

## SECOND ORDER NONSMOOTH MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEM INVOLVING SUPPORT FUNCTIONS

PALLAVI KHARBANDA\*, DIVYA AGARWAL AND DEEPA SINHA

**ABSTRACT.** In this paper, we have considered a class of constrained non-smooth multiobjective fractional programming problem involving support functions under generalized convexity. Also, second order Mond Weir type dual and Schaible type dual are discussed and various weak, strong and strict converse duality results are derived under generalized class of second order  $(F, \alpha, \rho, d)$ -V-type I functions. Also, we have illustrated through non-trivial examples that class of second order  $(F, \alpha, \rho, d)$ -V-type I functions extends the definitions of generalized convexity appeared in the literature.

AMS Mathematics Subject Classification : 90C30, 90C46, 90C25, 65K05.  
*Key words and phrases* : Nonsmooth programming,  $(F, \alpha, \rho, d)$ -V-type I functions, Support functions, Efficient solution.

### 1. Introduction

Investigation on sufficiency and duality results in the fractional optimization problems with multiple-objective functions has been one of the most attracting topics in the recent past. Schaible and Ibaraki [22] and Craven [2] have given many direct and indirect applications of fractional programming problems. In general, a fractional programming problem is non-convex. Therefore various generalizations of convexity notions have been proposed by many authors. Hanson and Mond [9] introduced  $F$ -convex functions which were generalized to  $(F, \rho)$  convex functions by Preda [21]. Liang et al. [15, 16] introduced a unified formulation of generalized convex functions, called  $(F, \alpha, \rho, d)$  convex functions and obtained sufficient optimality conditions and duality results of the single-objective and multiobjective fractional programming problem. Hachimi and Aghezzaf [6] gave the concept of  $(F, \alpha, \rho, d)$  type I functions which were further generalized to  $(F, \alpha, \rho, d)$ -V-type I functions by Gulati et al. [5].

Second order duality provides tighter bounds for the value of objective function of the primal problem when approximations are used because there are more parameters involved and therefore we apply second order duality to get a lower bound of the value of the primal when first order duality does not apply. Mangasarian [17] first formulated the second order dual for a nonlinear programming problem by introducing an additional vector  $p \in R^n$ . Instead of imposing explicit conditions on  $p$ , Mond [19] included  $p$  in a second order type convexity. Hanson [8] defined second order invexity for differentiable functions which were extended to second order pseudo type I, quasi type I by Mishra [18] and second order  $(F, \rho_i, \sigma_j)$ -type I functions by Srivastava and Govil [24]. Hachimi and Aghezzaf [7] introduced second order  $(F, \alpha, \rho, p, d)$  type I functions which were extended to second order  $(F, \alpha, \rho, d)$ -V-type I functions by Gulati and Agarwal [4]. Further, Husain et al. [12] discussed two types of second order dual models and derived various duality results for a class of nondifferentiable minimax programming problem under generalized convexity assumptions.

For nondifferentiable programs, Zhang and Mond [25] discussed duality results under generalized invexity. Ahmad et al. [1] obtained duality results under generalized second order  $(F, \alpha, \rho, d)$  convex functions for fractional programming problem involving positive semi-definite symmetric matrices. Jayswal et al. [13] obtained duality results for second order Mangasarian type and general Mond-Weir type duals assuming the objective and constraint functions to be second order  $(F, \alpha, \rho, d)$ -V-type I functions for a nondifferentiable multiobjective programming problem. Recently, sufficient optimality conditions and duality theorems are derived for three type of dual models related to multiobjective fractional programming problem involving  $(p, r) - \rho - (\eta, \theta)$  invex functions by Jayswal et al. [14].

In this paper, we have considered a multiobjective fractional programming problem in which support function appears in the numerator and denominator of the objective function and in each constraint. Also, the second order Mond-Weir type dual and Schaible type dual are formulated and various weak, strong and strict converse duality theorems under generalized class of second order  $(F, \alpha, \rho, d)$ -V-type I functions are established.

## 2. Preliminaries and Definitions

The following convention of vectors in  $R^n$  will be followed throughout this paper: For  $x, y \in R^n$ ,  $x \geq y \Leftrightarrow x_i \geq y_i$ ;  $x \geq y \Leftrightarrow x \geq y, x \neq y$ ;  $x > y \Leftrightarrow x_i > y_i, i = 1, 2, \dots, n$ . Let  $D$  be a non-empty subset of  $R^n$ .

Consider the multiobjective programming problem:

(MP) Minimize  $f(x)$  subject to  $h(x) \leq 0$ ,

where  $x \in D$ ,  $X = \{x \in D : h(x) \leq 0\}$  be the set of feasible solutions of (MP).

Also,  $f : D \rightarrow R^k$  and  $h : D \rightarrow R^m$  are second order differentiable functions.

**Definition 2.1** ([20]). Let  $C$  be a compact convex set in  $R^n$ . The support function of  $C$  at  $x \in R^n$  is  $S(x|C) = \max\{x^T y : y \in C\}$ .

The subdifferential of  $S(y|C)$  is given by  $\partial S(y|C) = \{z \in C : z^T y = S(y|C)\}$ . For any set  $A \subseteq R^n$ , the normal cone to  $A$  at any point  $x \in A$  is  $N_A(x) = \{y \in R^n : y^T(z - x) \leq 0, \forall z \in A\}$ . Also  $y \in N_C(x)$  iff  $S(y|C) = x^T y$ .

**Definition 2.2.** A functional  $F : D \times D \times R^n \rightarrow R$  is said to be sublinear in the third variable if for any  $x, u \in D \subseteq R^n$ ,

$$F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2), \quad \forall a_1, a_2 \in R^n,$$

$$F(x, u; \alpha a) = \alpha F(x, u; a), \quad \forall \alpha \in R, \alpha \geq 0, \forall a \in R^n.$$

Let the functions  $f = (f_1, \dots, f_k) : D \rightarrow R^k$  and  $h = (h_1, \dots, h_m) : D \rightarrow R^m$  are second order differentiable at  $u \in D$ . Also, let  $\alpha, \tilde{\alpha}$  are the vectors in  $R^{k+m}$  whose components are the functions  $\alpha_i^1, \alpha_j^2 : X \times D \rightarrow R_+ \setminus \{0\}$  and  $\tilde{\alpha}_i^1, \tilde{\alpha}_j^2 : X \times D \rightarrow R_+ \setminus \{0\}$  respectively for  $i = \overline{1, k}, j = \overline{1, m}$ , while  $\rho = (\rho_1^1, \dots, \rho_k^1, \rho_1^2, \dots, \rho_m^2) \in R^{k+m}$  and  $\tilde{\rho} \in R^2$  whose components are in  $R$  and function  $d(.,.) : X \times D \rightarrow R_+$  with the property that  $d(x, y) = 0$  iff  $x = y$ .

**Definition 2.3** ([4]).  $(f, h)$  is said to be second order  $(F, \alpha, \rho, d)$ -V-type I function at  $u \in D$  w.r.t.  $p, q \in R^n$  if for all  $x \in X$  and  $i = \overline{1, k}, j = \overline{1, m}$ :

$$f_i(x) - f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u) p \geq F(x, u; \alpha_i^1(x, u) (\nabla f_i(u) + \nabla^2 f_i(u) p)) + \rho_i^1 d^2(x, u)$$

$$-h_j(u) + \frac{1}{2} q^T \nabla^2 h_j(u) q \geq F(x, u; \alpha_j^2(x, u) (\nabla h_j(u) + \nabla^2 h_j(u) q)) + \rho_j^2 d^2(x, u)$$

If the inequalities in  $f_i$  are strict (whenever  $x \neq u$ ), then  $(f, h)$  is said to be second order semi-strictly  $(F, \alpha, \rho, d)$ -V-type I function at  $u$ .

**Remark 2.1.**

- (i) If we take  $\alpha_i^1(x, u) = \alpha^1(x, u)$ ;  $\alpha_j^2(x, u) = \alpha^2(x, u)$  for all  $i = \overline{1, k}; j = \overline{1, m}$ , the above definitions become that of second order  $(F, \alpha, \rho, p, d)$ -type I function introduced by Hachimi and Aghezzaf [7].
- (ii) If  $\alpha_i^1(x, u) = \alpha_j^2(x, u) = 1$  for all  $i = \overline{1, k}; j = \overline{1, m}$ , we get definition of second order  $(F, \rho_i, \sigma_j)$ -type I by Srivastava and Govil [24] and second order  $(F, \rho_i)$  convex and second order  $(F, \sigma_j)$  convex functions by Srivastava and Bhatia [23].
- (iii) If in the above definition, we take  $p, q = 0$ , then the above definition reduce to that of  $(F, \alpha, \rho, d)$ -V-type I function given by Gulati et al. [5].
- (iv) If  $\alpha_i^1(x, u) = \alpha^1(x, u)$ ;  $\alpha_j^2(x, u) = \alpha^2(x, u)$  for all  $i = \overline{1, k}; j = \overline{1, m}$ , and  $p, q = 0$ , then we get the definition of  $(F, \alpha, \rho, d)$ -type I function introduced by Hachimi and Aghezzaf [6].
- (v) If sublinear functional is defined as  $F(x, u; a) = \eta(x, u)^T a$  where  $a \in R^n$  and  $\eta(x, u) : X \times D \rightarrow R^n$  is a vector function and  $\rho_i^1 = \rho_j^2 = 0$  for all  $i = \overline{1, k}; j = \overline{1, m}$  and  $p, q = 0$ , then above definition reduces to V-type I function (Hanson et al. [10]).

**Definition 2.4** ([4]).  $(f, h)$  is said to be second order quasi  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u \in D$  w.r.t.  $p, q \in R^n$  if for all  $x \in X$  and  $i = \overline{1, k}, j = \overline{1, m}$  :

$$\begin{aligned} \sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \left( f_i(x) - f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u) p \right) &\leq 0 \\ \Rightarrow F \left( x, u; \sum_{i=1}^k (\nabla f_i(u) + \nabla^2 f_i(u) p) \right) &\leq -\tilde{\rho}^1 d^2(x, u) \\ \sum_{j=1}^m \tilde{\alpha}_j^2(x, u) \left( -h_j(u) + \frac{1}{2} q^T \nabla^2 h_j(u) q \right) &\leq 0 \\ \Rightarrow F \left( x, u; \sum_{j=1}^m (\nabla h_j(u) + \nabla^2 h_j(u) q) \right) &\leq -\tilde{\rho}^2 d^2(x, u) \end{aligned}$$

If the second (implied) inequality in  $f$  is strict (whenever  $x \neq u$ ), then  $(f, h)$  is said to be second order semi-strictly quasi  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$ .

**Definition 2.5** ([4]).  $(f, h)$  is said to be second order pseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u \in D$  w.r.t.  $p, q \in R^n$  if for all  $x \in X$  and  $i = \overline{1, k}, j = \overline{1, m}$  :

$$\begin{aligned} F \left( x, u; \sum_{i=1}^k (\nabla f_i(u) + \nabla^2 f_i(u) p) \right) &\geq -\tilde{\rho}^1 d^2(x, u) \\ \Rightarrow \sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \left( f_i(x) - f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u) p \right) &\geq 0 \\ F \left( x, u; \sum_{j=1}^m (\nabla h_j(u) + \nabla^2 h_j(u) q) \right) &\geq -\tilde{\rho}^2 d^2(x, u) \\ \Rightarrow \sum_{j=1}^m \tilde{\alpha}_j^2(x, u) \left( -h_j(u) + \frac{1}{2} q^T \nabla^2 h_j(u) q \right) &\geq 0 \end{aligned}$$

If the second (implied) inequality in  $f$  (resp.  $h$ ) is strict (whenever  $x \neq u$ ), then  $(f, h)$  is said to be second order semi-strictly pseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function in  $f$  (resp.  $h$ ) and if the second (implied) inequality in both  $f$  and  $h$  are strict (whenever  $x \neq u$ ), then  $(f, h)$  is said to be second order strictly pseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$ .

**Definition 2.6** ([4]).  $(f, h)$  is said to be second order quasipseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u \in D$  w.r.t.  $p, q \in R^n$  if for all  $x \in X$  and  $i = \overline{1, k}, j = \overline{1, m}$  :

$$\begin{aligned} \sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \left( f_i(x) - f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u) p \right) &\leq 0 \\ \Rightarrow F \left( x, u; \sum_{i=1}^k (\nabla f_i(u) + \nabla^2 f_i(u) p) \right) &\leq -\tilde{\rho}^1 d^2(x, u) \\ F \left( x, u; \sum_{j=1}^m (\nabla h_j(u) + \nabla^2 h_j(u) q) \right) &\geq -\tilde{\rho}^2 d^2(x, u) \\ \Rightarrow \sum_{j=1}^m \tilde{\alpha}_j^2(x, u) \left( -h_j(u) + \frac{1}{2} q^T \nabla^2 h_j(u) q \right) &\geq 0 \end{aligned}$$

If the second (implied) inequality in  $h$  is strict (whenever  $x \neq u$ ), then  $(f, h)$  is said to be second order quasi strictly pseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$ .

**Definition 2.7** ([4]).  $(f, h)$  is said to be second order pseudoquasi  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u \in D$  w.r.t.  $p, q \in R^n$  if for all  $x \in X$  and  $i = \overline{1, k}, j = \overline{1, m}$ :

$$\begin{aligned} & F\left(x, u; \sum_{i=1}^k (\nabla f_i(u) + \nabla^2 f_i(u)p)\right) \geq -\tilde{\rho}^1 d^2(x, u) \\ \Rightarrow & \sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \left( f_i(x) - f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u)p \right) \geq 0 \\ & \sum_{j=1}^m \tilde{\alpha}_j^2(x, u) \left( -h_j(u) + \frac{1}{2} q^T \nabla^2 h_j(u)q \right) \leq 0 \\ \Rightarrow & F\left(x, u; \sum_{j=1}^m (\nabla h_j(u) + \nabla^2 h_j(u)q)\right) \leq -\tilde{\rho}^2 d^2(x, u) \end{aligned}$$

If the second (implied) inequality in  $f$  is strict (whenever  $x \neq u$ ), then  $(f, h)$  is said to be second order strictly pseudoquasi  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$ .

Consider the following nondifferentiable multiobjective fractional programming problem involving support functions.

(FP) Minimize  $\left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right)$   
subject to  $h_j(x) + S(x|E_j) \leq 0$ ,  $j = \overline{1, m}$ , where  $x \in D \subseteq R^n$ ,  $X = \{x \in D : h_j(x) + S(x|E_j) \leq 0\}$  be the set of feasible solutions of (FP) and for  $i = \overline{1, k}, j = \overline{1, m}$ ,  $f_i, g_i, h_j : D \rightarrow R$  are second order differentiable functions.  $f_i(\cdot) + S(\cdot|C_i) \geq 0, g_i(\cdot) - S(\cdot|D_i) > 0; C_i, D_i, E_j$  are compact convex sets in  $R^n$  and  $S(x|C_i), S(x|D_i), S(x|E_j)$  define their respective support functions.

### 3. Illustration

In this section, we illustrate through examples that the class of second order  $(F, \alpha, \rho, d)$ -V-type I functions contains many earlier studied classes as special cases.

**Example 3.1.** For (MP), let  $D = R$ ,  $f = (f_1, f_2) : D \rightarrow R^2$ ,  $h : D \rightarrow R$  such that  $f_1(x) = -x^2 - 1$ ,  $f_2(x) = 2x^2 + x + 1$ ,  $h(x) = 1 - x$

So, the feasible region is  $X = \{x \in D : x \geq 1\}$ .

Let  $F(x, u; a) = \frac{a}{9}(x^2 + u^2 - 1)$ ;  $\alpha_1^1(x, u) = 9$ ;  $\alpha_2^1(x, u) = 2$ ;  $\alpha^2(x, u) = 18$ ;  
 $\rho_1^1 = 3$ ;  $\rho_2^1 = 0$ ;  $\rho^2 = 2$ ;  $d(x, u) = |x - u|$ ;  $u = 1$ ;  $p = 1$ ;  $q = 2$ .

It is easy to see that for all  $x \in X$ ,

$$f_1(x) - f_1(u) + \frac{1}{2} p^T \nabla^2 f_1(u)p \geq F(x, u; \alpha_1^1(x, u)(\nabla f_1(u) + \nabla^2 f_1(u)p)) + \rho_1^1 d^2(x, u), \quad (3.1)$$

$$f_2(x) - f_2(u) + \frac{1}{2} p^T \nabla^2 f_2(u)p \geq F(x, u; \alpha_2^1(x, u)(\nabla f_2(u) + \nabla^2 f_2(u)p)) + \rho_2^1 d^2(x, u), \quad (3.2)$$

and  $-h(u) + \frac{1}{2} q^T \nabla^2 h(u)q \geq F(x, u; \alpha^2(x, u)(\nabla h(u) + \nabla^2 h(u)q)) + \rho^2 d^2(x, u)$   
which shows that  $(f, h)$  is second order  $(F, \alpha, \rho, d)$ -V-type I function at  $u = 1$ .

But, for the above defined problem, if we take

(a) (i)  $\alpha_1^1(x, u) = \alpha_2^1(x, u) = 2$ , then for all  $x \in X$ ,  
 $f_1(x) - f_1(u) + \frac{1}{2}p^T \nabla^2 f_1(u)p < F(x, u; \alpha_1^1(x, u)(\nabla f_1(u) + \nabla^2 f_1(u)p)) + \rho_1^1 d^2(x, u)$   
(ii)  $\alpha_1^1(x, u) = \alpha_2^1(x, u) = 9$ , then for all  $x \in X$ ,  
 $f_2(x) - f_2(u) + \frac{1}{2}p^T \nabla^2 f_1(u)p < F(x, u; \alpha_2^1(x, u)(\nabla f_2(u) + \nabla^2 f_2(u)p)) + \rho_2^1 d^2(x, u)$   
In fact, the inequality (3.1) is satisfied for all  $\alpha_1^1(x, u) \geq 9$  and the inequality (3.2) is satisfied for all  $\alpha_2^1(x, u) \leq 2$ . Therefore, the inequalities (3.1) and (3.2) can not be satisfied simultaneously for any value of  $\alpha^1(x, u)$  such that  $\alpha^1(x, u) = \alpha_1^1(x, u) = \alpha_2^1(x, u)$  and hence  $(f, h)$  is not second order  $(F, \alpha, \rho, p, d)$ -type I function at  $u \in D$  as introduced by Hachimi and Aghezzaf [7].  
(b) if  $\alpha_1^1(x, u) = \alpha_2^1(x, u) = 1$ , then for all  $x \in X$ ,  
 $f_1(x) - f_1(u) + \frac{1}{2}p^T \nabla^2 f_1(u)p < F(x, u; \alpha_1^1(x, u)(\nabla f_1(u) + \nabla^2 f_1(u)p)) + \rho_1^1 d^2(x, u)$   
which shows that  $(f, h)$  is not second order  $(F, \rho_i, \sigma_j)$ -type I function at  $u \in D$  as introduced by Srivastava and Govil. [24].  
(c) if  $p = 0$ , then at  $x = 3$   
 $f_1(x) - f_1(u) < F(x, u; \alpha_1^1(x, u)\nabla f_1(u)) + \rho_1^1 d^2(x, u)$   
which shows that  $(f, h)$  is not  $(F, \alpha, \rho, d)$ -V-type I function at  $u \in D$  as introduced by Gulati et al. [5].

Therefore the above example clearly illustrates that the class of second order  $(F, \alpha, \rho, d)$ -V-type I functions is more generalized than the class of second order  $(F, \alpha, \rho, p, d)$ -type I functions, second order  $(F, \rho_i, \sigma_j)$ -type I functions and  $(F, \alpha, \rho, d)$ -V-type I functions.

**Example 3.2.** For (MP), let  $D = R$ ,  $f = (f_1, f_2) : D \rightarrow R^2$ ,  $h = (h_1, h_2) : D \rightarrow R^2$  such that

$$f_1(x) = -8x^2 + 8x, \quad f_2(x) = -24x^4 + 8x^2 + 16, \quad h_1(x) = x, \quad h_2(x) = x - 1$$

So, the feasible region is  $X = \{x \in D : x \leq 0\}$ .

Let  $F(x, u; a) = |a|(x^2 + u^2)$ ;  $\tilde{\alpha}_1^1(x, u) = \frac{1}{2}$ ;  $\tilde{\alpha}_2^1(x, u) = \frac{1}{8}$ ;  $\tilde{\alpha}_1^2(x, u) = \frac{1}{6}$   
 $\tilde{\alpha}_2^2(x, u) = \frac{1}{4}$ ;  $\alpha_1^1(x, u) = 2$ ;  $\alpha_2^1(x, u) = 8$ ;  $\alpha_1^2(x, u) = 6$ ;  $\alpha_2^2(x, u) = 4$ ;  $d(x, u) = |x - u|$ ;  $u = 0$ ;  $p = 1$ ;  $\rho_1^1 = -16$ ;  $\rho_2^1 = 0$ ;  $\rho_1^2 = 6$ ;  $\rho_2^2 = 8$ ;  $\tilde{\rho}^1 = -8$ ;  $\tilde{\rho}^2 = 3$ . It is easy to see that for all  $x \in X$ ,

$$\sum_{i=1}^2 \tilde{\alpha}_i^1(x, u) \left( f_i(x) - f_i(u) + \frac{1}{2}p^T \nabla^2 f_i(u)p \right) = -3(x^4 + x^2 + p^2) + 4x \leq 0$$

$$\Rightarrow F \left( x, u; \sum_{i=1}^2 (\nabla f_i(u) + \nabla^2 f_i(u)p) \right) + \tilde{\rho}^1 d^2(x, u) = 8x^2 - 8x^2 \leq 0$$

and

$$F \left( x, u; \sum_{j=1}^2 (\nabla h_j(u) + \nabla^2 h_j(u)q) \right) + \tilde{\rho}^2 d^2(x, u) = 5x^2 \geq 0$$

$$\Rightarrow \sum_{j=1}^2 \tilde{\alpha}_j^2(x, u) (-h_j(u) + \frac{1}{2}q^T \nabla^2 h_j(u)q) = \frac{1}{4} \geq 0$$

which shows that  $(f, h)$  is second order quasi-pseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function for all  $x \in X$  at  $u$ .

However, for the above defined problem, if we take  $x = -1$ , then

$$\begin{aligned} f_1(x) - f_1(u) + \frac{1}{2}p^T \nabla^2 f_1(u)p &= -8x^2 + 8x - 8p^2 \\ &< F(x, u; \alpha_1^1(x, u)(\nabla f_1(u) + \nabla^2 f_1(u)p)) + \rho_1^1 d^2(x, u) \\ &= 2|8 - 16p|x^2 - 16x^2 \end{aligned}$$

which shows that  $(f, h)$  is not second order  $(F, \alpha, \rho, d)$ -V-type I function for all  $x \in X$  at  $u$ .

**Example 3.3.** For (MP), let  $D = R$ ,  $f = (f_1, f_2) : D \rightarrow R^2$ ,  $h = (h_1, h_2) : D \rightarrow R^2$  such that

$$f_1(x) = \frac{x^4}{4} + 2x^2, \quad f_2(x) = x^6 + \frac{x^4}{2} + x^2 - 2, \quad h_1(x) = x, \quad h_2(x) = x^3$$

The feasible region is  $X = \{x \in D : x \leq 0\}$ .

Let  $F(x, u; a) = |a|(x^2 + 4u^2)$ ;  $\tilde{\alpha}_1^1(x, u) = 4$ ;  $\tilde{\alpha}_2^1(x, u) = 2$ ;  $\tilde{\alpha}_1^2(x, u) = 1$ ;  $\tilde{\alpha}_2^2(x, u) = \frac{1}{2}$ ;  $\alpha_1^1(x, u) = \frac{1}{4}$ ;  $\alpha_2^1(x, u) = \frac{1}{2}$ ;  $\alpha_1^2(x, u) = 1$ ;  $\alpha_2^2(x, u) = 2$ ;  $d(x, u) = |x - u|$ ;  $u = 0$ ;  $p = 1$ ;  $\rho_1^1 = 4$ ;  $\rho_2^1 = 4$ ;  $\rho_1^2 = 1$ ;  $\rho_2^2 = -10$ ;  $\tilde{\rho}^1 = 24$ ;  $\tilde{\rho}^2 = -4$ . It is easy to see that for all  $x \in X$ ,

$$\begin{aligned} &F\left(x, u; \sum_{i=1}^2 (\nabla f_i(u) + \nabla^2 f_i(u)p)\right) + \tilde{\rho}^1 d^2(x, u) = |6p| x^2 + 24x^2 \geq 0 \\ \Rightarrow &\sum_{i=1}^2 \tilde{\alpha}_i^1(x, u)(f_i(x) - f_i(u) + \frac{1}{2}p^T \nabla^2 f_i(u)p) = 2x^6 + 2x^4 + 10x^2 + 10p^2 \geq 0 \\ &\sum_{j=1}^2 \tilde{\alpha}_j^2(x, u)(-h_j(u) + \frac{1}{2}q^T \nabla^2 h_j(u)q) = 0 \leq 0 \\ \Rightarrow &F\left(x, u; \sum_{j=1}^2 (\nabla h_j(u) + \nabla^2 h_j(u)q)\right) + \tilde{\rho}^2 d^2(x, u) = -3x^2 \leq 0 \end{aligned}$$

which shows that  $(f, h)$  is second order pseudo-quasi  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function for all  $x \in X$  at  $u$ .

However, for the above defined problem, if we take  $x = -1$ , then

$$\begin{aligned} f_1(x) - f_1(u) + \frac{1}{2}p^T \nabla^2 f_1(u)p &= \frac{x^4}{4} + 2x^2 + 2p^2 \\ &< F(x, u; \alpha_1^1(x, u)(\nabla f_1(u) + \nabla^2 f_1(u)p)) + \rho_1^1 d^2(x, u) \\ &= |p|x^2 + 4x^2 \end{aligned}$$

which shows that  $(f, h)$  is not second order  $(F, \alpha, \rho, d)$ -V-type I function for all  $x \in X$  at  $u$ .

Therefore the above examples clearly illustrate that the class of second order  $(F, \alpha, \rho, d)$ -V-type I functions is more generalized than the cited classes in literature.

#### 4. Second Order Mond-Weir Type Dual

In this section, we establish weak, strong and strict converse duality theorems for second order Mond-Weir type dual of (FP) under generalized class of second order  $(F, \alpha, \rho, d)$ -V-type I functions.  
(MFD) Maximize

$$\left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} \right) p, \dots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right) p \right)$$

subject to

$$\begin{aligned} & \nabla \left( \sum_{i=1}^k \lambda_i \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^m y_j (h_j(u) + u^T w_j) \right) \\ & + \nabla^2 \left( \sum_{i=1}^k \lambda_i \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p + \sum_{j=1}^m y_j (h_j(u) + u^T w_j) q \right) = 0, \end{aligned} \quad (4.1)$$

$$\sum_{j=1}^m y_j \left( (h_j(u) + u^T w_j) - \frac{1}{2} q^T \nabla^2 (h_j(u) + u^T w_j) q \right) \geq 0, \quad (4.2)$$

$$z_i \in C_i, \quad v_i \in D_i, \quad i = 1, 2, \dots, k, \quad \lambda \geq 0, \quad w_j \in E_j, \quad y_j \geq 0, \quad j = 1, 2, \dots, m.$$

**Theorem 4.1** (Weak Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively with  $\lambda_i > 0$ ,  $i = 1, 2, \dots, k$ . If*

$$(i) \quad \left( \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right) \text{ is second order}$$

$(F, \alpha, \rho, d)$ -V-type I function at  $u$  for  $i = 1, 2, \dots, k$ ,

$$(ii) \quad \sum_{i=1}^k \frac{\lambda_i \rho_i^1}{\alpha_i^1(x, u)} + \frac{\rho^2}{\alpha^2(x, u)} \geq 0$$

then the following cannot hold  $\left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right) \leq \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} \right) p, \dots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right) p \right)$ .

*Proof.* Suppose the contradiction holds. Since  $\lambda_i > 0$ ,  $x^T z_i \leq S(x|C_i)$ ,  $x^T v_i \leq S(x|D_i)$ ,  $\alpha_i^1(x, u) > 0$ ,  $i = 1, 2, \dots, k$ , we have

$$\sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \right) < 0 \quad (4.3)$$

Since hypothesis (i) holds, therefore for  $i = 1, 2, \dots, k$ , we have

$$\begin{aligned} & \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \\ & \geq F \left( x, u; \alpha_i^1(x, u) \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p \right) \right) \\ & \quad - \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p + \rho_i^1 d^2(x, u), \end{aligned} \quad (4.4)$$

$$\begin{aligned}
& - \sum_{j=1}^m y_j(h_j(u) + u^T w_j) \\
& \geq F\left(x, u; \alpha^2(x, u) \left( \nabla \sum_{j=1}^m y_j(h_j(u) + u^T w_j) + \nabla^2 \sum_{j=1}^m y_j(h_j(u) + u^T w_j) q \right) \right) \quad (4.5) \\
& - \frac{1}{2} \sum_{j=1}^m q^T \nabla^2 y_j(h_j(u) + u^T w_j) q + \rho^2 d^2(x, u)
\end{aligned}$$

Multiplying the inequality (4.4) by  $\frac{\lambda_i}{\alpha_i^1(x, u)}$  and taking summation for  $i = 1, 2, \dots, k$ , we get

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\
& \geq \sum_{i=1}^k \lambda_i F\left(x, u; \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p \right) \right) \quad (4.6) \\
& - \frac{1}{2} \sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p + \sum_{i=1}^k \frac{\lambda_i \rho_i^1}{\alpha_i^1(x, u)} d^2(x, u)
\end{aligned}$$

On adding the inequalities (4.5) and (4.6) and using sublinearity of  $F$  along with equation (4.1), the inequality (4.2) and the hypothesis (ii), we obtain

$$\sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \right) \geq 0$$

which is a contradiction to (4.3). Hence the proof.  $\square$

**Theorem 4.2** (Weak Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively. If*

$$(i) \left( \lambda_i \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j(h_j(\cdot) + (\cdot)^T w_j) \right) \text{ is second order pseudoquasi}$$

$(F, \tilde{\alpha}, \tilde{\rho}, d)$ - $V$ -type I function at  $u$  for  $i = 1, 2, \dots, k$ ,

$$(ii) \tilde{\rho}^1 + \tilde{\rho}^2 \geq 0,$$

then for  $i = 1, 2, \dots, k$ , the following cannot hold

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} < \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p.$$

*Proof.* Suppose the contradiction holds.

Since  $x^T z_i \leq S(x|C_i)$ ,  $x^T v_i \leq S(x|D_i)$ ,  $\tilde{\alpha}_i^1(x, u) > 0$ ,  $i = 1, 2, \dots, k$ ,  $\lambda \geq 0$ , therefore

$$\sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \lambda_i \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \right) < 0$$

Also, by feasibility of  $(u, z, v, w, y, \lambda, p, q)$  and  $\tilde{\alpha}^2(x, u) > 0$  imply

$$\tilde{\alpha}^2(x, u) \sum_{j=1}^m y_j \left( -(h_j(u) + u^T w_j) + \frac{1}{2} q^T \nabla^2 (h_j(u) + u^T w_j) q \right) \leq 0,$$

Therefore by using hypothesis (i), we have

$$F \left( x, u; \sum_{i=1}^k \lambda_i \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p \right) \right) < -\tilde{\rho}^1 d^2(x, u)$$

$$F \left( x, u; \sum_{j=1}^m y_j (\nabla (h_j(u) + u^T w_j) + \nabla^2 (h_j(u) + u^T w_j) q) \right) \leq -\tilde{\rho}^2 d^2(x, u)$$

Using hypothesis (ii) and sublinearity of  $F$ , the above inequalities reduce to

$$\begin{aligned} & F(x, u; \sum_{i=1}^k \lambda_i (\nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p) \\ & + \sum_{j=1}^m y_j (\nabla (h_j(u) + u^T w_j) + \nabla^2 (h_j(u) + u^T w_j) q)) < -(\tilde{\rho}^1 + \tilde{\rho}^2) d^2(x, u) \leq 0 \end{aligned}$$

which contradicts  $F(x, u, 0) = 0$ . Hence the proof.  $\square$

The proof of the following theorems run on the same lines as the proof of the above theorem.

**Theorem 4.3** (Weak Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively with  $\lambda_i > 0, i = 1, 2, \dots, k$ . If*

$$(i) \left( \lambda_i \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right) \text{ is second order pseudoquasi}$$

$(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$  for  $i = 1, 2, \dots, k$ ,

$$(ii) \tilde{\rho}^1 + \tilde{\rho}^2 \geq 0$$

then the following cannot hold

$$\begin{aligned} & \left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right) \\ & \leq \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} \right) p, \dots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right) p \right). \end{aligned}$$

**Theorem 4.4** (Weak Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively. If*

$$(i) \left( \lambda_i \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right) \text{ is second order strictly pseudo}$$

$(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$  for  $i = 1, 2, \dots, k$ ,

$$(ii) \tilde{\rho}^1 + \tilde{\rho}^2 \geq 0$$

then the following cannot hold

$$\left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right) \\ \leq \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} \right) p, \dots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right) p \right).$$

**Theorem 4.5** (Weak Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively. If*

$$(i) \left( \lambda_i \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right) \text{ is second order semi-strictly quasi}$$

$(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$  for  $i = 1, 2, \dots, k$ ,

$$(ii) \tilde{\rho}^1 + \tilde{\rho}^2 \geq 0$$

then the following cannot hold

$$\left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right) \\ \leq \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} \right) p, \dots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right) p \right).$$

Strong duality theorem for the given model can be established on the lines of [3,11].

**Theorem 4.6** (Strong Duality). *If  $u$  is an efficient solution of (FP) and constraint qualification [3, 11] is satisfied, then there exists  $\bar{\lambda} \in R^k, \bar{y} \in R^m, \bar{z}_i, \bar{v}_i, \bar{w}_j \in R^n, i = 1, 2, \dots, k; j = 1, 2, \dots, m$ , such that  $(u, \bar{z}, \bar{v}, \bar{w}, \bar{\lambda}, \bar{y}, p = 0, q = 0)$  is a feasible solution of (MFD) and the corresponding values of the objective functions are equal. Further if the conditions of weak duality theorem 4.1 are satisfied for each feasible solution of (FP) and (MFD), then  $(u, \bar{z}, \bar{v}, \bar{w}, \bar{\lambda}, \bar{y}, p = 0, q = 0)$  is an efficient solution of (MFD).*

**Theorem 4.7** (Strict Converse Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively. If*

$$(i) \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p, \quad i = 1, 2, \dots, k,$$

$$(ii) \left( \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right) \text{ is second order semi-strictly}$$

$(F, \alpha, \rho, d)$ -V-type I function at  $u$  for  $i = 1, 2, \dots, k$ ,

$$(iii) \sum_{i=1}^k \frac{\lambda_i \rho_i^1}{\alpha_i^1(x, u)} + \frac{\rho^2}{\alpha^2(x, u)} \geq 0$$

then  $x = u$ .

*Proof.* Suppose  $x \neq u$ . Since hypothesis (ii) holds, therefore for  $i = 1, 2, \dots, k$ , we have

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} >_F \left( x, u; \alpha_i^1(x, u) \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p \right) \right) \\ - \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p + \rho_i^1 d^2(x, u) \quad (4.7)$$

and

$$\begin{aligned}
& - \sum_{j=1}^m y_j(h_j(u) + u^T w_j) \\
& \geq F\left(x, u; \alpha^2(x, u) \left( \nabla \sum_{j=1}^m y_j(h_j(u) + u^T w_j) + \nabla^2 \sum_{j=1}^m y_j(h_j(u) + u^T w_j) q \right) \right) \quad (4.8) \\
& - \frac{1}{2} q^T \nabla^2 \sum_{j=1}^m y_j(h_j(u) + u^T w_j) q + \rho^2 d^2(x, u)
\end{aligned}$$

Multiplying the inequality (4.7) by  $\frac{\lambda_i}{\alpha_i^1(x, u)}$  for  $i = 1, 2, \dots, k$ , adding to the inequality (4.8) and using sublinearity of  $F$ , we obtain

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) - \sum_{j=1}^m \frac{y_j}{\alpha^2(x, u)} (h_j(u) + u^T w_j) \\
& > F(x, u; \sum_{i=1}^k \lambda_i \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p + \sum_{j=1}^m y_j (\nabla(h_j(u) + u^T w_j) \\
& + \nabla^2(h_j(u) + u^T w_j) q)) - \frac{1}{2} \sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \\
& - \frac{1}{2} q^T \sum_{j=1}^m \frac{y_j}{\alpha^2(x, u)} \nabla^2(h_j(u) + u^T w_j) q + \left( \sum_{i=1}^k \frac{\lambda_i \rho_i^1}{\alpha_i^1(x, u)} + \frac{\rho^2}{\alpha^2(x, u)} \right) d^2(x, u)
\end{aligned}$$

Using equation (4.1), the inequality (4.2) and hypothesis (iii) and the fact that  $F(x, u, 0) = 0$ , the above equation reduces to

$$\sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \right) > 0$$

But, as  $x^T z_i \leq S(x|C_i)$ ,  $x^T v_i \leq S(x|D_i)$ ,  $\tilde{\alpha}_i^1(x, u) > 0$ ,  $i = 1, 2, \dots, k$ ,  $\lambda \geq 0$ , therefore hypothesis (i) yields

$$\sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} \left( \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \right) \leq 0$$

Hence we arrive at a contradiction. Thus  $x = u$ .  $\square$

**Theorem 4.8** (Strict Converse Duality). *Let  $x$  and  $(u, z, v, w, y, \lambda, p, q)$  be the feasible solutions of (FP) and (MFD) respectively. If*

- (i)  $\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p$ ,  $i = 1, 2, \dots, k$ ,
- (ii)  $\left( \lambda_i \frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right)$  is second order quasi strictly pseudo  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u$  for  $i = 1, 2, \dots, k$ ,
- (iii)  $\tilde{\rho}^1 + \tilde{\rho}^2 \geq 0$ , then  $x = u$ .

*Proof.* Suppose  $x \neq u$ . Since  $x^T z_i \leq S(x|C_i)$ ,  $x^T v_i \leq S(x|D_i)$ ,  $\tilde{\alpha}_i^1(x, u) > 0$ ,  $i = 1, 2, \dots, k$ ,  $\lambda \geq 0$  and hypothesis (i) holds, therefore

$$\sum_{i=1}^k \lambda_i \tilde{\alpha}_i^1(x, u) \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \sum_{i=1}^k \lambda_i \tilde{\alpha}_i^1(x, u) \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - \frac{1}{2} p^T \nabla^2 \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) p \right)$$

As  $\tilde{\alpha}^2(x, u) > 0$  and  $u$  is a feasible solution of (MFD), therefore

$$\tilde{\alpha}^2(x, u) \sum_{j=1}^m y_j \left( -(h_j(u) + u^T w_j) + \frac{1}{2} q^T \nabla^2 (h_j(u) + u^T w_j) q \right) \leq 0,$$

Using hypothesis (ii), the above inequalities reduce to

$$F \left( x, u; \sum_{i=1}^k \lambda_i \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p \right) \right) \leq -\tilde{\rho}^1 d^2(x, u)$$

$$F \left( x, u; \sum_{j=1}^m y_j \left( \nabla (h_j(u) + u^T w_j) + \nabla^2 (h_j(u) + u^T w_j) q \right) \right) < -\tilde{\rho}^2 d^2(x, u)$$

Using hypothesis (iii) and sublinearity of  $F$ , the above inequalities reduce to

$$F(x, u; \sum_{i=1}^k \lambda_i \left( \nabla \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \nabla^2 \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} p \right) + \sum_{j=1}^m y_j (\nabla (h_j(u) + u^T w_j) + \nabla^2 (h_j(u) + u^T w_j) q)) < -(\tilde{\rho}^1 + \tilde{\rho}^2) d^2(x, u) \leq 0$$

which contradicts  $F(x, u, 0) = 0$ . Hence  $x = u$ .  $\square$

## 5. Second Order Schaible Type Dual

In this section, we formulate Schaible Type Dual of (FP) and derive weak, strong and strict converse duality theorems.

(SFD) Maximize  $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_k)$   
subject to

$$\begin{aligned} & \nabla \left( \sum_{i=1}^k \lambda_i (f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i)) + \sum_{j=1}^m y_j (h_j(u) + u^T w_j) \right) \\ & + \sum_{i=1}^k \lambda_i \nabla^2 (f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i)) p + \sum_{j=1}^m y_j \nabla^2 (h_j(u) + u^T w_j) q = 0, \end{aligned} \quad (5.1)$$

$$\begin{aligned} & \lambda_i (f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i)) - \frac{1}{2} p^T \nabla^2 (f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i)) p \\ & \geq 0, \quad i = \overline{1, k} \end{aligned} \quad (5.2)$$

$$\sum_{j=1}^m y_j \left( (h_j(u) + u^T w_j) - \frac{1}{2} q^T \nabla^2 (h_j(u) + u^T w_j) q \right) \geq 0, \quad (5.3)$$

$$z_i \in C_i, \quad v_i \in D_i, \quad \tilde{\beta}_i \geq 0, \quad i = 1, 2, \dots, k, \quad \lambda \geq 0,$$

$$w_j \in E_j, \quad y_j \geq 0, \quad j = 1, 2, \dots, m.$$

**Theorem 5.1** (Weak Duality). *Let  $x$  and  $(u, z, v, w, \tilde{\beta}, y, \lambda, p, q)$  be the feasible solutions of (FP) and (SFD) respectively with  $\lambda_i > 0$ ,  $i = 1, 2, \dots, k$ . If*

$$(i) (f_i(\cdot) + (\cdot)^T z_i, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j)) \text{ and } (-g_i(\cdot) + (\cdot)^T v_i, \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j))$$

*are second order  $(F, \alpha, \rho, d)$ -V-type I functions at  $u$  for  $i = 1, 2, \dots, k$ ,*

$$(ii) \sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \lambda_i \bar{\rho}_i^1 + \tilde{\alpha}^2(x, u) \rho^2 \geq 0 \text{ where } \bar{\rho}_i^1 = \rho_i^1 (1 + \tilde{\beta}_i).$$

*then the following cannot hold*

$$\left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right) \leq (\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_k).$$

*Proof.* Suppose the contradiction holds. Since  $\lambda_i > 0$ ,  $x^T z_i \leq S(x|C_i)$ ,  $x^T v_i \leq S(x|D_i)$ ,  $\tilde{\alpha}_i^1(x, u) > 0$ ,  $i = 1, 2, \dots, k$ , therefore

$$\sum_{i=1}^k \tilde{\alpha}_i^1(x, u) \lambda_i \left( f_i(x) + x^T z_i - \tilde{\beta}_i (g_i(x) - x^T v_i) \right) < 0 \quad (5.4)$$

By hypothesis (i), for  $i = 1, 2, \dots, k$ , we have

$$(f_i(x) + x^T z_i) - (f_i(u) + u^T z_i) \geq F(x, u; \alpha_i^1(x, u) (\nabla(f_i(u) + u^T z_i) + \nabla^2(f_i(u) + u^T z_i)p)) - \frac{1}{2} p^T \nabla^2(f_i(u) + u^T z_i)p + \rho_i^1 d^2(x, u), \quad (5.5)$$

$$\begin{aligned} & - (g_i(x) - x^T v_i) + (g_i(u) - u^T v_i) \\ & \geq F(x, u; \alpha_i^1(x, u) (-\nabla(g_i(u) - u^T v_i) - \nabla^2(g_i(u) - u^T v_i)p)) \\ & \quad + \frac{1}{2} p^T \nabla^2(g_i(u) - u^T v_i)p + \rho_i^1 d^2(x, u), \end{aligned} \quad (5.6)$$

$$\begin{aligned} & - \sum_{j=1}^m y_j (h_j(u) + u^T w_j) \\ & \geq F \left( x, u; \alpha^2(x, u) \left( \sum_{j=1}^m y_j \nabla(h_j(u) + u^T w_j) + \sum_{j=1}^m y_j \nabla^2(h_j(u) + u^T w_j)q \right) \right) \\ & \quad - \frac{1}{2} \sum_{j=1}^m y_j q^T \nabla^2(h_j(u) + u^T w_j)q + \rho^2 d^2(x, u) \end{aligned} \quad (5.7)$$

Multiplying (5.6) by  $\tilde{\beta}_i$ ,  $i = \overline{1, k}$ , adding in (5.5) and using sublinearity of  $F$ , we get

$$\begin{aligned} & (f_i(x) + x^T z_i - \tilde{\beta}_i (g_i(x) - x^T v_i)) - (f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i)) \\ & \geq F(x, u; \alpha_i^1(x, u) [\nabla(f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i)) \\ & \quad + \nabla^2(f_i(u) + u^T z_i - \tilde{\beta}_i (g_i(u) - u^T v_i))p]) - \frac{1}{2} p^T \nabla^2(f_i(u) + u^T z_i - \\ & \quad \tilde{\beta}_i (g_i(u) - u^T v_i))p + \bar{\rho}_i^1 d^2(x, u) \end{aligned} \quad (5.8)$$

where  $\bar{\rho}_i^1 = \rho_i^1(1 + \tilde{\beta}_i)$ . Multiplying (5.8) by  $\tilde{\alpha}_i^1(x, u)\lambda_i$ ,  $i = 1, 2, \dots, k$  and adding in (5.7), we obtain

$$\begin{aligned} & \sum_{i=1}^k \tilde{\alpha}_i^1(x, u)\lambda_i \left[ f_i(x) + x^T z_i - \tilde{\beta}_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \tilde{\beta}_i(g_i(u) - u^T v_i)) \right] \\ & - \tilde{\alpha}^2(x, u) \sum_{j=1}^m y_j (h_j(u) + u^T w_j) \geq F(x, u; \sum_{i=1}^k \lambda_i [\nabla (f_i(u) + u^T z_i - \tilde{\beta}_i(g_i(u) - u^T v_i)) \\ & + \nabla^2 (f_i(u) + u^T z_i - \tilde{\beta}_i(g_i(u) - u^T v_i)) p] + \sum_{j=1}^m y_j [\nabla (h_j(u) + u^T w_j) + \nabla^2 (h_j(u) + u^T w_j) q]) \\ & - \frac{1}{2} \sum_{i=1}^k \tilde{\alpha}_i^1(x, u)\lambda_i p^T \nabla^2 (f_i(u) + u^T z_i - \tilde{\beta}_i(g_i(u) - u^T v_i)) p - \frac{1}{2} \tilde{\alpha}^2(x, u) \sum_{j=1}^m y_j \\ & q^T \nabla^2 (h_j(u) + u^T w_j) q + \left( \sum_{i=1}^k \tilde{\alpha}_i^1(x, u)\lambda_i \bar{\rho}_i^1 + \tilde{\alpha}^2(x, u)\rho^2 \right) d^2(x, u) \end{aligned}$$

Using equation (5.1), the inequalities (5.2), (5.3), the hypothesis (ii) and the fact that  $F(x, u, 0) = 0$ , the above inequality reduces to

$$\sum_{i=1}^k \tilde{\alpha}_i^1(x, u)\lambda_i \left( f_i(x) + x^T z_i - \tilde{\beta}_i(g_i(x) - x^T v_i) \right) \geq 0$$

which is a contradiction to (5.4). Hence the proof.  $\square$

**Theorem 5.2** (Strong Duality). *Let  $u$  be an efficient solution of (FP) and a constraint qualification is satisfied, then there exists  $\bar{\lambda} \in R^k, \bar{y} \in R^m, \bar{z}_i, \bar{v}_i, \bar{w}_j \in R^n, i = 1, 2, \dots, k; j = 1, 2, \dots, m$ , such that  $(u, \bar{z}, \bar{v}, \bar{w}, \bar{\beta}, \bar{y}, \bar{\lambda}, p = 0, q = 0)$  is a feasible solution of (SFD). Further if the conditions of weak duality theorem 5.1 are satisfied for each feasible solution of (FP) and (SFD), then  $(u, \bar{z}, \bar{v}, \bar{w}, \bar{\beta}, \bar{y}, \bar{\lambda}, p = 0, q = 0)$  is an efficient solution of (SFD) and the corresponding values of the objective functions are equal.*

*Proof.* Following the lines of [3, 11], it can be shown that there exists  $\bar{\mu} \in R^k, \bar{y} \in R^m, \bar{z}_i, \bar{v}_i, \bar{w}_j \in R^n, i = 1, 2, \dots, k; j = 1, 2, \dots, m$ , such that

$$\nabla \left( \sum_{i=1}^k \bar{\mu}_i \left( \frac{f_i(u) + u^T \bar{z}_i}{g_i(u) - u^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{y}_j (h_j(u) + u^T \bar{w}_j) \right) = 0, \quad (5.9)$$

and

$$\sum_{j=1}^m \bar{y}_j (h_j(u) + u^T \bar{w}_j) = 0,$$

where  $u^T \bar{z}_i = S(u|C_i), u^T \bar{v}_i = S(u|D_i), \bar{z}_i \in C_i, \bar{v}_i \in D_i, i = 1, 2, \dots, k, \bar{\mu} \geq 0, u^T \bar{w}_j = S(u|E_j), \bar{w}_j \in E_j, \bar{y}_j \geq 0, j = 1, 2, \dots, m$ ,

Equation (5.9) can be written as

$$\sum_{i=1}^k \frac{\bar{\mu}_i}{g_i(u) - u^T \bar{v}_i} \left( \nabla (f_i(u) + u^T \bar{z}_i) - \frac{f_i(u) + u^T \bar{z}_i}{g_i(u) - u^T \bar{v}_i} \nabla (g_i(u) - u^T \bar{v}_i) \right) + \sum_{j=1}^m \bar{y}_j \nabla (h_j(u) + u^T \bar{w}_j) = 0,$$

Setting  $\bar{\lambda}_i = \frac{\bar{\mu}_i}{g_i(u) - u^T \bar{v}_i}$  and  $\tilde{\beta}_i = \frac{f_i(u) + u^T \bar{z}_i}{g_i(u) - u^T \bar{v}_i}$ ,  $i = 1, 2, \dots, k$ , we get

$$\nabla \left( \sum_{i=1}^k \bar{\lambda}_i \left( f_i(u) + u^T \bar{z}_i - \tilde{\beta}_i (g_i(u) - u^T \bar{v}_i) \right) + \sum_{j=1}^m \bar{y}_j (h_j(u) + u^T \bar{w}_j) \right) = 0,$$

$$f_i(u) + u^T \bar{z}_i - \tilde{\beta}_i (g_i(u) - u^T \bar{v}_i) = 0, \quad i = \overline{1, k}$$

$$\sum_{j=1}^m \bar{y}_j (h_j(u) + u^T \bar{w}_j) = 0,$$

$u^T \bar{z}_i = S(u|C_i)$ ,  $u^T \bar{v}_i = S(u|D_i)$ ,  $z_i \in C_i$ ,  $v_i \in D_i$ ,  $i = 1, 2, \dots, k$ ,  $\bar{\lambda} \geq 0$ ,  $u^T \bar{w}_j = S(u|E_j)$ ,  $w_j \in E_j$ ,  $\bar{y}_j \geq 0$ ,  $j = 1, 2, \dots, m$ .

Thus  $(u, \bar{z}, \bar{v}, \bar{w}, \bar{\beta}, \bar{y}, \bar{\lambda}, p = 0, q = 0)$  is a feasible solution of (SFD) and the corresponding values of the objective functions are equal.  $\square$

**Theorem 5.3** (Strict Converse Duality). *Let  $x_o$  and  $(u_o, z, v, w, \tilde{\beta}, y, \lambda, p, q)$  be the feasible solutions of (FP) and (SFD) respectively. If*

- (i)  $\frac{f_i(x_o) + S(x_o|C_i)}{g_i(x_o) - S(x_o|D_i)} \leq \tilde{\beta}_i$ ,  $i = 1, 2, \dots, k$ ,
  - (ii)  $\left( \lambda_i \left( f_i(\cdot) + (\cdot)^T z_i - \tilde{\beta}_i (g_i(\cdot) + (\cdot)^T v_i) \right), \sum_{j=1}^m y_j (h_j(\cdot) + (\cdot)^T w_j) \right)$  is second order semi-strictly quasi  $(F, \tilde{\alpha}, \tilde{\rho}, d)$ -V-type I function at  $u_o$ ,
  - (iii)  $\tilde{\rho}^1 + \tilde{\rho}^2 \geq 0$ ,
- then  $x_o = u_o$ .

*Proof.* Suppose  $x_o \neq u_o$ . Since  $\lambda \geq 0$ ,  $\tilde{\alpha}_i^1(x_o, u_o) > 0$ ,  $x_o^T z_i \leq S(x_o|C_i)$ ,  $x_o^T v_i \leq S(x_o|D_i)$ ,  $i = 1, 2, \dots, k$  and hypothesis (i) holds, therefore,

$$\sum_{i=1}^k \tilde{\alpha}_i^1(x_o, u_o) \lambda_i \left( f_i(x_o) + x_o^T z_i - \tilde{\beta}_i (g_i(x_o) - x_o^T v_i) \right) \leq 0 \quad (5.10)$$

As  $\tilde{\alpha}^2(x_o, u_o) > 0$ , and  $(u_o, z, v, w, y, \lambda, p, q)$  is the feasible solution of (SFD), therefore by feasibility condition (5.3), we get

$$\tilde{\alpha}^2(x_o, u_o) \sum_{j=1}^m y_j \left( -(h_j(u_o) + u_o^T w_j) + \frac{1}{2} q^T \nabla^2 (h_j(u_o) + u_o^T w_j) q \right) \leq 0,$$

which by hypothesis (ii) implies

$$F \left( x_o, u_o; \sum_{j=1}^m y_j \left[ \nabla (h_j(u_o) + u_o^T w_j) + \nabla^2 (h_j(u_o) + u_o^T w_j) q \right] \right) \leq -\tilde{\rho}^2 d^2(x_o, u_o)$$

As hypothesis (iii) hold, the above inequality alongwith feasibility condition (5.1) and sublinearity of  $F$  imply

$$\begin{aligned} & F\left(x_o, u_o; \sum_{i=1}^k \lambda_i \left[ \nabla \left( f_i(u_o) + u_o^T z_i - \tilde{\beta}_i(g_i(u_o) - u_o^T v_i) \right) \right. \right. \\ & \quad \left. \left. + \nabla^2(f_i(u_o) + u_o^T z_i - \tilde{\beta}_i(g_i(u_o) - u_o^T v_i))p \right] \right) \\ & \geq -F\left(x_o, u_o; \sum_{j=1}^m y_j \left( \nabla(h_j(u_o) + u_o^T w_j) + \nabla^2(h_j(u_o) + u_o^T w_j)q \right) \right) \\ & \geq \tilde{\rho}^2 d^2(x_o, u_o) \geq -\tilde{\rho}^1 d^2(x_o, u_o) \end{aligned}$$

which on applying hypothesis (ii) gives

$$\begin{aligned} & \sum_{i=1}^k \tilde{\alpha}_i^1(x_o, u_o) \lambda_i [f_i(x_o) + x_o^T z_i - \tilde{\beta}_i(g_i(x_o) - x_o^T v_i) \\ & \quad - (f_i(u_o) + u_o^T z_i - \tilde{\beta}_i(g_i(u_o) - u_o^T v_i))] \\ & \quad + \frac{1}{2} p^T \nabla^2 (f_i(u_o) + u_o^T z_i - \tilde{\beta}_i(g_i(u_o) - u_o^T v_i)) p > 0 \end{aligned}$$

Using feasibility condition (5.2), we obtain

$$\sum_{i=1}^k \tilde{\alpha}_i^1(x_o, u_o) \lambda_i (f_i(x_o) + x_o^T z_i - \tilde{\beta}_i(g_i(x_o) - x_o^T v_i)) > 0$$

which is a contradiction to (5.10). Hence  $x_o = u_o$ .  $\square$

## REFERENCES

1. I. Ahmad, Z. Husain and S. Al. Homidan, *Second order duality in nondifferentiable fractional programming*, Nonlinear Analysis: Real world Applications 12(2011), 1103-1110.
2. B.D. Craven, *Fractional programming*, in: Sigma series in Applied Mathematics, 4(1988), Heldermann Verlag, Berlin, Germany, 145.
3. R.R. Egudo, *Multiobjective fractional duality*, Bull. Austral. Math. Soc. 37(1988), 367-378.
4. T.R. Gulati and D. Agarwal, *Second order duality in multiobjective programming involving  $(F, \alpha, \rho, d)$ -V-type I functions*, Numer. Funct. Anal. Optim. 28(2007), 1263-1277.
5. T.R. Gulati, I. Ahmad and D. Agarwal, *Sufficiency and duality in multiobjective programming under generalized type I functions*, J. Optim. Theory App. 135(2007), 411-427.
6. M. Hachimi and B. Aghezzaf, *Sufficiency and duality in differentiable multiobjective programming involving generalized type I functions*, J. Math. Anal. Appl. 296(2004), 382-392.
7. M. Hachimi and B. Aghezzaf, *Second order duality in multiobjective programming involving generalized type I functions*, Numer. Funct. Anal. Optim. 25(2004), 725-736.
8. M.A. Hanson, *Second order invexity and duality in mathematical programming*, Opsearch 30(1993), 313-320.
9. M.A. Hanson and B. Mond, *Further generalizations of convexity in mathematical programming*, J. Info. Optimization Sci. 3(1982), 25-32.

10. M.A. Hanson, R. Pini and C. Singh, *Multiobjective Programming under Generalized Type I Inconvexity*, J. Math. Anal. Appl. 261(2001), 562-577.
11. I. Husain and Z. Jabeen, *On fractional programming containing support functions*, J. Appl. Math. Comput. 18(2005), 361-376.
12. Z. Husain, A. Jayswal and I. Ahmad, *Second order duality for nondifferentiable minimax programming problems with generalized convexity*, J. Global. Optim. 44(2009), 593-608.
13. A. Jayswal, D. Kumar and R. Kumar, *Second order duality for nondifferentiable multiobjective programming problem involving  $(F, \alpha, \rho, d)$ -V-type I functions*, Optim. Lett. 4(2010), 211-226.
14. A. Jayswal, R. Kumar and D. Kumar, *Multiobjective fractional programming problems involving  $(p, r) - \rho - (\eta, \theta)$ - invex function*, J. Appl. Math. Comput. 39(2012), 35-51.
15. Z.A. Liang, H.X. Huang and P.M. Pardalos, *Optimality conditions and duality for a class of nonlinear fractional programming problems*, J. Optim. Theory App. 110(2001), 611-619.
16. Z.A. Liang, H.X. Huang and P.M. Pardalos, *Efficiency conditions and duality for a class of multiobjective fractional programming problems*, J. Global Optim. 27(2003), 447-471.
17. O.L. Mangasarian, *Second and higher duality in nonlinear programming*, J. Math. Anal. Appl. 51(1975), 607-620.
18. S.K. Mishra, *Second order generalized invexity and duality in mathematical programming*, Optimization 42(1997), 51-69.
19. B. Mond, *Second order duality for nonlinear programs*, Opsearch 11(1974), 90-99.
20. B. Mond and M. Schechter, *Nondifferentiable symmetric duality*, Bull. Austral. Math. Soc. 53(1996), 177-188.
21. V. Preda, *On efficiency and duality for multiobjective programs*, J. Math. Anal. Appl. 166(1992), 365-377.
22. S. Schaible and T. Ibaraki, *Fractional programming*, Eur. J. Oper. Res. 12(1983), 325-338.
23. M. Srivastava and M. Bhatia, *Symmetric duality for multiobjective programming using second order  $(F, \rho)$  convexity*, Opsearch 43(2006), 274-295.
24. M. Srivastava and M. Govil, *Second order duality for multiobjective programming involving  $(F, \rho, \sigma)$ -type I functions*, Opsearch 37(2000), 310-326.
25. J. Zhang and B. Mond, *Duality for a nondifferentiable programming problem*, Bull. Aust. Math. Soc. 55(1997), 29-44.

**Pallavi Kharbanda** is pursuing her Ph.D. from Centre for Mathematical Sciences, Banasthali University, Rajasthan. She did her M.Sc., M.Phil. from Kurukshetra University, India and since then she has been serving several Institutions.

Research Scholar, Centre for Mathematical Sciences, Banasthali University, Rajasthan, India.

e-mail: pallavimath@yahoo.co.in

**Divya Agarwal** is an expert in the area of Mathematical Programming, Nonlinear Programming Problem. She has done her Ph.D. from I.I.T. Roorkee and M.Sc. from I.I.T. Delhi. She has several papers published in international journals with good impact factor.

Department of Applied Sciences and Humanities, itm University, Gurgaon, India.

e-mail: divyaagarwal@itmindia.edu

**Deepa Sinha** Deepa Sinha has done her Ph.D. from Delhi University and M.Sc. from I.I.T. Kanpur. She has guided many students in Discrete Mathematics in M.Sc., M. Phil, and Ph.D level. Also, she has refereed several research papers for National and international Journals of high impact factor.

Department of Mathematics, South Asian University, New Delhi, India.

e-mail: deepasinha2001@gmail.com