

YEH CONVOLUTION OF WHITE NOISE FUNCTIONALS[†]

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ABSTRACT. In this paper, we study the Yeh convolution of white noise functionals. We first introduce the notion of Yeh convolution of test white noise functionals and prove a dual property of the Yeh convolution. By applying the dual object of the Yeh convolution, we study the Yeh convolution of generalized white noise functionals, which is a non-trivial extension. Finally, we study relations between the Yeh convolution and Fourier-Gauss, Fourier-Mehler transform.

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1. Introduction

Since Yeh in [14] introduced the convolution $\phi * \psi$, called the Yeh convolution, of Wiener functionals ϕ and ψ by

$$\phi * \psi(y) \equiv \int_{C_0[0,1]} \phi\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) \psi\left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right) m(dx) \quad (1)$$

for $y \in C_0[0, 1]$, whenever the integral exists, where $(C_0[0, 1], m)$ is the (standard) Wiener space with the Wiener measure m , several authors studied the Yeh convolution with relations between first variations and various transforms (see [5] and its references).

On the other hand, the white noise theory initiated by Hida [4] to give rigorous meaning of white noise as the time derivative of the Brownian motion has been extensively developed with wide applications to stochastic calculus, mathematical finance and mathematical physics, etc. The convolution product by Kuo [8] and Fourier-Gauss transforms [2, 8] of white noise functionals are important

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applications to infinite dimensional (harmonic) analysis. Also, the convolution products of white noise operators were studied in [6].

In this paper, we study the Yeh convolution of white noise functionals. Since the test white noise functionals have nice regular properties, the Yeh convolution of test white noise functionals is well defined (see [5]). However, from singular properties of the generalized white noise functionals, the extension of Yeh convolution to the generalized white noise functionals is not trivial. Hence main purpose of this paper is to develop a method to give a rigorous meaning of the Yeh convolution of generalized white noise functionals. For our purpose, we study a dual property (see Theorem 3.4) of the Yeh convolution of test white noise functionals and then as an application of the dual object, we study the Yeh convolution of generalized white noise functionals. We also study relations between the Yeh convolution and Fourier-Gauss, Fourier-Mehler transform (see Theorems 3.3 and 4.4).

This paper is organized as follows: In Section 2 we recall basic notions and well-known results in white noise theory, which are necessary for our study, see [12, 8]. In Section 3 we introduce the notion of Yeh convolution of test white noise functionals and prove a dual property of the Yeh convolution. We study a relations between the Yeh convolution and Fourier-Gauss transform. In Section 4 we study the extension of the Yeh convolution to the convolution of generalized white noise functionals. Finally, we study a relation between the Yeh convolution and Fourier-Mehler transform.

2. Preliminaries

Let $H_{\mathbb{R}}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let A be a positive selfadjoint operator on $H_{\mathbb{R}}$ satisfying that there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and an increasing sequence $\{l_n\}_{n=1}^{\infty}$ with $l_1 > 1$ and $\sum_{n=1}^{\infty} l_n^{-2} < \infty$ such that

$$Ae_n = l_n e_n, \quad n = 1, 2, \dots$$

We note that $\rho := \|A^{-1}\|_{\text{OP}} = l_1^{-1} < 1$ and $\|A^{-1}\|_{\text{HS}}^2 < \infty$. Then by the standard construction from $H_{\mathbb{R}}$ and A (see [8, 12]), we have a Gelfand triple:

$$E_{\mathbb{R}} \subset H_{\mathbb{R}} \subset E_{\mathbb{R}}^*, \quad (2)$$

where $E_{\mathbb{R}}^*$ is the strong dual space of $E_{\mathbb{R}}$. In fact, the topology of $E_{\mathbb{R}}$ is defined by the Hilbertian norms $\{|\cdot|_p \equiv |A^p \cdot|_{p \geq 0}\}$, where $|\cdot|$ is the norm generated by $\langle \cdot, \cdot \rangle$ and then $E_{\mathbb{R}}$ becomes a countable Hilbert nuclear space. By taking complexification of (2) we have the complex Gelfand triple:

$$E \subset H \subset E^*,$$

where $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$ and $H = H_{\mathbb{R}} + iH_{\mathbb{R}}$. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$ again.

The (Boson) Fock space $\Gamma(H)$ over H is defined by

$$\Gamma(H) = \left\{ \phi = (f_n)_{n=0}^\infty : f_n \in H^{\widehat{\otimes} n}, \quad \|\phi\|^2 = \sum_{n=0}^\infty n! |f_n|_0^2 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n -fold symmetric tensor product of H and $H^{\widehat{\otimes} 0} = \mathbb{C}$. Let $\Gamma(A)$ be the second quantization of the operator A defined by

$$\Gamma(A)\phi = (A^{\otimes n} f_n)_{n=0}^\infty, \quad \phi = (f_n)_{n=0}^\infty \in \Gamma(H),$$

and then $\Gamma(A)$ is a positive selfadjoint operator with $\|\Gamma(A)^{-1}\|_{\text{OP}} < 1$ and $\|\Gamma(A)^{-1}\|_{\text{HS}}^2 < \infty$. By the standard construction from $\Gamma(H)$ and $\Gamma(A)$, we have a Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*.$$

In fact, the (projective) topology of (E) is determined by the family $\{\|\cdot\|_p\}_{p \geq 0}$ of norms defined by

$$\|\phi\|_p^2 = \sum_{n=0}^\infty n! |f_n|_p^2, \quad \phi = (f_n)_{n=0}^\infty \in (E).$$

It is known that for each $\Phi \in (E)^*$ there exists a unique sequence $(F_n)_{n=0}^\infty$ with $F_n \in (E^{\otimes n})_{\text{sym}}^*$ such that

$$\langle \langle \Phi, \phi \rangle \rangle = \sum_{n=0}^\infty n! \langle F_n, f_n \rangle, \quad \phi = (f_n)_{n=0}^\infty \in (E),$$

in this case, Φ is denoted by $\Phi = (F_n)_{n=0}^\infty$.

It follows from the Bochner-Minlos theorem that there exists a unique probability measure μ on $E_{\mathbb{R}}^*$ such that its characteristic function is given by

$$\exp\left(-\frac{1}{2} |\xi|_0^2\right) = \int_{E_{\mathbb{R}}^*} e^{i\langle x, \xi \rangle} \mu(dx), \quad \xi \in E_{\mathbb{R}}.$$

The above probability measure μ is called the *standard Gaussian measure* on $E_{\mathbb{R}}^*$ and the probability space $(E_{\mathbb{R}}^*, \mu)$ is referred to as the (standard) *Gaussian space*. The unitary isomorphism between $L^2(E_{\mathbb{R}}^*, \mu; \mathbb{C})$ and $\Gamma(H)$, called the Wiener-Itô-Segal isomorphism, is uniquely determined by the correspondence:

$$\begin{aligned} \Gamma(H) \ni \phi_\xi &= \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots \right) \\ \longleftrightarrow \phi_\xi(x) &= e^{\langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle} \in L^2(E_{\mathbb{R}}^*, \mu; \mathbb{C}), \end{aligned}$$

where ϕ_ξ is called an *exponential vector* (or *coherent state*) and $\phi_\xi(x)$ is called the Gaussianization of ϕ_ξ .

Since $\{\phi_{\xi_1} \otimes \dots \otimes \phi_{\xi_m} : \xi_i \in E, i = 1, 2, \dots, m\}$ spans a dense subspace of $(E)^{\otimes m}$, every $\Xi \in \mathcal{L}((E)^{\otimes m}, ((E)^{\otimes n})^*)$ is uniquely determined by the function $G : E^{m+n} \rightarrow \mathbb{C}$ defined by

$$G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n) = \langle \langle \Xi(\phi_{\xi_1} \otimes \dots \otimes \phi_{\xi_m}), \phi_{\eta_1} \otimes \dots \otimes \phi_{\eta_n} \rangle \rangle \quad (3)$$

for $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in E$. In particular, for $\Xi \in \mathcal{L}((E), (E)^*)$, the form given as in (3) is denoted by $\widehat{\Xi}$ and called the symbol of Ξ .

Theorem 2.1 ([7]). *A Gâteaux-entire function $G : E^{\otimes m+n} \rightarrow \mathbb{C}$ is expressed in the form (3) with $\Xi \in \mathcal{L}((E)^{\otimes m}, ((E)^{\otimes n})^*)$ if and only if there exist constant numbers $C \geq 0, K \geq 0$ and $p \geq 0$ such that*

$$|G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n)|^2 \leq C e^{K(\sum_{j=1}^m |\xi_j|_p^2 + \sum_{k=1}^n |\eta_k|_p^2)}$$

for any $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in E$. Moreover $\Xi \in \mathcal{L}((E)^{\otimes m}, (E)^{\otimes n})$ if and only if for any $\epsilon > 0$ and $p \geq 0$ there exist constant numbers $C \geq 0$ and $q \geq 0$ such that

$$|G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n)|^2 \leq C e^{\epsilon(\sum_{j=1}^m |\xi_j|_{p+q}^2 + \sum_{k=1}^n |\eta_k|_{-p}^2)} \tag{4}$$

for any $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in E$.

For more study of analytic characterization theorems in white noise theory, we refer to [13, 9, 11, 1].

For each $\kappa_{l,m} \in (E^{\otimes l+m})^*$, by applying Theorem 2.1 we can see that there exists an operator $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E)^*)$, called an *integral kernel operator*, such that

$$\widehat{\Xi}(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E.$$

Note that $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E))$ if and only if $\kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^*$. For $f \in E^*$, we write

$$\Xi_{0,1}(f) = a(f), \quad \Xi_{1,0}(f) = a^*(f).$$

The operator $a(f)$ and $a^*(f)$ are called the *annihilation* and *creation operator*, respectively.

Let τ be the trace corresponding to the identity operator $I \in \mathcal{L}(E, E)$ under the canonical isomorphism $\mathcal{L}(E, E^*) \cong (E \otimes E)^*$ by the kernel theorem, that is,

$$\langle \tau, \eta \otimes \xi \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in E.$$

The Gross Laplacian [3] $\Delta_G \in \mathcal{L}((E), (E))$ is defined by $\Delta_G = \Xi_{0,2}(\tau)$ and then we have

$$\widehat{\Delta}_G(\xi, \eta) = \langle \xi, \xi \rangle e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E.$$

3. Convolutions of Test White Noise Functionals

In this section, we study the Yeh convolution of test white noise functionals.

3.1. Translation and Dilation Operators. For each $y \in E_{\mathbb{R}}^*$, the translation operator \mathcal{T}_y on (E) is defined by

$$\mathcal{T}_y \phi(x) = \phi(x + y), \quad x \in E_{\mathbb{R}}^*.$$

In fact, for $\xi \in E$ we have

$$\mathcal{T}_y \phi_\xi = e^{\langle y, \xi \rangle} \phi_\xi = e^{a(y)} \phi_\xi, \quad \widehat{\mathcal{T}}_y(\xi, \eta) = e^{\langle y, \xi \rangle + \langle \xi, \eta \rangle}.$$

Therefore, by applying Theorem 2.1, we can see that $\mathcal{T}_y \in \mathcal{L}((E), (E))$.

For each $\alpha \in \mathbb{C}$, by applying Theorem 2.1 we can easily see that there exists an operator $\mathcal{D}_\alpha \in \mathcal{L}((E), (E))$ such that

$$\widehat{\mathcal{D}}_\alpha(\xi, \eta) = e^{\alpha\langle \xi, \eta \rangle + \frac{\alpha^2 - 1}{2}\langle \xi, \xi \rangle}, \quad \xi, \eta \in E,$$

which implies that

$$\mathcal{D}_\alpha \phi_\xi(x) = e^{\frac{\alpha^2 - 1}{2}\langle \xi, \xi \rangle} \phi_{\alpha\xi}(x) = \phi_\xi(\alpha x), \quad x \in E_{\mathbb{R}}^*, \quad \xi \in E.$$

Since the exponential vectors ϕ_ξ span a dense subspace of (E) and \mathcal{D}_α is continuous,

$$\mathcal{D}_\alpha \phi(x) = \phi(\alpha x), \quad x \in E_{\mathbb{R}}^*, \quad \phi \in (E).$$

Therefore, \mathcal{D}_α is called the dilation. Moreover, we have

$$\mathcal{D}_\alpha = \Gamma(\alpha I) e^{\frac{1}{2} \Delta_G((\alpha^2 - 1)I)}.$$

For each $\alpha \in \mathbb{C}$ and $y \in E^*$, put

$$\mathcal{R}_{\alpha, y} = \mathcal{D}_\alpha \mathcal{T}_y.$$

Then for any $\xi \in E$, we have

$$\mathcal{R}_{\alpha, y} \phi_\xi(x) = \phi_\xi(\alpha x + y) = \phi_{\alpha\xi}(x) \phi_\xi(y) e^{\frac{\alpha^2}{2}\langle \xi, \xi \rangle}, \tag{5}$$

which implies that

$$\widehat{\mathcal{R}}_{\alpha, y}(\xi, \eta) = e^{\alpha\langle \xi, \eta \rangle + \frac{\alpha^2 - 1}{2}\langle \xi, \xi \rangle + \langle y, \xi \rangle}, \quad \xi, \eta \in E.$$

Thus we have the following expression:

$$\mathcal{R}_{\alpha, y} = \Gamma(\alpha I) e^{\frac{1}{2} \Delta_G((\alpha^2 - 1)I)} e^{a(y)} \in \mathcal{L}((E), (E)). \tag{6}$$

3.2. Convolutions of Test White Noise Functionals. We start with the following lemma for the existence of the operator C^l .

Lemma 3.1. *There exists a unique operator $C^l \in \mathcal{L}((E) \otimes (E), (E))$ such that for any $\xi_1, \xi_2, \eta_1 \in E$,*

$$\langle \langle C^l(\phi_{\xi_1} \otimes \phi_{\xi_2}), \phi_{\eta_1} \rangle \rangle = e^{\frac{1}{\sqrt{2}}\langle \xi_1 - \xi_2, \eta_1 \rangle}. \tag{7}$$

Proof. For any $p, q \geq 0$, we obtain that

$$\begin{aligned} \left| \frac{1}{\sqrt{2}} \langle \xi, \eta \rangle \right| &\leq \frac{1}{\sqrt{2}} |\xi|_p |\eta|_{-p} \\ &\leq \frac{1}{\sqrt{2}} \rho^q |\xi|_{p+q} |\eta|_{-p} \\ &\leq \frac{1}{8\epsilon} \rho^{2q} |\xi|_{p+q}^2 + \epsilon |\eta|_{-p}^2 \end{aligned} \tag{8}$$

for any $\epsilon > 0$. Since for any $\epsilon > 0$, there exists $q \geq 0$ such that $\rho^{2q} \leq 8\epsilon^2$. The right hand side of (7) is denoted by $G(\xi_1, \xi_2, \eta_1)$. Then by applying (8), we can

see that $G(\xi_1, \xi_2, \eta_1)$ satisfies (4) with $m = 2$ and $n = 1$. Therefore, by Theorem 2.1, there exists a unique operator $C^l \in \mathcal{L}((E) \otimes (E), (E))$ such that

$$\langle\langle C^l(\phi_{\xi_1} \otimes \phi_{\xi_2}), \phi_{\eta_1} \rangle\rangle = G(\xi_1, \xi_2, \eta_1), \quad \xi_1, \xi_2, \eta_1 \in E,$$

which gives the proof. □

Lemma 3.2. *We have*

$$C^l(\phi_{\xi_1} \otimes \phi_{\xi_2})(y) = \left\langle\left\langle \mathcal{R}_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}y} \phi_{\xi_1}, \mathcal{R}_{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}y} \phi_{\xi_2} \right\rangle\right\rangle, \quad y \in E_{\mathbb{R}}^*$$

for $\xi_1, \xi_2 \in E$.

Proof. For any $\xi_1, \xi_2 \in E$, by applying (5), we see that

$$\left\langle\left\langle \mathcal{R}_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}y} \phi_{\xi_1}, \mathcal{R}_{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}y} \phi_{\xi_2} \right\rangle\right\rangle = \phi_{\frac{1}{\sqrt{2}}(\xi_1 - \xi_2)}(y), \tag{9}$$

which gives the proof. □

From Lemma 3.2, for any $\phi, \psi \in (E)$, we write

$$\phi *_{\mathbb{Y}}^l \psi = C^l(\phi \otimes \psi),$$

which is called the Yeh convolution of ϕ and ψ and coincides with the convolution defined in (1).

For each $\alpha, \beta \in \mathbb{C}$, by applying Theorem 2.1 we can easily see that there exists a unique operator $\mathcal{G}_{\alpha, \beta} \in \mathcal{L}((E), (E))$ such that

$$\widehat{\mathcal{G}_{\alpha, \beta}}(\xi, \eta) = e^{\frac{\alpha}{2}\langle \xi, \xi \rangle + \beta\langle \xi, \eta \rangle}, \quad \xi, \eta \in E.$$

The operator $\mathcal{G}_{\alpha, \beta}$ is called the Fourier-Gauss transform [2] and its adjoint operator $\mathcal{F}_{\alpha, \beta} = \mathcal{G}_{\alpha, \beta}^* \in \mathcal{L}((E)^*, (E)^*)$ is called the Fourier-Mehler transform. Then we have

$$\mathcal{G}_{\alpha, \beta} = \Gamma(\beta I) e^{\frac{\alpha}{2} \Delta_G} \tag{10}$$

and the integral representation:

$$\mathcal{G}_{\alpha, \beta} \phi(x) = \int_{E_{\mathbb{R}}^*} \phi\left(\beta x + \sqrt{\alpha + 1 - \beta^2} y\right) \mu(dy), \quad x \in E_{\mathbb{R}}^*$$

(see [10, 2, 5]). Then the following theorem gives a relation between the Yeh convolution and the Fourier-Gauss transform.

Theorem 3.3. *Let $\alpha \in \mathbb{C}$. Then for any $\phi, \psi \in (E)$, we have*

$$\mathcal{G}_{\alpha^2, \alpha}(\phi *_{\mathbb{Y}}^l \psi) = \left(\mathcal{G}_{\frac{\alpha^2}{2}, \frac{\alpha}{\sqrt{2}}} \phi\right) \left(\mathcal{G}_{\frac{\alpha^2}{2}, -\frac{\alpha}{\sqrt{2}}} \psi\right), \tag{11}$$

where the right hand side is the pointwise multiplication.

Proof. For any $\xi, \eta \in E$, by applying (9), we obtain that

$$\begin{aligned} \mathcal{G}_{\alpha^2, \alpha}(\phi_{\xi} *_{\mathbb{Y}}^l \phi_{\eta}) &= \mathcal{G}_{\alpha^2, \alpha} \phi_{\frac{1}{\sqrt{2}}(\xi - \eta)} \\ &= e^{\frac{\alpha^2}{4}\langle \xi - \eta, \xi - \eta \rangle} \phi_{\frac{\alpha}{\sqrt{2}}(\xi - \eta)} \end{aligned}$$

$$\begin{aligned}
 &= e^{\frac{\alpha^2}{4} \langle \xi, \xi \rangle + \frac{\alpha^2}{4} \langle \eta, \eta \rangle} \phi_{\frac{\alpha}{\sqrt{2}} \xi} \phi_{-\frac{\alpha}{\sqrt{2}} \eta} \\
 &= \left(\mathcal{G}_{\frac{\alpha^2}{2}, \frac{\alpha}{\sqrt{2}}} \phi_{\xi} \right) \left(\mathcal{G}_{\frac{\alpha^2}{2}, -\frac{\alpha}{\sqrt{2}}} \phi_{\eta} \right),
 \end{aligned}$$

which gives the proof of (11). □

Theorem 3.4. For any $\phi, \psi, \varphi \in (E)$, we have

$$\langle \langle \phi *_Y^l \psi, \varphi \rangle \rangle = \langle \langle \phi(y), \langle \langle \psi, \mathcal{R}_{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}y} \varphi \rangle \rangle \rangle \rangle,$$

where the right hand side is understood as the bilinear form for the functions of variable y .

Proof. For any $\xi_2, \eta \in E$, by applying (5), we obtain that

$$\langle \langle \phi_{\xi_2}, \mathcal{R}_{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}y} \phi_{\eta} \rangle \rangle = e^{-\frac{1}{\sqrt{2}} \langle \xi_2, \eta \rangle} \phi_{\frac{1}{\sqrt{2}} \eta}(y),$$

which implies that

$$\begin{aligned}
 \langle \langle \phi_{\xi_1}(y), \langle \langle \phi_{\xi_2}, \mathcal{R}_{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}y} \phi_{\eta} \rangle \rangle \rangle \rangle &= e^{\frac{1}{\sqrt{2}} \langle \xi_1 - \xi_2, \eta \rangle} \\
 &= \langle \langle \phi_{\xi_1} *_Y^l \phi_{\xi_2}, \phi_{\eta} \rangle \rangle.
 \end{aligned}$$

Therefore, the proof is immediate from the continuities since exponential vectors span a dense subspace of (E) . □

From Theorem 3.4, we write

$$\psi *_Y^r \varphi(y) = \langle \langle \psi, \mathcal{R}_{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}y} \varphi \rangle \rangle, \quad y \in E_{\mathbb{R}}^*.$$

Then we have

$$\langle \langle \phi *_Y^l \psi, \varphi \rangle \rangle = \langle \langle \phi, \psi *_Y^r \varphi \rangle \rangle$$

for any $\phi, \psi, \varphi \in (E)$.

4. Convolution of Generalized White Noise Functionals

In this section, we study an extension of convolution operator to generalized white noise functionals.

Theorem 4.1. There exists an operator $C^r \in \mathcal{L}((E)^* \otimes (E), (E))$ such that

$$C^r(\psi \otimes \varphi) = \psi *_Y^r \varphi$$

for $\psi, \varphi \in (E)$.

Proof. Consider the function $G : E \times E \times E \rightarrow E$ defined by

$$G(\xi_1, \eta_1, \eta_2) = e^{\frac{1}{\sqrt{2}} \langle \eta_1 - \eta_2, \xi_1 \rangle}, \quad \xi_1, \eta_1, \eta_2 \in E.$$

Then by applying similar arguments used in the proof of Lemma 3.1 and Theorem 2.1, there exists a unique operator $\Xi \in \mathcal{L}((E), (E) \otimes (E))$ such that

$$\langle \langle \Xi(\phi_{\xi_1}), \phi_{\eta_1} \otimes \phi_{\eta_2} \rangle \rangle = G(\xi_1, \eta_1, \eta_2), \quad \xi_1, \eta_1, \eta_2 \in E.$$

On the other hand, by the kernel theorem, we have the following topological isomorphisms:

$$\mathcal{L}((E)^* \otimes (E), (E)) \xrightarrow{J_1} (E) \otimes (E) \otimes (E)^* \xrightarrow{J_2} \mathcal{L}((E), (E) \otimes (E)).$$

In fact, for any $\Xi \in \mathcal{L}((E)^* \otimes (E), (E))$ and $\Phi, \Psi \in (E)^*$, $\phi \in (E)$, we obtain that

$$\begin{aligned} \langle \langle \Xi(\Phi \otimes \phi), \Psi \rangle \rangle &= \langle \langle J_1 \Xi, \Psi \otimes \Phi \otimes \phi \rangle \rangle \\ &= \langle \langle (J_2 J_1 \Xi)(\phi), \Psi \otimes \Phi \rangle \rangle. \end{aligned}$$

Put

$$C^r = J_1^{-1} (J_2^{-1}(\Xi)).$$

Then for any $\xi_1, \eta_1, \eta_2 \in E$, we obtain that

$$\begin{aligned} \langle \langle C^r(\phi_{\eta_2} \otimes \phi_{\xi_1}), \phi_{\eta_1} \rangle \rangle &= \langle \langle J_1(C^r), \phi_{\eta_1} \otimes \phi_{\eta_2} \otimes \phi_{\xi_1} \rangle \rangle \\ &= \langle \langle J_2(J_1(C^r))(\phi_{\xi_1}), \phi_{\eta_1} \otimes \phi_{\eta_2} \rangle \rangle \\ &= \langle \langle \Xi(\phi_{\xi_1}), \phi_{\eta_1} \otimes \phi_{\eta_2} \rangle \rangle \\ &= e^{\frac{1}{\sqrt{2}} \langle \eta_1 - \eta_2, \xi_1 \rangle} \\ &= \langle \langle C^l(\phi_{\eta_1} \otimes \phi_{\eta_2}), \phi_{\xi_1} \rangle \rangle. \end{aligned}$$

Therefore, for any $\xi_1, \eta_1, \eta_2 \in E$, we have

$$\langle \langle C^r(\phi_{\eta_2} \otimes \phi_{\xi_1}), \phi_{\eta_1} \rangle \rangle = \langle \langle \phi_{\eta_1}, \phi_{\eta_2} *_{Y}^r \phi_{\xi_1} \rangle \rangle = \langle \langle \phi_{\eta_2} *_{Y}^r \phi_{\xi_1}, \phi_{\eta_1} \rangle \rangle,$$

which implies the proof. □

For each $\Phi \in (E)^*$ and $\phi \in (E)$, from Theorem 4.1, the convolution $\Phi *_{Y}^r \phi \in (E)$ is defined by

$$\Phi *_{Y}^r \phi = C^r(\Phi \otimes \phi).$$

Theorem 4.2. *The operator C^l can be extended to $(E)^* \otimes (E)^*$ as a continuous linear operator in $\mathcal{L}((E)^* \otimes (E)^*, (E)^*)$ of which the extension is denoted by the same symbol.*

Proof. By the dual property, the proof is enough to see that $(C^l)^* \in \mathcal{L}((E), (E) \otimes (E))$. For any ξ_1, η_1, η_2 , we have

$$\begin{aligned} \langle \langle (C^l)^*(\phi_{\xi_1}), \phi_{\eta_1} \otimes \phi_{\eta_2} \rangle \rangle &= \langle \langle C^l(\phi_{\eta_1} \otimes \phi_{\eta_2}), \phi_{\xi_1} \rangle \rangle \\ &= \langle \langle \phi_{\eta_1} *_{Y}^l \phi_{\eta_2}, \phi_{\xi_1} \rangle \rangle \\ &= e^{\frac{1}{\sqrt{2}} \langle \eta_1 - \eta_2, \xi_1 \rangle}. \end{aligned}$$

Then by applying similar arguments used in the proof of Lemma 3.1, we see that $(C^l)^* \in \mathcal{L}((E), (E) \otimes (E))$. Therefore, $C^l \in \mathcal{L}((E)^* \otimes (E)^*, (E)^*)$. □

For each $\Phi, \Psi \in (E)^*$, the Yeh convolution $\Phi *_{Y}^l \Psi \in (E)^*$ is defined by

$$\Phi *_{Y}^l \Psi = C^l(\Phi \otimes \Psi).$$

Theorem 4.3. For any $\Phi, \Psi \in (E)^*$ and $\varphi \in (E)$, we have

$$\langle\langle \Phi *^l_Y \Psi, \varphi \rangle\rangle = \langle\langle \Phi, \Psi *^r_Y \varphi \rangle\rangle.$$

Proof. The proof is immediate by the definitions of the convolutions $*^l_Y$ and $*^r_Y$. \square

The following theorem gives a relation between the Yeh convolution and the Fourier-Mehler transform.

Theorem 4.4. Let $\alpha \in \mathbb{C}$. Then for any $\Phi, \Psi \in (E)^*$, we have

$$(\mathcal{F}_{\alpha,1}\Phi) *^l_Y (\mathcal{F}_{\alpha,1}\Psi) = \mathcal{F}_{\alpha,1} (\Phi *^l_Y \Psi). \tag{12}$$

Proof. Note that for any $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ and $\xi \in E$,

$$\Gamma(\beta I)e^{\alpha\Delta_G}\phi_\xi = e^{\alpha\langle\xi,\xi\rangle}\phi_{\beta\xi} = e^{\alpha\beta^{-2}\Delta_G}\Gamma(\beta I)\phi_\xi,$$

which implies that

$$\Gamma(\beta I)e^{\alpha\Delta_G} = e^{\alpha\beta^{-2}\Delta_G}\Gamma(\beta I).$$

Also, we note that for any $\alpha \in \mathbb{C}$ and $y \in E^*$, $e^{a(y)}e^{\alpha\Delta_G} = e^{\alpha\Delta_G}e^{a(y)}$. Therefore, by (6) and (10), we obtain that

$$\begin{aligned} \Psi *^r_Y \mathcal{G}_{\alpha,1}\phi &= \left\langle\left\langle \Psi, \Gamma\left(-\frac{1}{\sqrt{2}}I\right) e^{-\frac{1}{4}\Delta_G} e^{\frac{1}{\sqrt{2}}a(y)} e^{\frac{\alpha}{2}\Delta_G} \phi \right\rangle\right\rangle \\ &= \left\langle\left\langle \Psi, e^{\alpha\Delta_G} \Gamma\left(-\frac{1}{\sqrt{2}}I\right) e^{-\frac{1}{4}\Delta_G} e^{\frac{1}{\sqrt{2}}a(y)} \phi \right\rangle\right\rangle \\ &= (\mathcal{F}_{2\alpha,1}\Psi) *^r_Y \phi, \end{aligned}$$

and

$$\mathcal{G}_{\alpha,1}(\phi_\xi *^r_Y \phi_\eta) = e^{-\frac{1}{\sqrt{2}}\langle\xi,\eta\rangle + \frac{\alpha}{4}\langle\eta,\eta\rangle} \phi_{\frac{1}{\sqrt{2}}\eta} = \phi_\xi *^r_Y (\mathcal{G}_{\frac{\alpha}{2},1}\phi_\eta),$$

which implies that

$$\mathcal{G}_{\alpha,1}(\Psi *^r_Y \varphi) = \Psi *^r_Y (\mathcal{G}_{\frac{\alpha}{2},1}\varphi), \quad \Psi \in (E)^*, \quad \varphi \in (E).$$

Therefore, we obtain that

$$\begin{aligned} \langle\langle \mathcal{F}_{\alpha,1}\Phi *^l_Y \mathcal{F}_{\alpha,1}\Psi, \varphi \rangle\rangle &= \langle\langle \Phi, \mathcal{G}_{\alpha,1}(\mathcal{F}_{\alpha,1}\Psi *^r_Y \varphi) \rangle\rangle \\ &= \langle\langle \Phi, (\Psi *^r_Y \mathcal{G}_{\frac{\alpha}{2},1}\mathcal{G}_{\frac{\alpha}{2},1}\varphi) \rangle\rangle \\ &= \langle\langle \Phi, (\Psi *^r_Y \mathcal{G}_{\alpha,1}\varphi) \rangle\rangle \\ &= \langle\langle \mathcal{F}_{\alpha,1}(\Phi *^l_Y \Psi), \varphi \rangle\rangle, \end{aligned}$$

which gives the proof of (12). \square

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