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NONLINEAR FRACTIONAL PROGRAMMING PROBLEM WITH INEXACT PARAMETER

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Abstract. In this paper a methodology is developed to solve a nonlinear fractional programming problem, whose objective function and constraints are interval valued functions. Interval valued convex fractional programming problem is studied. This model is transformed to a general convex programming problem and relation between the original problem and the transformed problem is established. These theoretical developments are illustrated through a numerical example.

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1. Introduction

In most of the real-life optimization models, the parameters in the objective function and the constraints are not exactly known due to the presence of improper information in the data set. If these parameters vary in closed intervals then the corresponding optimization model is an interval optimization model. Nonlinear interval optimization problems are studied by many researchers in several directions during last few decades (see [1, 2, 5, 6, 8, 9, 10, 11, 14, 15]). Most of these models are quadratic programming problems with interval parameters. These methodologies usually convert the model to a pair of two level mathematical programming problems yielding two individual optimal solutions, corresponding to the optimal value range. An interval optimization problem, where the objective function appears as a ratio of two interval valued functions, is an interval fractional programming problem (IFP). Such type of situation appears when the decision maker needs to decide the expenditure cost per unit time, the ratio of profit and expenditure cost, the ratio of the amount of the components in a mixture etc., subject to the condition that, these costs vary

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and other parameters within some lower and upper bounds. Here we explain two real life situations where the objective function is the ratio of two interval valued functions.

Example 1 (Linear (IFP)). *Suppose, a homogeneous product is to be transported from m number of sources to n number of destinations. The i^{th} source can provide a_i units of a certain product and the j^{th} destination has a demand for b_j units of same product. From the historical data, it is observed that due to traffic jam, climate change, shortage of vehicles etc., the transportation cost and deterioration cost for the transportation of one unit of the product from i^{th} source to j^{th} destination varies in the bounds c_{ij}^L, c_{ij}^R and d_{ij}^L, d_{ij}^R respectively. The objective is to minimize the ratio of transportation cost to deterioration cost. If x_{ij} is the number of units transported from source i to destination j, then the mathematical model of the conventional transportation problem becomes*

$$\begin{aligned}
\min \quad & \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}[c_{ij}^L, c_{ij}^R]x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}[d_{ij}^L, d_{ij}^R]x_{ij}} \\
\textit{subject to} \quad & \sum_{j=1}^{n} x_{ij} \le a_i,\ i = 1, 2, \ldots, m, \\
& \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n, \\
& x_{ij} \ge 0,\ \forall\ i, j.
\end{aligned}$$

Example 2 (Nonlinear (IFP)). *Consider a portfolio management problem which has n number of risky assets and one risk-less asset. The Sharpe ratio of n number of risky assets portfolio is the excess return per unit deviation. Due to the uncertainty in the market, the returns of the assets cannot be predicted exactly. From the historical data, one can estimate the upper and lower bound of the parameters of the return in a fixed time period. Hence the Sharpe ratio can be described in terms of interval parameters for a fixed time period. Let x_j be the proportion of the total funds invested on j^{th} asset and the expected return of j^{th} asset lies between r_j^L and r_j^R. The expected return of risk-less asset lies between r_f^L and r_f^R. Since the standard deviation and the expected return of all risky assets are lying in intervals, so the correlation between any two of them will also lie in an interval. The interval matrix, $\mathbf{Q}_M = (\mathbf{Q}_{ij})_{n\times n}$ be a $n \times n$ symmetric covariance interval matrix, where $\mathbf{Q}_{ij} = [q_{ij}^L, q_{ij}^R]$. In this circumstance, the expected return and the variance of the resulting portfolio $x = (x_1, x_2, \ldots, x_n)^T$ are $\sum_{i=1}^{n}[r_i^L, r_i^R]x_i$ and $\sum_{i,j}[q_{ij}^L, q_{ij}^R]x_i x_j$, respectively. Since the variance is always non-negative, so we assume that $\sum_{i,j}[q_{ij}^L, q_{ij}^R]x_i x_j > 0$.*
In order to maximize the Sharpe ratio of the portfolio $(x_1, x_2, \ldots, x_n)^T$, it is

necessary to solve the interval fractional programming problem

$$\max \quad \frac{\sum_{i=1}^{n}[r_i^L, r_i^R]x_i \ominus [r_f^L, r_f^R]}{(\sum_{i,j}[q_{ij}^L, q_{ij}^R]x_i x_j)^{\frac{1}{2}}}$$

$$\textit{subject to} \quad \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0.$$

Hladik [4] considered a generalized linear fractional programming problem with interval data and studied sensitivity analysis of the model. However nonlinear interval fractional programming problem has not been addressed so far. In this paper we concentrate on nonlinear interval fractional programming problem and studied the existence of it's solution. The basic difference between Hladik's approach and our approach is that, Hladik's methodology computes the range of optimal values when all the parameters vary in intervals and conversely for given bounds of the optimal values, tolerance intervals for the coefficients are calculated, whereas our proposed methodology provides efficient solution of interval fractional programming problem.

In this paper, we propose a methodology to derive the solution of a nonlinear interval fractional programming problem and justified the existence of an efficient solution in place of two individual solutions corresponding to the optimal range. The proposed model includes both linear and nonlinear interval valued functions. Since any interval valued function is a set valued mapping, so interval fractional programming problem is treated as a multi-objective programming problem and converted to a general optimization problem which is free from uncertainty. Section 2 provides some notations and preliminaries on interval analysis. In Section 3 existence of solution of (IFP) is established. Further the original problem is transformed to a convex programming problem and relationship between the solution of the original problem and the transformed problem is studied. The methodology is verified through a numerical example.

2. Interval Analysis in Parametric Form

Throughout the paper, the following notations are used. Bold capital letters denote closed intervals.

$I(R)$= The set of all closed intervals in R.

$(I(R))^k$= The product space $\underbrace{I(R) \times I(R) \times \ldots \times I(R)}_{k \text{ times}}$.

$\mathbf{C}_v^k$= k dimensional column whose elements are intervals. $\mathbf{C}_v^k \in (I(R))^k$, $\mathbf{C}_v^k = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_k)^T$, $\mathbf{C}_j = [c_j^L, c_j^R],\ j = 1, 2, \ldots, k$.

2.1. Algebraic operations and order relation in $I(R)$ in parametric form.

Let $* \in \{+, -, \cdot, /\}$ be a binary operation on the set of real numbers. The binary operation $\circledast$ between two intervals $\mathbf{A} = [a^L, a^R]$ and $\mathbf{B} = [b^L, b^R]$ in $I(R)$, denoted by $\mathbf{A} \circledast \mathbf{B}$ is the set $\{a * b : a \in \mathbf{A}, b \in \mathbf{B}\}$. In the case of division, $\mathbf{A} \oslash \mathbf{B}$, it is assumed that $0 \notin \mathbf{B}$. These interval operations can also

be expressed in terms of parameters. Mahapatra and Mandal [12] defined the parametric form for positive interval $[a^L, a^R]$ as $(a^L)^t(a^R)^{1-t}$, where $t \in [0,1]$. However, this parametric form holds for the interval $\mathbf{A}$ with $a^L > 0$. To avoid this restriction, we consider a liner expression of $[a^L, a^R]$. Any point in $\mathbf{A}$ may be expressed as $a(t) = a^L + t(a^R - a^L)$, $t \in [0,1]$. An interval $\mathbf{A}$ is said to be a positive interval if $a(t)$ is positive for every $t \in [0,1]$.
The algebraic operations of intervals in the classical form are defined in terms of either lower and upper bound or, mean and spread of the intervals. The algebraic operations of intervals and other properties can also be explained in parametric form as follows.

$$\mathbf{A} \circledast \mathbf{B} = \{a(t_1) * b(t_2) |\ t_1, t_2 \in [0,1]\}. \tag{1}$$

An interval vector $\mathbf{C}_v^k \in (I(R))^k$, $\mathbf{C}_v^k = (\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)^T$, can be expressed in terms of parameters as

$$\begin{aligned}\mathbf{C}_v^k = \Big\{ & c(t) |\ c(t) = (c_1(t_1), c_2(t_2), \dots, c_k(t_k))^T,\ c_j(t_j) \in \mathbf{C}_j, c_j(0) = c_j^L, \\ & c_j(1) = c_j^R, t = (t_1, t_2, \dots, t_k)^T,\ 0 \le t_j \le 1, j = 1, 2, \dots, k \Big\}\end{aligned} \tag{2}$$

The product of a real number k and an interval $\mathbf{A}$ is

$$k\mathbf{A} = \mathbf{A}k = \{a(t)k |\ a(t) \in \mathbf{A}\}.$$

The product of a real vector $d \in R^k$ and an interval vector $\mathbf{C}_v^k \in (I(R))^k$ is denoted by $\mathbf{C}_v^k \diamond d$ and is equals to $\sum_{j=1}^k \mathbf{C}_j d_j$.

The following properties hold in $I(R)$, whose proofs are easy and follow from the definition of algebraic operations of intervals in terms of parameters as in Expression (1).

Lemma 2.1. *(a) For $\alpha, \beta \in R$ and $\mathbf{A} \in I(R)$, $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} \oplus \beta\mathbf{A}$.*
(b) For $a, b \in R^n$ and $\mathbf{A}_v^n \in (I(R))^n$, $\mathbf{A}_v^n \diamond (a + b) = (\mathbf{A}_v^n \diamond a) \oplus (\mathbf{A}_v^n \diamond b)$.

The set of intervals $I(R)$, is not a totally order set. Several partial orderings in $I(R)$ exist in literature. Moore [13] defined two type of order relations between $\mathbf{A}$, $\mathbf{B} \in I(R)$, one of which is explained as the extension of " $<$ " on real line ("$\mathbf{A} < \mathbf{B}$ iff $a_R < b_L$") and another is the extension of set inclusion ("$\mathbf{A} \subseteq \mathbf{B}$ iff $a_L \ge b_L$ and $a_R \le b_R$"). These order relations can not explain the ranking between two partially overlapping intervals. So Ishibuchi and Tanaka [7] defined partial order relations $\preceq_{LR}$, $\preceq_{LC}$, $\preceq_{RC}$ to rank between the partially overlapping intervals. Bhurjee and Panda [1] redefined the following specific partial ordering in terms of parameters, which reduces to all the above partial order relations in particular cases.

Definition 2.2 ([1])**.** For $\mathbf{A}, \mathbf{B} \in I(R)$,

(*i*) $\mathbf{A} \preceq \mathbf{B}$ if $a(t) \le b(t)\ \forall\ t \in [0,1]$.

(*ii*) $\mathbf{A} \ne \mathbf{B}$ if $a(t) \ne b(t)$ for at least one $t \in [0,1]$.

2.2. Interval valued function. Interval valued function is defined in several ways by many authors. (See [3, 13, 15]) Some of these are functions of one or more interval arguments onto intervals and others are interval extension of real valued functions. The interval valued function in the parametric form, introduced by [1] is as follows.
For $c(t) \in \mathbf{C}_v^k$, let $f_{c(t)} : R^n \to R$. Then for a given interval vector $\mathbf{C}_v^k$, we define an interval valued function $\mathbf{F}_{\mathbf{C}_v^k} : R^n \to I(R)$ by

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \Big\{ f_{c(t)}(x) \;\Big|\; f_{c(t)} : R^n \to R, c(t) \in \mathbf{C}_v^k \Big\}.$$

For every fixed x, if $f_{c(t)}(x)$ is continuous in t then $\min_{t\in[0,1]^k} f_{c(t)}(x)$ and $\max_{t\in[0,1]^k} f_{c(t)}(x)$, exist. In that case

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \Big[\min_{t\in[0,1]^k} f_{c(t)}(x), \max_{t\in[0,1]^k} f_{c(t)}(x) \Big].$$

If $f_{c(t)}(x)$ is linear in t then $\min_{t\in[0,1]^k} f_{c(t)}(x)$ and $\max_{t\in[0,1]^k} f_{c(t)}(x)$ exist in the set of vertices of $\mathbf{C}_v^k$. If $f_{c(t)}(x)$ is monotonically increasing in t then $\mathbf{F}_{\mathbf{C}_v^k}(x) = [f_{c(0)}(x), f_{c(1)}(x)]$.

2.3. Interval valued convex function. Property of interval valued convex function plays an important role for the existence of the solution of interval optimization problem.

Definition 2.3 ([1]). An interval valued function $\mathbf{F}_{\mathbf{C}_v^k} : R^n \to I(R)$ is said to be convex function with respect to $\preceq$ on a convex set $D \subseteq R^n$ if for every $x_1,\ x_2 \in D$ and $0 \le \lambda \le 1$,

$$\mathbf{F}_{\mathbf{C}_v^k}(\lambda x_1 + (1-\lambda)x_2) \preceq \lambda \mathbf{F}_{\mathbf{C}_v^k}(x_1) \oplus (1-\lambda)\mathbf{F}_{\mathbf{C}_v^k}(x_2).$$

Remark 2.1. From Definition (2.3), one may observe that $\mathbf{F}_{\mathbf{C}_v^k}$ is convex with respect to $\preceq$ means

$$f_{c(t)}(\lambda x_1 + (1-\lambda)x_2) \le \lambda f_{c(t)}(x_1) + (1-\lambda) f_{c(t)}(x_2), \tag{3}$$

for all $t \in [0,1]^k$; the value of t is same on both sides of this inequality. So one can conclude that $\mathbf{F}_{\mathbf{C}_v^k}$ is convex with respect to $\preceq$ if and only if $f_{c(t)}(x)$ is a convex function on D for every t.

3. Interval Fractional Programming Problem

We consider a general interval fractional programming problem (IFP) as

$$(IFP): \qquad \min \qquad \frac{\mathbf{F}_{\mathbf{C}_v^k}(x)}{\mathbf{G}_{\mathbf{D}_v^l}(x)}$$

$$\text{subject to} \quad \mathbf{H}^j_{\mathbf{B}_v^{m_j}}(x) \preceq \mathbf{A}_j,\ j \in J,$$

where the interval valued functions$\mathbf{F}_{\mathbf{C}_v^k}$, $\mathbf{G}_{\mathbf{D}_v^l}$, $\mathbf{H}^j_{\mathbf{B}_v^{m_j}} : R^n \to I(R)$ are the sets, $\mathbf{F}_{\mathbf{C}_v^k}(x) = \Big\{ f_{c(t)}(x) |\ c(t) \in \mathbf{C}_v^k \Big\}$, $\mathbf{G}_{\mathbf{D}_v^l}(x) = \Big\{ g_{d(t')}(x) > 0 |\ d(t') \in \mathbf{D}_v^l \Big\}$,

$\mathbf{H}^j_{\mathbf{B}_v^{m_j}}(x) = \Big\{h^j_{b_j(t''_j)}(x) |\, b_j(t''_j) \in \mathbf{B}_v^{m_j}\Big\}$, $\mathbf{A}_j \in I(R)$, $\mathbf{A}_j = [a_j^L, a_j^R]$ and $J = \{1, 2, \ldots, p\}$.

Throughout this section, we consider $t \in [0,1]^k, t' \in [0,1]^l, t''_j \in [0,1], j \in J$.

Following the partial ordering in Definition (2.3), the feasible region of (IFP) can be expressed as the set,

$$\begin{aligned} S &= \{x \in R^n : \mathbf{H}^j_{\mathbf{B}_v^{m_j}}(x) \preceq \mathbf{A}_j, j \in J\} \\ &= \{x \in R^n : h^j_{b_j(t''_j)}(x) \leq a_j(t''_j), a_j(t''_j) \in \mathbf{A}_j, j \in J\} \end{aligned} \tag{4}$$

The objective function of (IFP), which is a ratio of two interval valued functions, can be expressed in the form of a set of real valued functions as follows.

$$\frac{\mathbf{F}_{\mathbf{C}_v^k}(x)}{\mathbf{G}_{\mathbf{D}_v^l}(x)} = \Big\{\frac{f_{c(t)}(x)}{g_{d(t')}(x)}\Big|\ c(t) \in \mathbf{C}_v^k, d(t') \in \mathbf{D}_v^l, g_{d(t')}(x) > 0\Big\} \tag{5}$$

So

$$\min_{x\in S} \frac{\mathbf{F}_{\mathbf{C}_v^k}(x)}{\mathbf{G}_{\mathbf{D}_v^l}(x)} = \min_{x\in S} \Big\{\frac{f_{c(t)}(x)}{g_{d(t')}(x)}\Big|\ g_{d(t')}(x) > 0, c(t) \in \mathbf{C}_v^k, d(t') \in \mathbf{D}_v^l\Big\}$$

For different pairs (t, t'), $\frac{f_{c(t)}(x)}{g_{d(t')}(x)}$ represents different functions of x. Hence (IFP) can be treated as a multi-objective problem. Assuming that for every pair $(c(t), d(t'))$, the optimization problem $\min_{x\in S} \frac{f_{c(t)}(x)}{g_{d(t')}(x)}$ has a solution, we define the solution of (IFP) (which is parallel to the concept of efficient solution in case of multi-objective programming problem), as follows.

Definition 3.1. $x^* \in S$ is called an efficient solution of (IFP) if there is no $x \in S$ with
$\frac{f_{c(t)}(x)}{g_{d(t')}(x)} \leq \frac{f_{c(t)}(x^*)}{g_{d(t')}(x^*)}$ $\forall\ (t, t')$ and for at least one (t, t'), $\frac{f_{c(t)}(x)}{g_{d(t')}(x)} < \frac{f_{c(t)}(x^*)}{g_{d(t')}(x^*)}$.

Definition 3.2. $x^* \in S$ is called a properly efficient solution of (IFP) if
(i) x^* is an efficient solution and
(ii) if there is a real number $\mu > 0$ such that, for every $(t, t') \in [0,1]^{k+l}$, we have

$$\frac{f_{c(t)}(x^*)}{g_{d(t')}(x^*)} - \frac{f_{c(t)}(x)}{g_{d(t')}(x)} \leq \mu\Big(\frac{f_{c(\mathbf{t})}(x)}{g_{d(\mathbf{t'})}(x)} - \frac{f_{c(\mathbf{t})}(x^*)}{g_{d(\mathbf{t'})}(x^*)}\Big)$$

for some $(\mathbf{t}, \mathbf{t'}) \neq (t, t')$ such that $\frac{f_{c(\mathbf{t})}(x)}{g_{d(\mathbf{t'})}(x)} > \frac{f_{c(\mathbf{t})}(x^*)}{g_{d(\mathbf{t'})}(x^*)}$ whenever $x \in S$ and $\frac{f_{c(t)}(x)}{g_{d(t')}(x)} < \frac{f_{c(t)}(x^*)}{g_{d(t')}(x^*)}$.

3.1. Interval convex fractional programming problem. In the light of a general convex optimization problem, one can define interval convex fractional programming problem. An interval fractional programming problem (IFP) is said to be an interval convex fractional programming problem if $\frac{\mathbf{F}_{\mathbf{C}_v^k}(x)}{\mathbf{G}_{\mathbf{D}_v^l}(x)}$ is an interval valued convex function on a convex feasible region. It is very difficult to impose conditions on $\mathbf{F}_{\mathbf{C}_v^k}, \mathbf{G}_{\mathbf{D}_v^l}$ to justify that $\frac{\mathbf{F}_{\mathbf{C}_v^k}(x)}{\mathbf{G}_{\mathbf{D}_v^l}(x)}$ is an interval convex function. So we study some particular type (IFP) which can be transformed to a general convex programming problem and free from interval uncertainties.

For $x \in R^n$, let

$$\frac{1}{\mathbf{G}_{\mathbf{D}_v^l}(x)} = [\tau^L, \tau^R] = \Big\{\tau \in R_+ \mid \tau \in [\tau^L, \tau^R]\Big\},$$

where $\tau^L = \min_{t'} \frac{1}{g_{d(t')}(x)}, \tau^R = \max_{t'} \frac{1}{g_{d(t')}(x)}$. Denote, $\frac{1}{g_{d(t')}(x)} = \tau$. Since $g_{d(t')}(x) > 0$ so $\tau > 0$ and $\min_{t'} g_{d(t')}(x) \leq g_{d(t')}(x) \leq \max_{t'} g_{d(t')}(x)\ \forall\ t' \in [0,1]^l$. Then

$$\tau \min_{t'} g_{d(t')}(x) \leq 1 \text{ and } \tau \max_{t'} g_{d(t')}(x) \geq 1 \tag{6}$$

If we put $y = \tau x$, then the feasible set S defined in (4) is transformed to the following set χ.

$$\chi = \Big\{(y,\tau) \mid \tau \min_{t'} g_{d(t')}(y/\tau) \leq 1,\ \tau \max_{t'} g_{d(t')}(y/\tau) \geq 1,$$
$$\tau h^j_{b_j(t''_j)}(y/\tau) \leq \tau a_j(t''_j), j \in J\Big\} \tag{7}$$

With the help of above transformation, Expression (5) may be written as,

$$\frac{\mathbf{F}_{\mathbf{C}_v^k}(x)}{\mathbf{G}_{\mathbf{D}_v^l}(x)} = \Big\{\tau f_{c(t)}(x)\Big|\ g_{d(t')}(x)\tau = 1,\ c(t) \in \mathbf{C}_v^k, d(t') \in \mathbf{D}_v^m\Big\} \tag{8}$$

To take care all real valued functions $\tau f_{c(t)}(x)$ in the objective function, select a weight function $w : [0,1]^k \to R_+$, and consider the following optimization problem $(IFP)_I$.

$$(IFP)_I : \qquad \min \int_k w(t)\tau f_{c(t)}\Big(\frac{y}{\tau}\Big)dT \tag{9}$$
$$\text{subject to} \qquad (y,\tau) \in \chi,$$

where $\int_k = \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{k \text{ times}}$, $dT = dt_1 dt_2 \dots dt_k,\ t = (t_1, t_2, \dots, t_k)^T,\ y = \tau x,$ $x \in S$. The optimization problem $(IFP)_I$ is free from the interval uncertainty and partial order relations. In the objective function of $(IFP)_I$ one may consider $w(t)$ in such a way that $\int_k w(t)dT = 1$, but it may not be mandatory. This weight may be provided by the decision maker.

Since $t_1, t_2, \dots, t_k$ are mutually independent continuous variables in $[0,1]$ and $w(t)$ is real valued function in t_i's, so the objective function of $(IFP)_I$ can be simplified by integrating with respect to each t_i. Then the objective function is a function of y and τ only and we denote this by $\rho(y,\tau)$. Hence $(IFP)_I$ is equivalent to

$$\begin{aligned}(IFP)_I: \quad & \min \quad \rho(y,\tau) \\ & \text{subject to} \quad \Phi_j(y,\tau,t''_j) \leq \tau a_j(t''_j), && (10)\\ & \max_{t'} \Psi(y,\tau,t') \geq 1, && (11)\\ & \min_{t'} \Psi(y,\tau,t') \leq 1, && (12)\end{aligned}$$

where $\rho(y,\tau) = \int_k w(t)\tau f_{c(t)}\left(\frac{y}{\tau}\right)dT$; $\Phi_j(y,\tau,t''_j) = \tau h^j_{b_j(t''_j)}\left(\frac{y}{\tau}\right), j \in J$; $\Psi(y,\tau,t') = \tau g_{d(t')}\left(\frac{y}{\tau}\right)$.

Theorem 3.3. *If the interval valued functions $\mathbf{H}^j_{\mathbf{B}_v^{m_j}} : D \to I(R), j \in J$ are convex functions with respect to $\preceq$ on the open convex set $D \subseteq R^n$ then the feasible set of (IFP) is a convex set.*

Proof. Suppose $\mathbf{H}^j_{\mathbf{B}_v^{m_j}}$ $\forall\ j$ are interval valued convex functions with respect to $\preceq$ on the open convex set $D \subseteq R^n$. From Definition (**??**) and Remark (2.1), for $x_1, x_2 \in S$ (S is defined in (4)) and $0 \leq \sigma \leq 1$, $j \in J$; we have $h^j_{b_j(t''_j)}(x_1) \leq a_j(t''_j)$, $h^j_{b_j(t''_j)}(x_2) \leq a_j(t''_j)$ and

$$\begin{aligned} h^j_{b_j(t''_j)}(\sigma x_1 + (1-\sigma)x_2) &\leq \sigma h^j_{b_j(t''_j)}(x_1) + (1-\sigma)h^j_{b_j(t''_j)}(x_2) \\ &\leq \sigma a_j(t''_j) + (1-\sigma)a_j(t''_j) = a_j(t''_j) \end{aligned}$$

holds for every t''_j. This implies that $\mathbf{H}^j_{\mathbf{B}_v^{m_j}}(\sigma x_1 + (1-\sigma)x_2) \preceq \mathbf{A}_j, j \in J$. Hence $\sigma x_1 + (1-\sigma)x_2 \in S$, that is, the feasible set S of (IFP) is a convex set. □

Definition 3.4. For $\mathbf{C}_v^{n+1} \in (I(R))^{n+1}$, with $\mathbf{C}_v^{n+1} = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n, \mathbf{B})^T$, an interval valued function $\mathbf{F}_{\mathbf{C}_v^{n+1}} : D \to I(R)$, $D \subseteq R^n$ is an affine function if it is written in the following form,

$$\mathbf{F}_{\mathbf{C}_v^{n+1}}(x) = \mathbf{C}_1 x_1 \oplus \mathbf{C}_2 x_2 \oplus \ldots \oplus \mathbf{C}_n x_n \oplus \mathbf{B}.$$

In the light of the discussion in Section 2, we may write

$$\begin{aligned}\mathbf{F}_{\mathbf{C}_v^{n+1}}(x) = \mathbf{C}_v^n \diamond x \oplus \mathbf{B} &= \mathbf{C}_1 x_1 \oplus \mathbf{C}_2 x_2 \oplus \ldots \oplus \mathbf{C}_n x_n \oplus \mathbf{B} \\ &= \{f_{\alpha(\hat{t})}(x) | \alpha(\hat{t}) = (c_1(t_1), c_2(t_2), \ldots, c_n(t_n), b(t_{n+1}))^T\}\end{aligned}$$

where $\hat{t} = (t_1, t_2, \ldots, t_{n+1})^T \in [0,1]^{n+1}$, $f_{\alpha(\hat{t})}(x) = \sum_{j=1}^n c_j(t_j)x_j + b(t_{n+1})$.
For every $x, y \in D$ and $0 \leq \lambda \leq 1$,

$$\begin{aligned}\mathbf{F}_{\mathbf{C}_v^{n+1}}(\lambda x + (1-\lambda)y) &= \mathbf{C}_v^n \diamond (\lambda x + (1-\lambda)y) \oplus \mathbf{B} \\ &= \Big\{\sum_{j=1}^n c_j(t_j)(\lambda x_j + (1-\lambda)y_j) + b(t_{n+1}) | t_j \in [0,1], \forall j\Big\} \\ &= \lambda\Big\{\sum_{j=1}^n c_j(t_j)x_j + b(t_{n+1})\Big\} + (1-\lambda)\Big\{\sum_{j=1}^n c_j(t_j)y_j + b(t_{n+1})\Big\}\end{aligned}$$

$$\begin{aligned} &= \lambda(\mathbf{C}_v^n \diamond x \oplus \mathbf{B}) + (1-\lambda)(\mathbf{C}_v^n \diamond y \oplus \mathbf{B}) \\ &= \lambda \mathbf{F}_{\mathbf{C}_v^{n+1}}(x) \oplus (1-\lambda)\mathbf{F}_{\mathbf{C}_v^{n+1}}(y). \end{aligned}$$

Hence, if $\mathbf{F}_{\mathbf{C}_v^{n+1}} : D \to I(R)$ is an interval valued affine function on $D \subseteq R^n$, $\mathbf{F}_{\mathbf{C}_v^{n+1}}(x) = \mathbf{C}_v^n \diamond x \oplus \mathbf{B}$ then for every $x, y \in D$ and $0 \le \lambda \le 1$,

$$\mathbf{F}_{\mathbf{C}_v^{n+1}}(\lambda x + (1-\lambda)y) = \lambda \mathbf{F}_{\mathbf{C}_v^{n+1}}(x) \oplus (1-\lambda)\mathbf{F}_{\mathbf{C}_v^{n+1}}(y) \tag{13}$$

Remark 3.1. From (13), one may observe that $\mathbf{F}_{\mathbf{C}_v^k}$ is affine if and only if $f_{\alpha(\hat{t})}(x)$ is an affine function on D for every $\hat{t}$.

Theorem 3.5. *If $\mathbf{F}_{\mathbf{C}_v^k}, \mathbf{H}^j_{\mathbf{B}_v^{m_j}} : D \to I(R)$ are interval valued convex functions with respect to the order relation $\preceq$ on the convex set $D \subseteq R^n$; and $\mathbf{G}_{\mathbf{D}_v^l} : D \to I(R)$ is an interval valued affine function on D then $(IFP)_I$ is a convex programming problem.*

Proof. Let $\mathbf{F}_{\mathbf{C}_v^k}$ and $\mathbf{H}^j_{\mathbf{B}_v^{m_j}}$ are interval valued convex functions with respect to the order relation $\preceq$ on D; and $\mathbf{G}_{\mathbf{D}_v^l}$ is an interval valued affine function on convex set D.
First we need to see that the feasible set for $(IFP)_I$ is a convex set. That is, if $(y_1, \tau_1), (y_2, \tau_2)$ are feasible points then for $\lambda \in [0,1]$, $(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2)$ satisfies the conditions (10),(11) and (12).
For $t''_j, j \in J$, suppose $\Phi_j(y_1, \tau_1, t''_j) \le \tau_1 a_j(t''_j)$, $\Phi_j(y_2, \tau_2, t''_j) \le \tau_2 a_j(t''_j)$. Now

$$\begin{aligned} &\Phi_j(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2, t''_j) \\ &= (\lambda\tau_1 + (1-\lambda)\tau_2) h^j_{b_j(t''_j)}\Big(\frac{\lambda y_1 + (1-\lambda)y_2}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big) \\ &= (\lambda\tau_1 + (1-\lambda)\tau_2) h^j_{b_j(t''_j)}\Big[\frac{\lambda\tau_1}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big(\frac{y_1}{\tau_1}\Big) + \frac{(1-\lambda)\tau_2}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big(\frac{y_2}{\tau_2}\Big)\Big] \end{aligned} \tag{14}$$

Since $\mathbf{H}^j_{\mathbf{B}_v^{m_j}}, j \in J$ is a convex interval valued function with respect to $\preceq$, so from Remark (2.1), $h^j_{b_j(t''_j)}, j \in J$ is a convex function for each t''_j on $S' = \{\frac{y}{\tau} | y = \tau x, x \in D, \tau \in R, \tau > 0\}$. Then (14) becomes,

$$\begin{aligned} \Phi_j(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2, t''_j) &\le \lambda\tau_1 h^j_{b_j(t''_j)}\Big(\frac{y_1}{\tau_1}\Big) + (1-\lambda)\tau_2 h^j_{b_j(t''_j)}\Big(\frac{y_2}{\tau_2}\Big) \\ &= \lambda\Phi_j(y_1, \tau_1, t''_j) + (1-\lambda)\Phi_j(y_2, \tau_2, t''_j) \\ &\le \lambda\tau_1 a_j(t''_j) + (1-\lambda)\tau_2 a_j(t''_j) \\ &= (\lambda\tau_1 + (1-\lambda)\tau_2) a_j(t''_j). \end{aligned}$$

Hence Inequality (10) holds for $(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2)$.
Similarly,

$$\begin{aligned} &\Psi(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2, t') \\ &= (\lambda\tau_1 + (1-\lambda)\tau_2) g_{d(t')}\Big(\frac{\lambda y_1 + (1-\lambda)y_2}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big) \\ &= (\lambda\tau_1 + (1-\lambda)\tau_2) g_{d(t')}\Big[\frac{\lambda\tau_1}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big(\frac{y_1}{\tau_1}\Big) + \frac{(1-\lambda)\tau_2}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big(\frac{y_2}{\tau_2}\Big)\Big] \end{aligned} \tag{15}$$

Since $\mathbf{G}_{\mathbf{D}_v^l}$ is an interval valued affine function, so for each t', $g_{d(t')}$ is an affine function on S'. Then Equation (15) becomes,

$$\begin{aligned}\Psi(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2, t') &= \lambda\Big[\tau_1 g_{d(t')}\Big(\frac{y_1}{\tau_1}\Big)\Big] + (1-\lambda)\Big[\tau_2 g_{d(t')}\Big(\frac{y_2}{\tau_2}\Big)\Big] \\ &= \lambda\Psi(y_1,\tau_1,t') + (1-\lambda)\Psi(y_2,\tau_2,t') \qquad (16)\end{aligned}$$

Since $\max_{t'}\Psi(y_1,\tau_1,t') \geq 1, \max_{t'}\Psi(y_2,\tau_2,t') \geq 1$ and $\min_{t'}\Psi(y_1,\tau_1,t') \leq 1, \min_{t'}\Psi(y_2,\tau_2,t') \leq 1$, we obtain the following two relations from (16).

$$\max_{t'}\Psi(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2, t') \geq 1 \qquad (17)$$

$$\text{and } \min_{t'}\Psi(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2, t') \leq 1. \qquad (18)$$

So Inequalities (11) and (12) hold for $(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2)$. Hence the feasible set for $(IFP)_I$ is a convex set.
Next to see that the objective function of the problem $(IFP)_I$ is a convex function.

$$\begin{aligned}&\rho(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2) \\ = &\int_k w(t)(\lambda\tau_1 + (1-\lambda)\tau_2) f_{c(t)}\Big(\frac{\lambda y_1 + (1-\lambda)y_2}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big)dT \\ = &\int_k w(t)(\lambda\tau_1 + (1-\lambda)\tau_2) f_{c(t)}\left[\frac{\lambda\tau_1}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big(\frac{y_1}{\tau_1}\Big) + \frac{(1-\lambda)\tau_2}{\lambda\tau_1 + (1-\lambda)\tau_2}\Big(\frac{y_2}{\tau_2}\Big)\right]dT\end{aligned}$$

Since $\mathbf{F}_{\mathbf{C}_v^k}$ is an interval valued convex function with respect to $\preceq$, so for each t, $f_{c(t)}$ is convex function on S'. Hence the above equation reduces to

$$\begin{aligned}&\rho(\lambda y_1 + (1-\lambda)y_2, \lambda\tau_1 + (1-\lambda)\tau_2) \\ \leq\ &\lambda\int_k w(t)\tau_1 f_{c(t)}\Big(\frac{y_1}{\tau_1}\Big)dT + (1-\lambda)\int_k w(t)\tau_2 f_{c(t)}\Big(\frac{y_2}{\tau_2}\Big)dT \\ =\ &\lambda\rho(y_1,\tau_1) + (1-\lambda)\rho(y_2,\tau_2)\end{aligned}$$

From the above discussion, we conclude that $(IFP)_I$ is a convex programming problem. $\square$

3.2. Existence of solution of (IFP)**.** From the discussions of previous subsection one can conclude that $(IFP)_I$ is a general convex programming problem under certain assumptions on the interval valued functions of (IFP). Then $(IFP)_I$ can be solved using general convex programming techniques. However, to find the solution of (IFP), a relation between the solution of (IFP) and $(IFP)_I$ should be established. This is the central idea of the entire development. The following result establishes such a relation.

Theorem 3.6. *If* $(y^*, \tau^*) \in \chi$ *is an optimal solution of* $(IFP)_I$ *then* $x^* = \frac{y^*}{\tau^*} \in S$ *is a properly efficient solution of* (IFP).

Proof. Let $(y^*, \tau^*) \in \chi$ be an optimal solution of $(IFP)_I$. Then x^* is an efficient solution of (IFP). Otherwise there will be $x' \in \chi$, with $x' = \frac{y'}{\tau'}$ for some y' and

τ' such that $\tau' f_{c(t)}(x') \leq \tau^* f_{c(t)}(x^*)$ for all t and $\tau' f_{c(t)}(x') \neq \tau^* f_{c(t)}(x^*)$ for at least one t holds. Since $w(t) \geq 0$, this implies

$$\int_k w(t)\tau' f_{c(t)}(x')\, dt_1 dt_2 \ldots dt_k \leq \int_k w(t)\tau^* f_{c(t)}(x^*)\, dt_1 dt_2 \ldots dt_k$$

This contradicts the optimality of (y^*, τ^*).
Assume that x^* is not a properly efficient solution of (IFP). From Definition (3.2) it follows that for some $t \in [0,1]^k$ and some $x = \frac{y}{\tau} \in S$ with $\tau f_{c(t)}(x) < \tau^* f_{c(t)}(x^*)$, one can choose $\mu = \max\left\{\frac{w(\mathbf{t})}{w(t)}\right\}$, $t \neq \mathbf{t};\ t, \mathbf{t} \in [0,1]^k$, $w(t) > 0$, such that

$$\frac{\tau^* f_{c(t)}(x^*) - \tau f_{c(t)}(x)}{\tau f_{c(\mathbf{t})}(x) - \tau^* f_{c(\mathbf{t})}(x^*)} > \mu, \tag{19}$$

with $\tau f_{c(\mathbf{t})}(x) > \tau^* f_{c(\mathbf{t})}(x^*)$ holds. (μ always exists since $w(t) > 0$ in $[0,1]^k$). This implies

$$\begin{aligned} \tau^* f_{c(t)}(x^*) - \tau f_{c(t)}(x) &> \frac{w(\mathbf{t})}{w(t)}\Big(\tau f_{c(\mathbf{t})}(x) - \tau^* f_{c(\mathbf{t})}(x^*)\Big) \\ \Rightarrow w(t)\tau^* f_{c(t)}\Big(\frac{y^*}{\tau^*}\Big) + w(\mathbf{t})\tau^* f_{c(\mathbf{t})}\Big(\frac{y^*}{\tau^*}\Big) &> w(\mathbf{t})\tau f_{c(\mathbf{t})}\Big(\frac{y}{\tau}\Big) + w(t)\tau f_{c(t)}\Big(\frac{y}{\tau}\Big). \end{aligned}$$

This implies

$$\begin{gathered} \int_k w(t)\tau^* f_{c(t)}\Big(\frac{y^*}{\tau^*}\Big)\, dt_1 dt_2 \ldots dt_k + \int_k w(\mathbf{t})\tau^* f_{c(\mathbf{t})}\Big(\frac{y^*}{\tau^*}\Big)\, d\mathbf{t}_1 d\mathbf{t}_2 \ldots d\mathbf{t}_k \\ > \\ \int_k w(\mathbf{t})\tau f_{c(\mathbf{t})}\Big(\frac{y}{\tau}\Big)\, d\mathbf{t}_1 d\mathbf{t}_2 \ldots d\mathbf{t}_k + \int_k w(t)\tau f_{c(t)}\Big(\frac{y}{\tau}\Big)\, dt_1 dt_2 \ldots dt_k. \end{gathered}$$

Hence

$$\int_k w(t)\tau^* f_{c(t)}\Big(\frac{y^*}{\tau^*}\Big) dT > \int_k w(t)\tau f_{c(t)}\Big(\frac{y}{\tau}\Big) dT \Rightarrow \rho(y^*, \tau^*) > \rho(y, \tau).$$

This contradicts to the assumption that (y^*, τ^*) is an optimal solution of $(IFP)_I$. □

3.3. Solution procedure for $(IFP)_I$. Let $\Psi(y, \tau, t')$ has the maximum and the minimum value over t' at t'^* and t'_*, respectively. Then the optimization problem $(IFP)_I$ becomes,

$$\begin{aligned} (IFP)_I: \qquad & \min \qquad \rho(y, \tau) \\ & \text{subject to} \quad \Psi(y, \tau, t'^*) \geq 1, \Psi(y, \tau, t'_*) \leq 1, \\ & \qquad\qquad\quad \Phi_j(y, \tau, t''_j) \leq \tau a_j(t''_j), \\ & \qquad\qquad\quad y \in R^n,\ \tau > 0. \end{aligned}$$

Under the assumption of Theorem (3.5), it is true that the above problem is a convex programming problem. So KKT optimality conditions are sufficient for the existence of it's optimal solution. The Lagrangian function for $(IFP)_I$ is

$$\begin{aligned} L(y,\tau,\eta,\mu,\nu) &= \rho(y,\tau) + \eta_1(1-\tau g_{d(t'^*)}\Big(\frac{y}{\tau}\Big)) + \eta_2(\tau g_{d(t'_*)}\Big(\frac{y}{\tau}\Big) - 1) \\ & \quad + \sum_{j=1}^{p} \mu_j(\tau h_{jb_j(t''_j)}\Big(\frac{y}{\tau}\Big) - \tau a_j(t''_j)) - \nu(\tau), \end{aligned}$$

where $\eta_1, \eta_2 \in R_+,\ \mu_j \in R_+,\ j \in J, \nu \in R_+$ are dual variables. The KKT optimality conditions for $(IFP)_I$ are

$$\int_k w(t)\nabla_y f_{c(t)}\Big(\frac{y}{\tau}\Big)dT - eta_1 \nabla_y g_{d(t'^*)}\Big(\frac{y}{\tau}\Big)L$$

$$+\eta_2 l \nabla_y g_{d(t'_*)}\Big(\frac{y}{\tau}\Big) + \sum_{j=1}^{p}\mu_j \nabla_y h_{jb_j(t''_j)}\Big(\frac{y}{\tau}\Big) = 0,$$

$$\int_k w(t) f_{c(t)}\Big(\frac{y}{\tau}\Big)dT - \Big(\frac{y}{\tau}\Big)\int_k w(t)\partial_\tau f_{c(t)}\Big(\frac{y}{\tau}\Big)dT$$

$$+\eta_1\Big[\Big(\frac{y}{\tau}\Big)\partial_\tau g_{d(t'^*)}\Big(\frac{y}{\tau}\Big) - g_{d(t'^*)}\Big(\frac{y}{\tau}\Big)\Big] + \eta_2\Big[-\Big(\frac{y}{\tau}\Big)\partial_\tau g_{d(t'_*)}\Big(\frac{y}{\tau}\Big) + g_{d(t'_*)}\Big(\frac{y}{\tau}\Big)\Big]$$

$$+\sum_{j=1}^{p}\mu_j\Big[h_{jb_j(t''_j)}\Big(\frac{y}{\tau}\Big) - \Big(\frac{y}{\tau}\Big)\partial_\tau h_{jb_j(t''_j)}\Big(\frac{y}{\tau}\Big) - a_j(t''_j)] = \nu,$$

$$\eta_1(1-\tau g_{d(t'^*)}\Big(\frac{y}{\tau}\Big)) = 0, \eta_2(\tau g_{d(t'_*)}\Big(\frac{y}{\tau}\Big) - 1) = 0,$$

$$\mu_j(h_{jb_j(t''_j)}\Big(\frac{y}{\tau}\Big) - a_j(t''_j)) = 0, y \in R^n,\ \tau > 0,$$

where $\partial_\tau = \frac{\partial}{\partial \tau}$. Methodology of this section is explained in the following numerical example.

Example 3. Consider the following interval fractional quadratic programming problem as,

$$\begin{aligned} (IFP)\ \min\ & \frac{[-10,-6]x_1 \oplus [2,3]x_2 \oplus [4,10]x_1^2 \oplus [-1,1]x_1x_2 \oplus [10,20]x_2^2}{[-5,-3]x_1 \oplus [1,2]x_2} \\ \text{s.t. } & [1,2]x_1 \oplus 3x_2 \succeq [1,10],\ [-2,8]x_1 \oplus [4,6]x_2 \succeq [4,6], x_1, x_2 \geq 0. \end{aligned}$$

(IFP) is equivalent to,

$$\begin{aligned} \min \quad & \Big\{\frac{f_{c(t)}(x_1,x_2)}{g_{d(t')}(x_1,x_2)}\Big| t = (t_1,t_2,\ldots,t_5), t' = (t_6,t_7)\Big\} \\ \text{subject to} \quad & (1+t_8)x_1 + 3x_2 \geq (1+9t_8), \\ & (-2+10t_9)x_1 + (4+2t_9)x_2 \geq (4+2t_9), \\ & x_1 \geq 0,\ x_2 \geq 0, t_i \in [0,1], i = 1,2,\ldots,9. \end{aligned}$$

where $f_{c(t)}(x_1,x_2) = (-10+4t_1)x_1 + (2+t_2)x_2 + (4+6t_3)x_1^2 + (-1+2t_4)x_1x_2 + (10+10t_5)x_2^2$ and $g_{d(t)}(x_1,x_2) = (-5+2t_6)x_1 + (1+t_7)x_2$.
Let $\tau = \frac{1}{(-5+2t_6)x_1+(1+t_7)x_2}$ and $y_i = \tau x_i\ i = 1,2$. For some $w : [0,1]^5 \to R_+$, the transformed problem $(IFP)_I$ is

$$(IFP)_I \ \min \int_5 \frac{w(t)}{\tau}\Big\{(-10+4t_1)\tau y_1 + (2+t_2)\tau y_2 + (4+6t_3)y_1^2 + (-1+2t_4)y_1y_2 + (10+10t_5)y_2^2\Big\}dt_1dt_2\ldots,dt_5$$

$$\text{subject to } -5y_1+y_2 \le 1, -3y_1+2y_2 \ge 1, (1+t_8)y_1+3y_2 \ge (1+9t_8)\tau,$$
$$(-2+10t_9)y_1+(4+2t_9)y_2 \ge (4+2t_9)\tau, y_1 \ge 0, y_2 \ge 0, \tau > 0.$$

In particular for $w(t_1,t_2,t_3,t_4,t_5) = t_1+t_3$, $(IFP)_I$ is simplified.

$$(IFP)_I \qquad \min \qquad \tau\Big\{-\frac{23}{3}\frac{y_1}{\tau} + \frac{5}{2}\frac{y_2}{\tau} + \frac{15}{2}\Big(\frac{y_1}{\tau}\Big)^2 + 15\Big(\frac{y_2}{\tau}\Big)^2\Big\}$$
$$\text{subject to} \quad \tau\Big(-5\frac{y_1}{\tau} + \frac{y_2}{\tau}\Big) \le 1, \tau\Big(-3\frac{y_1}{\tau} + 2\frac{y_2}{\tau}\Big) \ge 1,$$
$$\tau\Big((1+t_8)\frac{y_1}{\tau} + 3\frac{y_2}{\tau}\Big) \ge (1+9t_8)\tau,$$
$$\tau\Big((-2+10t_9)\frac{y_1}{\tau} + (4+2t_9)\frac{y_2}{\tau}\Big) \ge (4+2t_9)\tau,$$
$$y_1 \ge 0, y_2 \ge 0, \tau > 0.$$

Since the numerator of the objective function of (IFP) is an interval valued convex function and its denominator is an interval valued affine function, so by Theorem (3.5), $(IFP)_I$ is a convex optimization problem. The KKT conditions for $(IFP)_I$ are,

$$-\frac{23}{3} + 15\frac{y_1}{\tau} - 5\eta_1 + 3\eta_2 - (1+t_8)\mu_1 - (-2+10t_9)\mu_2 - \nu_1 = 0,$$
$$\frac{5}{2} + 30\frac{y_2}{\tau} + \eta_1 - 2\eta_2 + 3\mu_1 + (4+2t_9)\mu_2 - \nu_2 = 0,$$
$$-\frac{15}{2}(\frac{y_1}{\tau^2})^2 - 15(\frac{y_1}{\tau^2})^2 + \eta_1(-5\frac{y_1}{\tau} + \frac{y_2}{\tau}) - \eta_2(-3\frac{y_1}{\tau} + 2\frac{y_2}{\tau})$$
$$+\mu_1(1+9t_8) + \mu_2(4+2t_9) - \nu_3 = 0,$$
$$\eta_1(-5y_1+y_2-1) = 0, \eta_2(1+3y_1-2y_2) = 0,$$
$$\mu_1(\tau(1+9t_8) - (1+t_8)y_1 - 3y_2) = 0,$$
$$\mu_2(\tau(4+2t_9) - (-2+10t_9)y_1 + (4+2t_9)y_2) = 0,$$
$$\nu_1y_1 = 0, \nu_2y_2 = 0, \tau > 0.$$

This nonlinear system of equations has solution as $y_1^* = 0.2451372 \times 10^{-6}$, $y_2^* = 0.1537882 \times 10^{-7}$, $\tau^* = 0.3422284 \times 10^{-6}$. By Theorem (3.6), the efficient solution of (IFP) is $x_1^* = 0.71629707$, $x_2^* = 0.0449373$.
This methodology can be applied to large scale computational also. Since the problem in our example has less number of variables, so we have solved in LINGO

directly. However, in case of large number of variables it can solved in MATLAB, MATHEMATICA or other optimization softwares.

4. Conclusion

In this paper, nonlinear interval fractional programming problem is converted to a general nonlinear programming problem, which is free from interval uncertainty and partial order relations. This problem becomes a convex programming problem under some assumptions. We have proved that an optimal solution of the deterministic equivalent is an efficient solution of the original problem. This methodology may be used to solve generalized nonlinear fractional programming models with interval parameters.

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