$, we have $\mid t^{\sigma_1} f(t,y_1) - t^{\sigma_1} f(t,y_2) \mid < \varepsilon, \forall t \in [0,1], y_1, y_2 \in [0,c]$. Obviously, if $\|v - v_0\| < \delta$, then $v_0(t), v(t) \in [0,c]$ and $|v(t) - v_0(t)| < \delta, \forall t \in [0,1]$. Hence, we have

$$\mid t^{\sigma_1} f(t,v(t)) - t^{\sigma_1} f(t,v_0(t)) \mid < \varepsilon, \tag{5}$$

for all $t \in [0,1], v \in P_1, \|v - v_0\| < \delta$. It follows from (6), we can get

$$\begin{aligned} \|A_1 v - A_1 v_0\| &= \max_{t\in[0,1]} \mid A_1 v(t) - A_1 v_0(t) \mid \\ &\le \max_{t\in[0,1]} \int_0^1 G_{1\alpha}(t,s) s^{-\sigma_1} \mid s^{\sigma_1} f(s,v(s)) - s^{\sigma_1} f(s,v_0(s)) \mid \mathrm{d}s \\ &< \varepsilon \int_0^1 G_{1\alpha}(t,s) s^{-\sigma_1} \mathrm{d}s \end{aligned}$$

$$\begin{aligned} &\leq \varepsilon\int_0^1\Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]s(1-s)^{\alpha-1}s^{-\sigma_1}\mathrm{d}s \\ &= \varepsilon\Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]\int_0^1(1-s)^{\alpha-1}s^{1-\sigma_1}\mathrm{d}s \\ &= \varepsilon\Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]B(2-\sigma_1,\alpha). \end{aligned}$$

Owing to the arbitrariness of v_0, we know that $A_1 : P_1 \to P_1$ is continuous. Similarly, we can get that $A_2 : P_1 \to P_1$ is continuous. So we proved $A : P \to P$ is continuous. □

Let $M \subset P$ be bounded. That is to say there exists a constant $l > 0$ such that $\|(u,v)\| \leq l, \forall(u,v) \in M$. Since $t^{\sigma_1}f(t,y), t^{\sigma_2}f(t,y)$ are continuous on $[0,1]\times [0,+\infty)$, let $L = \max_{t\in[0,1],(u,v)\in M}$
$\{t^{\sigma_1}f(t,v(t)), t^{\sigma_2}g(t,u(t))\} + 1$. Then for each $(u,v) \in M$, we have

$$\begin{aligned} \mid A_1v(t)\mid &\leq \int_0^1 G_{1\alpha}(t,s)s^{-\sigma_1}\mid s^{\sigma_1}f(s,v(s))\mid \mathrm{d}s \\ &\leq L\int_0^1\Big[\frac{\alpha-1}{\Gamma(\alpha)}+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big]s(1-s)^{\alpha-1}s^{-\sigma_1}\mathrm{d}s \\ &= L\Big[\alpha-1+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big]\frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}. \end{aligned}$$

Hence, we have

$$\|A_1v\| = \max_{t\in[0,1]}|A_1v(t)| \leq L\Big[\alpha-1+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big]\frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}.$$

Similarly, we have

$$\|A_2u\| = \max_{t\in[0,1]}|A_2u(t)| \leq L\Big[\beta-1+\frac{\frac{bn}{\beta}c^{\beta-1}}{(1-\frac{bc^\beta}{\beta})}\Big]\frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-\sigma_2)}.$$

Thus,

$$\begin{aligned} \|A(u,v)\| &= \max_{t\in[0,1]}\{|A_1v|,|A_2u|\} \\ &\leq \max\Big\{\Big[\alpha-1+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big]\frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}, \\ &\qquad \Big[\beta-1+\frac{\frac{bn}{\beta}c^{\beta-1}}{(1-\frac{bc^\beta}{\beta})}\Big]\frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-\sigma_2)}\Big\}L. \end{aligned}$$

Therefore, $A(M)$ is bounded.

Next, we prove that A is equicontinuous.

Let $\delta = \min\left\{\frac{\varepsilon(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(1+\alpha-\sigma_1)}{2^{n+1}(1+\frac{\mu}{\alpha})L\Gamma(1-\sigma_1)}, \frac{\varepsilon(1-\frac{bc^\beta}{\beta})\Gamma(1+\beta-\sigma_2)}{2^{n+1}(1+\frac{b}{\beta})L\Gamma(1-\sigma_2)}\right\}$, for $\forall\varepsilon > 0$. Then for all $(u,v) \in M, t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $0 < t_2 - t_1 < \delta$, we have

$$\begin{aligned}
& \mid A_1v(t_2) - A_1v(t_1) \mid \\
= & \mid \int_0^1 G_\alpha(t_2,s)f(s,v(s))\mathrm{d}s - \int_0^1 G_\alpha(t_1,s)f(s,v(s))\mathrm{d}s \mid \\
= & \mid -\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \\
& + \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \\
& - \frac{1}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^\eta \frac{\frac{\mu}{\alpha}(\eta-s)^{\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \\
& - \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \mid \\
\leq & \mid \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \mid \\
& + \mid \frac{1+\frac{\mu}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \mid \\
& + \mid \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \mid \\
\leq & L\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}\mathrm{d}s \\
& + \frac{L(1+\frac{\mu}{\alpha})(t_2^{\alpha-1}-t_1^{\alpha-1})}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}\mathrm{d}s \\
& + L\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1}\mathrm{d}s \\
= & \frac{L(t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1})}{\Gamma(\alpha)}B(1-\sigma_1,\alpha) + \frac{(1+\frac{\mu}{\alpha})L(t_2^{\alpha-1}-t_1^{\alpha-1})}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}B(1-\sigma_1,\alpha) \\
\leq & \frac{(1+\frac{\mu}{\alpha})L\Gamma(1-\sigma_1)}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(1+\alpha-\sigma_1)}(t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}+t_2^{\alpha-1}-t_1^{\alpha-1}).
\end{aligned}$$

Similarly,

$$\mid A_2u(t_2) - A_2u(t_1) \mid \leq \frac{(1+\frac{b}{\beta})L\Gamma(1-\sigma_2)}{(1-\frac{bc^\beta}{\beta})\Gamma(1+\beta-\sigma_2)}(t_2^{\beta-\sigma_2}-t_1^{\beta-\sigma_2}+t_2^{\beta-1}-t_1^{\beta-1}).$$

Hence we figure on $t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}, t_2^{\alpha-1}-t_1^{\alpha-1}, t_2^{\beta-\sigma_2}-t_1^{\beta-\sigma_2}, t_2^{\beta-1}-t_1^{\beta-1}$ in the following three cases.

Case 1. If $0 \leq t_1 < \delta, 0 \leq t_2 < 2\delta$, then

$t_2^{\alpha-\sigma_1} - t_1^{\alpha-\sigma_1} \leq t_2^{\alpha-\sigma_1} \leq (2\delta)^{\alpha-\sigma_1} < 2^n\delta, t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} \leq (2\delta)^{\alpha-1} < 2^n\delta.$
Case 2. If $0 \leq t_1 < t_2 \leq \delta$, then
$t_2^{\alpha-\sigma_1} - t_1^{\alpha-\sigma_1} \leq t_2^{\alpha-\sigma_1} \leq \delta^{\alpha-\sigma_1} < 2^n\delta, t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} \leq \delta^{\alpha-1} < 2^n\delta.$
Case 3. If $\delta \leq t_1 < t_2 \leq 1$, then
$t_2^{\alpha-\sigma_1} - t_1^{\alpha-\sigma_1} = (\alpha-\sigma_1)\int_{t_1}^{t_2} x^{\alpha-\sigma_1-1}\mathrm{d}x \leq (\alpha-\sigma_1)(t_2-t_1)t_2^{\alpha-\sigma_1-1} \leq (\alpha-\sigma_1)\delta < 2^n\delta,$
$t_2^{\alpha-1} - t_1^{\alpha-1} = (\alpha-1)\int_{t_1}^{t_2} x^{\alpha-2}\mathrm{d}x \leq (\alpha-1)(t_2-t_1)t_2^{\alpha-2} \leq (\alpha-1)\delta < 2^n\delta.$
Hence,
$| A_1v(t_2) - A_1v(t_1) | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$
Similarly,
$| A_2u(t_2) - A_2u(t_1) | < \varepsilon.$
Therefore, $A(M)$ is equicontinuous, and by Arzelà-Ascoli's theorem, we obtain that $A(M)$ is a relatively compact set, then we prove operator $A : P \to P$ is completely continuous.

Now we give the three results of this paper.

Theorem 3.3. *Let $n-1 < \alpha, \beta \leq n$. Let $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and satisfy $\lim_{t\to 0+} f(t,\cdot) = +\infty, \lim_{t\to 0+} g(t,\cdot) = +\infty$. Assume that there exists $0 < \sigma_1, \sigma_2 < 1$ such that $t^{\sigma_1}f(t,y), t^{\sigma_2}g(t,y)$ are continuous on $[0,1]\times[0,+\infty)$ and there exist $t_0 \in (0,1)$ and two positive constants ρ, ξ subjecting to $\rho > \max\{\frac{\xi(\alpha-1)}{m_1 t_0^{\alpha-1}}, \frac{\xi(\beta-1)}{m_2 t_0^{\beta-1}}\}$, where*

$$m_1 = \frac{\frac{\mu\eta^\alpha}{\alpha}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\int_{t_0}^1 s^{1-\sigma_1}(1-s)^{\alpha-1}\mathrm{d}s, m_2 = \frac{\frac{bc^\beta}{\beta}}{(1-\frac{bc^\beta}{\beta})}\int_{t_0}^1 s^{1-\sigma_2}(1-s)^{\beta-1}\mathrm{d}s.$$

Further suppose
$(i)\forall(t,y) \in [0,1]\times[0,\xi], t^{\sigma_1}f(t,y) \geq \frac{\xi\Gamma(\alpha)}{m_1 t_0^{\alpha-1}}, t^{\sigma_2}g(t,y) \geq \frac{\xi\Gamma(\beta)}{m_2 t_0^{\beta-1}}$;
$(ii)\forall(t,y) \in [0,1]\times[0,\rho]$,
$t^{\sigma_1}f(t,y) \leq \frac{\rho\Gamma(2+\alpha-\sigma_1)}{[\alpha-1+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}]\Gamma(2-\sigma_1)}, t^{\sigma_2}g(t,y) \leq \frac{\rho\Gamma(2+\beta-\sigma_2)}{[\beta-1+\frac{\frac{bn}{\beta}c^{\beta-1}}{(1-\frac{bc^\beta}{\beta})}]\Gamma(2-\sigma_2)}.$
Then BVP (1) has at least one positive solution.

Proof. From the conditions we obtain $\rho > \max\{\frac{\xi(\alpha-1)}{m_1 t_0^{\alpha-1}}, \frac{\xi(\beta-1)}{m_2 t_0^{\beta-1}}\} > \xi$. We divide the demonstration into two steps.

Step 1. Let $\Omega_1 = \{(u,v) \in P \mid \|u\| < \xi, \|v\| < \xi\}$ such that $0 \leq u(t), v(t) \leq \xi$ for $(u,v) \in P \cap \partial\Omega_1$ and $\forall t \in [0,1]$. By condition (i) and Lemma 2.4, we get

$$\begin{aligned} A_1v(t_0) &= \int_0^1 G_\alpha(t_0,s)s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \\ &\geq \int_{t_0}^1 G_\alpha(t_0,s)s^{-\sigma_1}s^{\sigma_1}f(s,v(s))\mathrm{d}s \end{aligned}$$

$$\geq \frac{\xi\Gamma(\alpha)}{m_1 t_0^{\alpha-1}} \int_{t_0}^{1} \frac{\frac{\mu\eta^\alpha}{\alpha}}{\Gamma(\alpha)(1-\frac{\mu\eta^\alpha}{\alpha})} t_0^{\alpha-1} s(1-s)^{\alpha-1} s^{-\sigma_1} \mathrm{d}s = \xi.$$

Hence,

$$\|A_1 v\| = \max_{t\in[0,1]} |A_1 v(t)| \geq \xi, \forall v \in P_1 \cap \partial\Omega_1.$$

Similarly,

$$\|A_2 u\| = \max_{t\in[0,1]} |A_2 u(t)| \geq \xi, \forall u \in P_1 \cap \partial\Omega_1.$$

Therefore, $\|A(u,v)\| \geq \xi = \|(u,v)\|$.

Step 2. Let $\Omega_2 = \{(u,v) \in P \mid \|u\| < \rho, \|v\| < \rho\}.\forall (u,v) \in P \cap \partial\Omega_2, t \in [0,1]$, we have that $0 \leq u(t), v(t) \leq \rho$. By condition (i) and Lemma 2.4, we get

$$\begin{aligned}
& A_1 v(t) \\
= & \int_0^1 G_{1\alpha}(t,s) s^{-\sigma_1} s^{\sigma_1} f(s, v(s)) \mathrm{d}s \\
\leq & \frac{\rho\Gamma(2+\alpha-\sigma_1)}{\Big[\alpha - 1 + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big]\Gamma(2-\sigma_1)} \int_0^1 \Big[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big] \cdot \\
& s(1-s)^{\alpha-1} s^{-\sigma_1} \mathrm{d}s \\
= & \rho.
\end{aligned}$$

Then we obtain

$$\|A_1 v\| \leq \rho, \forall v \in P_1 \cap \partial\Omega_2.$$

Similarly,

$$\|A_2 u\| = \max_{t\in[0,1]} |A_2 u(t)| \leq \rho, \forall u \in P_1 \cap \partial\Omega_2.$$

Therefore, $\|A(u,v)\| \leq \rho = \|(u,v)\|$.

Besides,by Lemma 3.2,operator $A : P \to P$ is completely continuous. Then with Lemma 2.5, our proof is complete. □

Theorem 3.4. *Let $n-1 < \alpha, \beta \leq n$. Let $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and satisfy $\lim_{t\to 0+} f(t,\cdot) = +\infty, \lim_{t\to 0+} g(t,\cdot) = +\infty$. Assume that there exists $0 < \sigma_1, \sigma_2 < 1$ such that $t^{\sigma_1} f(t,y), t^{\sigma_2} g(t,y)$ are continuous on $[0,1] \times [0,+\infty)$. Suppose they satisfy the following conditions (iii) there exist two continuous and nondecreasing functions $\varphi, \psi : [0,+\infty) \to (0,+\infty)$ such that*

$$t^{\sigma_1} f(t,y) \leq \varphi(y), t^{\sigma_2} g(t,y) \leq \psi(y), \forall (t,y) \in [0,1] \times [0,+\infty);$$

(iv)there exists an $r > 0$, yielding
$\frac{r}{\max\{\varphi(r),\psi(r)\}} > \max\Big\{\Big[\alpha - 1 + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big]\frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}, \Big[\beta - 1 + \frac{\frac{bn}{\beta}c^{\beta-1}}{(1-\frac{bc^\beta}{\beta})}\Big]\frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-\sigma_2)}\Big\}.$
Then the BVP (1) has a positive solution.

Proof. Let $U = \{(u,v) \in P \mid \|u\| < r, | \|v\| < r\}$, so that $U \subset P$. By Lemma 3.2, we get to know that operator $A : \overline{U} \to P$ is completely continuous. And if there exists $(u,v) \in \partial U$ and $\lambda \in (0,1)$, we have $(u,v) = \lambda A(u,v)$, then by (iii) for $t \in [0,1]$, we obtain

$$
\begin{aligned}
u(t) &= \lambda A_1 v(t) = \lambda \int_0^1 G_\alpha(t,s) f(s, v(s)) \mathrm{d}s \\
&\leq \int_0^1 G_\alpha(t,s) s^{-\sigma_1} s^{\sigma_1} f(s, v(s)) \mathrm{d}s \\
&\leq \int_0^1 G_\alpha(t,s) s^{-\sigma_1} \varphi(v(s)) \mathrm{d}s \\
&\leq \varphi(\|v\|) \int_0^1 G_{1\alpha}(t,s) s^{-\sigma_1} \mathrm{d}s \\
&\leq \varphi(\|v\|) \Big[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big] \int_0^1 (1-s)^{\alpha-1} s^{1-\sigma_1} \mathrm{d}s \\
&= \varphi(\|v\|) \Big[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})\Gamma(\alpha)}\Big] B(2-\sigma_1, \alpha).
\end{aligned}
$$

Hence,

$$\|u\| \leq \varphi(\|(u,v)\|) \Big[\alpha - 1 + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big] \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)},$$

i.e.,

$$\frac{\|u\|}{\varphi(\|(u,v)\|)} \leq \Big[\alpha - 1 + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big] \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}.$$

Similarly,

$$\frac{\|v\|}{\psi(\|(u,v)\|)} \leq \Big[\beta - 1 + \frac{\frac{\mu n}{\beta}\eta^{\beta-1}}{(1-\frac{\mu\eta^\beta}{\beta})}\Big] \frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-\sigma_2)}.$$

Consequently,

$$\frac{\|(u,v)\|}{\max\{\varphi(\|(u,v)\|), \psi(\|(u,v))\}} \leq \max\Big\{\Big[\alpha - 1 + \frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^\alpha}{\alpha})}\Big] \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}, \Big[\beta - 1 + \frac{\frac{bn}{\beta}c^{\beta-1}}{(1-\frac{bc^\beta}{\beta})}\Big] \frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-\sigma_2)}\Big\}.$$

Again by (iv) we know $\|(u,v)\| \neq r$ which contradicts that $(u,v) \in \partial U$. Then based on Lemma 2.6, there is a fixed point $(u,v) \in \overline{U}$. Therefore the BVP (1) has a positive solution. □

Example 3.1. For any $n-1 < \alpha, \beta \leq n$, take $t_0 = \frac{1}{4}, \xi > 0$ and $\rho > 0$ with $\rho > \max\{\frac{4^{\alpha-1}\xi(\alpha-1)}{m_1}, \frac{4^{\beta-1}\xi(\beta-1)}{m_2}\}$. Choose $\sigma_1 = \sigma_2 = \frac{1}{2}$. Consider the boundary value problem to the singular system of fractional equations

$$
\begin{cases}
D_{0+}^{\alpha} u(t) + \frac{c_1+v}{\sqrt{t}} = 0, 0 < t < 1, \\
D_{0+}^{\beta} v(t) + \frac{c_2+u}{\sqrt{t}} = 0, 0 < t < 1, \\
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = \frac{1}{2} \displaystyle\int_0^{\frac{1}{2}} u(s) \mathrm{d}s, \\
v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, v(1) = \frac{1}{2} \displaystyle\int_0^{\frac{1}{2}} v(s) \mathrm{d}s,
\end{cases}
\tag{6}
$$

where c_1, c_2 are constants satisfying

$$\frac{4^{\alpha-1}\xi\Gamma(\alpha)}{m_1} \leq c_1 \leq \frac{\rho\Big[\Gamma(2+\alpha-\sigma_1)-(\alpha-1+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^{\alpha}}{\alpha})})\Gamma(2-\sigma_1)\Big]}{[\alpha-1+\frac{\frac{\mu n}{\alpha}\eta^{\alpha-1}}{(1-\frac{\mu\eta^{\alpha}}{\alpha})}]\Gamma(2-\sigma_1)},$$

$$\frac{4^{\beta-1}\xi\Gamma(\beta)}{m_2} \leq c_2 \leq \frac{\rho\Big[\Gamma(2+\beta-\sigma_2)-(\beta-1+\frac{\frac{\mu n}{\beta}\eta^{\beta-1}}{(1-\frac{\mu\eta^{\beta}}{\beta})})\Gamma(2-\sigma_2)\Big]}{[\beta-1+\frac{\frac{\mu n}{\beta}\eta^{\beta-1}}{(1-\frac{\mu\eta^{\beta}}{\beta})}]\Gamma(2-\sigma_2)}.$$

Denote $f(t,y) = \frac{c_1+y}{\sqrt{t}}, g(t,y) = \frac{c_2+y}{\sqrt{t}}$. Then f, g are continuous on $(0,1] \times [0,+\infty)$ and $\lim_{t\to 0+} f(t,\cdot) = +\infty, \lim_{t\to 0+} g(t,\cdot) = +\infty$. All conditions of Theorem 3. hold. Therefore, BVP (8) has at least one positive solution.

Example 3.2. Consider the boundary value problem to the following singular system of fractional equations

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t) + \frac{(t-\frac{1}{2})^2\ln(2+v(t))}{\sqrt{t}} = 0, 0 < t < 1, \\ D_{0+}^{\frac{5}{2}}v(t) + \frac{(t-\frac{1}{2})^2\ln(2+u(t))}{\sqrt{t}} = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, u(1) = \frac{1}{2}\displaystyle\int_0^{\frac{1}{2}} u(s)\mathrm{d}s, \\ v(0) = v'(0) = 0, v(1) = \frac{1}{2}\displaystyle\int_0^{\frac{1}{2}} v(s)\mathrm{d}s, \end{cases} \tag{7}$$

Denote f, g are continuous on $(0,1]\times[0,+\infty)$ and $\lim_{t\to 0+} f(t,\cdot) = +\infty, \lim_{t\to 0+} g(t,\cdot) = +\infty$. Choose $\sigma_1 = \sigma_2 = \frac{1}{2}$ and $\varphi(y) = \psi(y) = \ln(2+y)$, then we have $\sqrt{t}\frac{(t-\frac{1}{2})^2\ln(2+v(t))}{\sqrt{t}} \leq \ln(2+v(t)), \forall(t,y) \in [0,1]\times[0,+\infty)$. $\varphi, \psi : [0,+\infty) \to (0,+\infty)$ are continuous, nondecreasing functions, so condition (iii) of Theorem 3.2 holds. Next set $r = 1$. Then condition (iv) of Theorem 3.2 holds. Therefore, BVP (8) has at least one positive solution.

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