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FUNCTIONAL ITERATIVE METHODS FOR SOLVING TWO-POINT BOUNDARY VALUE PROBLEMS†

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ABSTRACT. In this paper, we first propose a new technique of the functional iterative methods VIM (Variational iteration method) and NHPM (New homotopy perturbation method) for solving two-point boundary value problems, and then we compare their numerical results with those of the finite difference method (FDM).

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1. Introduction

The numerical solution of two-point boundary value problems (BVPs) is very important because its wide application in scientific research. Thus many scientists try to find more accurate approximate solutions of BVPs. Numerical methods such as Finite difference method (FDM) [17] and Finite element method (FEM) are computationally expensive and have less convergence speed and low accuracy, which may produce inaccurate results. Therefore, many researchers attempt to propose new methods for solving BVPs. In general, the functional equations are very difficult to solve and their exact solutions are difficult to obtain. Therefore, recently some various functional iterative methods have been developed such as Variational iteration method (VIM) [3, 5, 6, 13, 14, 16], Homotopy perturbation method (HPM) [7, 8, 9, 10, 11, 12] and New homotopy perturbation method (NHPM) [4, 15] to solve the various functional equations. In recent years, Computer Algebra Systems (CAS) such as Maple and Mathematica are developed. Consequently by using CAS, we can calculate approximate solutions of functional equations more easily and quickly.

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The purpose of this paper is to provide more sophisticated functional iterative methods VIM and NHPM for solving two-point boundary value problems by using CAS. This paper is organized as follows. In Section 2, we introduce brief description of the VIM and NHPM. In Section 3, we first propose a new technique of the VIM and NHPM for solving two-point boundary value problems, and then we compare their numerical results with those of the FDM. In Section 4, some conclusion are drawn.

2. Brief Description of the VIM and NHPM

In this section, we give a brief description of the functional iterative methods VIM and NHPM for two-point boundary value problems (BVPs).

2.1. VIM (Variational Iteration Method). We consider the following two-point BVPs:

$$-\frac{d^2u(x)}{dx^2} + \sigma(x)u(x) = f(x), \quad a < x < b, \tag{1}$$

with the two-point boundary conditions

$$u(a) = \alpha; \quad u(b) = \beta \tag{2}$$

where α and β are given real constants, and $f(x)$ and $\sigma(x)$ are given real continuous functions on $a \leq x \leq b$.

To illustrate the basic concepts of the VIM, Eq. (1) can be rewritten as

$$Lu(x) + Nu(x) = g(x), \tag{3}$$

where L is a linear operator, N is a non-linear operator, and $g(x)$ is a nonhomogeneous term. Then, we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\{Lu_n(\xi) + N\widetilde{u}_n(\xi) - g(\xi)\}d\xi, \tag{4}$$

where $\lambda(\xi)$ is a general Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right is called the correction term and $\widetilde{u}_n$ is considered as a restricted variation, i.e., $\delta\widetilde{u}_n = 0$.

Once the Lagrange multiplier $\lambda(\xi)$ is obtained by using variational theory and $\delta\widetilde{u}_n = 0$, the following variational iterative method is obtained :

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\{Lu_n(\xi) + Nu_n(\xi) - g(\xi)\}d\xi \ (n \geq 0) \tag{5}$$

where u_0 is a suitably chosen approximate function. The above functional iteration can be easily computed by using Computer Algebra System (CAS) such as Mathematica and Maple.

2.2. NHPM(New Homotopy Perturbation Method). We consider the following nonlinear BVP :

$$A(u(r)) - f(r) = 0,\ r \in \Omega, \tag{6}$$

with the boundary conditions

$$B\left(u(r), \frac{\partial u(r)}{\partial n}\right) = 0,\ r \in \Gamma, \tag{7}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts, L and N, when L is a linear and N is a nonlinear operator. Therefore, Eq. (6) can be rewritten as

$$L(u(r)) + N(u(r)) - f(r) = 0. \tag{8}$$

By the homotopy technique, we construct a homotopy $U(r,p) : \Omega \times [0,1] \to \mathbb{R}$, which satisfies

$$H(U,p) = (1-p)[L(U) - u_0] + p[A(U) - f(r)] = 0,\ p \in [0,1],\ r \in \Omega, \tag{9}$$

or equivalently,

$$H(U,p) = L(U) - u_0 + pu_0 + p[N(U) - f(r)] = 0, \tag{10}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of solution of Eq. (6). Clearly, we have

$$\begin{aligned} H(U,0) &= L(U) - u_0 = 0, \\ H(U,1) &= A(U) - f(r) = 0. \end{aligned} \tag{11}$$

The chainging process of p from 0 to 1 is just that of $U(r,p)$ from $L^{-1}(u_0)$ to $u(r)$, where $u(r)$ is the exact solution of Eq. (6). According to the HPM, the solutions of Eq. (10) can be represented as a power series in p as

$$U \equiv U(r,p) = \sum_{n=0}^{\infty} p^n U_n. \tag{12}$$

Now let us write the Eq. (10) in the following form

$$L(U) = u_0 + p[f(r) - u_0 - N(U)]. \tag{13}$$

By applying the inverse operator L^{-1} to both sides of Eq. (13), we have

$$U = L^{-1}(u_0) + p[L^{-1}(f(r)) - L^{-1}(u_0) - L^{-1}N(U)]. \tag{14}$$

Suppose that the initial approximation of Eq. (6) has the form

$$u_0 = \sum_{n=0}^{\infty} a_n P_n, \tag{15}$$

where $a_0, a_1, a_2, \cdots$ are unknown coefficients and $P_0, P_1, P_2, \cdots$ are specific functions depending on the problem. Now by substituting Eqs. (12) and (15) into the Eq. (14), we get

$$\begin{aligned}\sum_{n=0}^{\infty} p^n U_n = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n\right) \\ +p\left[L^{-1}(f(r)) - L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n\right) - L^{-1}N\left(\sum_{n=0}^{\infty} p^n U_n\right)\right]\end{aligned} \tag{16}$$

Comparing coefficients of terms with identical powers of p leads to

$$\begin{aligned}&p^0 : U_0 = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n\right), \\ &p^1 : U_1 = L^{-1}(f(r)) - L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n\right) - L^{-1}N(U_0(x)), \\ &p^n : U_n = -L^{-1}N(U_0, U_1, ..., U_{n-1}) \ (n \geq 2),\end{aligned} \tag{17}$$

Now if we can solve these equations in such a way that $U_1 = 0$, then Eq. (17) results in $U_2 = U_3 = \cdots = U_n = \cdots = 0$. Therefore, the exact solution may be obtained by using

$$u(r) = \lim_{p \to 1} U(r,p) = U_0 = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n\right). \tag{18}$$

3. A New Technique of the VIM and NHPM

In this section, we first propose a new technique of the functional iterative methods VIM and NHPM for solving two-point boundary value problems (BVPs), and then we compare their numerical results with those of the finite difference method (FDM).

Example 3.1. We consider the one-dimensional two-point BVP in [16]:

$$\begin{aligned}&u''(x) - 40x\, u(x) = 2 \\ &u(-1) = u(1) = 0\end{aligned} \tag{19}$$

Method 1 (VIM) : According to the VIM, we can construct a correct functional as follows :

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\{u_n''(\xi) - 40\xi\widetilde{u}_n(\xi) - 2\}d\xi. \tag{20}$$

Using the variational theory and $\delta\widetilde{u}_n = 0$ in order to identify the Lagrange multiplier $\lambda(\xi)$, one has

$$\begin{aligned}\delta u_{n+1}(x) &= \delta u_n(x) + \delta\int_0^x \lambda(\xi)\{u_n''(\xi) - 40\xi\widetilde{u}_n(\xi) - 2\}d\xi \\ &= \delta u_n(x) + \int_0^x \lambda(\xi)\delta u_n''(\xi)d\xi \\ &= \delta u_n(x) + \lambda(x)\delta u_n'(x) - \lambda'(x)\delta u_n(x) + \int_0^x \lambda''(\xi)\delta u_n''(\xi)d\xi \\ &= (1-\lambda'(x))\delta u_n(x) + \lambda'(x)\delta u_n(x) + \int_0^x \lambda''(\xi)\delta u_n''(\xi)d\xi = 0.\end{aligned} \tag{21}$$

Then we have $\lambda(\xi) = \xi - x$, leading to the following variational iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\{u_n''(\xi) - 40\xi u_n(\xi) - 2\}d\xi. \tag{22}$$

We begin with an arbitrary initial approximation

$$u_0(x) = a + bx, \tag{23}$$

where a and b are constants to be determined.

Using Mathematica with $u_0(x) = a + bx$, we can obtain

$$\begin{aligned}u_1(x) &= a + bx + x^2 + \frac{20a}{3}x^3 + \frac{10b}{3}x^4 \\ u_2(x) &= a + bx + x^2 + \frac{20a}{3}x^3 + \frac{10b}{3}x^4 + 2x^5 + \frac{80a}{9}x^6 + \frac{200b}{63}x^7 \\ &\vdots\end{aligned} \tag{24}$$

To determine a and b, we impose two boundary conditions $u(-1) = u(1) = 0$ on $u_{10}(x)$ which is not written here because of too many terms. Solving the linear system $u_{10}(-1) = 0$ and $u_{10}(1) = 0$ by Mathematica, we can obtain an approximate value of a and b :

$$\begin{aligned}a &= -\frac{62613093136318865494387350228474 3}{524429902094147271787256918719793 3} \approx -0.119393 \\ b &= -\frac{998741641694663420636287315299931 8}{406460058772763903950492563319462 49} \approx -0.245717.\end{aligned} \tag{25}$$

Lastly, we substitute these approximate value of a and b into $u_{10}(x)$ which is obtained by the iteration formula (22). The numerical results are listed in Table 1 under the column VIM.

Lu [16] has used $u_1(1) = 0$ and $u_1(-1) = 0$ to find approximate values of a and b, which yields worse approximate solution than those in this paper. As can be seen in Table 1, our technique of VIM performs much better than Lu [16].

Method 2 (NHPM) : By the NHPM, we construct the homotopy

$$(1-p)\left[U''(x) - u_0(x)\right] + p\left[U''(x) - 40xU(x) - 2\right] = 0,$$

or equivalently,

$$U''(x) = u_0(x) - pu_0(x) - 40pxU(x) + 2p. \tag{26}$$

Applying the inverse operator $L^{-1} = \int_{-1}^{x}\int_{-1}^{t}(\cdot)d\tau dt$ to both side of Eq. (26), we have

$$\begin{aligned}U(x) =&U(-1) + U'(-1)(x+1) + \int_{-1}^{x}\int_{-1}^{t} u_0(\tau)d\tau dt \\ &- p\int_{-1}^{x}\int_{-1}^{t} u_0(\tau)d\tau dt + p\int_{-1}^{x}\int_{-1}^{t} 40\tau U(\tau)d\tau dt + p(x+1)^2.\end{aligned} \tag{27}$$

Suppose the solution $U(x)$ of Eq. (27) is represented as

$$U(x) = U_0(x) + pU_1(x) + p^2U_2(x) + \cdots, \tag{28}$$

where $U_i(x)$ are functions which should be determined. Substituting Eq. (28) into Eq. (27) and comparing coefficients of terms with identical powers of p, we obtain

$$\begin{aligned}&p^0 : U_0(x) = U'(-1)(x+1) + \int_{-1}^{x}\int_{-1}^{t} u_0(\tau)d\tau dt \\ &p^1 : U_1(x) = (x+1)^2 - \int_{-1}^{x}\int_{-1}^{t} u_0(\tau)d\tau dt + \int_{-1}^{x}\int_{-1}^{t} 40\tau U_0(\tau)d\tau dt \\ &p^n : U_n(x) = \int_{-1}^{x}\int_{-1}^{t} 40\tau U_{n-1}(\tau)d\tau dt \quad (n \geq 2).\end{aligned}$$

Since the boundary condition is given at $x = -1$, we can let

$$u_0(x) = \sum_{n=0}^{\infty} a_n(x+1)^n. \tag{29}$$

Let $U'(-1) = \alpha$, where α is a constant to be determined. Then we have

$$\begin{aligned}U_0(x) &= \alpha(x+1) + \int_{-1}^{x}\int_{-1}^{t}\sum_{n=0}^{\infty} a_n(\tau+1)^n d\tau dt \\ &= \alpha(x+1) + \sum_{n=0}^{\infty}\frac{a_n}{(n+1)(n+2)}(x+1)^{n+2}\end{aligned} \tag{30}$$

and

$$\begin{aligned}U_1(x) = &(x+1)^2 - \int_{-1}^{x}\int_{-1}^{t}\sum_{n=0}^{\infty} a_n(\tau+1)^n d\tau dt \\ &+ \int_{-1}^{x}\int_{-1}^{t} 40\tau\left[\alpha(x+1) + \sum_{n=0}^{\infty}\frac{a_n}{(n+1)(n+2)}(x+1)^{n+2}\right]d\tau dt \\ = &(x+1)^2 - \sum_{n=0}^{\infty}\frac{a_n}{(n+1)(n+2)}(x+1)^{n+2} \\ &+ \frac{10}{3}\alpha(x+1)^4 - \frac{20}{3}\alpha(x+1)^3 \\ &+ \sum_{n=0}^{\infty}\frac{40a_n}{(n+1)(n+2)(n+4)(n+5)}(x+1)^{n+5} \\ &- \sum_{n=0}^{\infty}\frac{40a_n}{(n+1)(n+2)(n+3)(n+4)}(x+1)^{n+4}\end{aligned} \tag{31}$$

Letting $U_1(x) = 0$, one obtains

$$\begin{aligned} &1 - \frac{a_0}{2} = 0 \\ &a_1 + 40\alpha = 0 \\ &a_2 - 40\alpha + 20a_0 = 0 \\ &a_k = \frac{40a_{k-3}}{(k-2)(k-1)} - \frac{40a_{k-2}}{(k-1)k} \quad (k \geq 3). \end{aligned} \tag{32}$$

From Eq. (32), it follows

$$\begin{aligned} &a_0 = 2 \\ &a_1 = -40\alpha \\ &a_2 = 40(\alpha - 1) \\ &a_k = \frac{40}{k-1}\left(\frac{a_{k-3}}{k-2} - \frac{a_{k-2}}{k}\right) \quad (k \geq 3). \end{aligned} \tag{33}$$

From Eqs. (30) and (33),

$$u(x) = U_0(x) = \alpha(x+1) + \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)}(x+1)^{n+2}. \tag{34}$$

To find an approximate value of α, we apply the boundary condition $u(1) = 0$ which is never used in the above computational process. For this paper, the α is approximated by solving the following equation

$$u_{15}(1) = 0, \tag{35}$$

where $u_{15}(x) = \alpha(x+1) + \sum_{n=0}^{15} \frac{a_n}{(n+1)(n+2)}(x+1)^{n+2}$. Solving Eq. (35) with Mathematica, we can obtain approximate value of α

$$\alpha \approx 1.40778. \tag{36}$$

Lastly, substituting this approximate value of α into Eq. (34) yields the following approximate solution

$$\begin{aligned} \widehat{u}(x) =& 1.40778(x+1) + 2(x+1)^2 - 56.3112(x+1)^3 + 16.3112(x+1)^4 \\ &+ \sum_{n=3}^{\infty} \left[\left(\frac{a_{n-3}}{n-2} - \frac{a_{n-2}}{n}\right) \frac{40}{(n-1)(n+1)(n+2)}\right](x+1)^{n+2}. \end{aligned} \tag{37}$$

To increase the accuracy of α, we can use the condition $u_k(1) = 0$ for values of k which is greater than 15. *This technique is never used before by any other researchers.*

In Tables 1 and 2, we provide numerical results obtained by the VIM, NHPM and FDM as compared with those of Adomian for $u = \Phi_{12}$ in [2] because we can't find the exact solution of this problem, where FDM(n) means an approximate solution obtained by the finite difference method with the mesh size $h = \frac{2}{n}$. As can be seen in Table 2, the VIM and NHPM are more accurate than FDM(n).

TABLE 1. Numerical results for Example 3.1

x	Φ_{12}	VIM	NHPM	FDM(80)	FDM(160)
-1.0	0.000000	2.28983×10^{-16}	0.000000	0.000000	0.000000
-0.8	0.254206	0.254206	0.254206	0.254677	0.254323
-0.6	0.296396	0.296396	0.296396	0.296369	0.296389
-0.4	0.146349	0.146349	0.146349	0.145632	0.146170
-0.2	-0.025886	-0.025886	-0.025886	-0.026773	-0.026107
0.0	-0.119393	-0.135649	-0.119393	-0.026773	-0.119551
0.2	-0.135649	-0.135649	-0.135649	-0.135973	-0.135729
0.4	-0.113969	-0.113969	-0.113969	-0.114092	-0.114000
0.6	-0.083321	-0.083322	-0.083322	-0.083349	-0.083328
0.8	-0.050944	-0.050944	-0.050944	-0.050931	-0.050941
1.0	0.000000	-9.1073×10^{-17}	0.000000	0.000000	0.000000

TABLE 2. Absolute errors of numerical methods for Example 3.1 compared to Φ_{12}

x	VIM	NHPM	FDM(80)	FDM(160)
-1.0	0.000000	0.000000	0.000000	0.000000
-0.8	0.000000	0.000000	0.000471	0.000117
-0.6	0.000000	0.000000	0.000027	0.000007
-0.4	0.000000	0.000000	0.000717	0.000179
-0.2	0.000000	0.000000	0.000887	0.000221
0.0	0.000000	0.000000	0.000636	0.000158
0.2	0.000000	0.000000	0.000324	0.000080
0.4	0.000000	0.000000	0.000123	0.000031
0.6	0.000001	0.000001	0.000028	0.000007
0.8	0.000000	0.000000	0.000013	0.000003
1.0	0.000000	0.000000	0.000000	0.000000

Example 3.2. We consider the two-point BVP in [16]:

$$\begin{aligned} &u''(x) + \frac{1}{x}u'(x) + u(x) - \frac{5}{4} - \frac{x^2}{16} = 0 \\ &u'(0) = 0, \;\; u(1) = \frac{17}{16} \end{aligned} \tag{38}$$

Method 1 (VIM) : According to the VIM, we can construct a correct functional as follows :

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left[u''(\xi) + \frac{1}{\xi}\widetilde{u}'(\xi) + \widetilde{u}(\xi) - \frac{5}{4} - \frac{\xi^2}{16} \right] d\xi. \tag{39}$$

Using the variational theory and $\delta\widetilde{u}_n = 0$ in order to identify the Lagrange multiplier $\lambda(\xi)$, one has

$$\begin{aligned}
\delta u_{n+1}(x) &= \delta u_n(x) + \delta \int_0^x \lambda(\xi)\left[u''(\xi) + \frac{1}{\xi}\widetilde{u}'(\xi) + \widetilde{u}(\xi) - \frac{5}{4} - \frac{\xi^2}{16}\right] d\xi \\
&= \delta u_n(x) + \int_0^x \lambda(\xi)\delta u_n''(\xi) d\xi \\
&= \delta u_n(x) + \lambda(x)\delta u_n'(x) - \lambda'(x)\delta u_n(x) + \int_0^x \lambda''(\xi)\delta u_n''(\xi) d\xi \\
&= (1-\lambda'(x))\delta u_n(x) + \lambda'(x)\delta u_n(x) + \int_0^x \lambda''(\xi)\delta u_n''(\xi) d\xi = 0.
\end{aligned} \tag{40}$$

Then we have $\lambda(\xi) = \xi - x$, leading to the following variational iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left[u''(\xi) + \frac{1}{\xi}u'(\xi) + u(\xi) - \frac{5}{4} - \frac{\xi^2}{16}\right] d\xi \tag{41}$$

We begin with an arbitrary initial approximation

$$u_0(x) = a, \tag{42}$$

where a is a constant to be determined. Using Mathematica with $u_0(x) = a$, we can obtain

$$\begin{aligned}
u_1(x) &= a + \frac{a}{2}x^2 + \frac{1}{192}x^4 \\
u_2(x) &= a - \frac{7}{144}x^2 + \frac{a}{24}x^4 - \frac{1}{5760}x^6 \\
&\vdots
\end{aligned} \tag{43}$$

To determine a, we impose the boundary condition $u(1) = \frac{17}{16}$ to $u_{20}(x)$ which is not written here because of too many terms. Solving equation $u_{20}(1) = \frac{17}{16}$ with Mathematica, we can obtain an approximate value of a

$$a \approx 1.06742. \tag{44}$$

Lastly, we substitute this approximate value of a into $u_{20}(x)$ which is obtained by the iteration formula (41). The numerical results are listed in Table 3 under the column VIM.

Method 2 (NHPM) : By the NHPM, we construct the homotopy

$$(1-p)\left[U''(x) - u_0(x)\right] + p\left[U''(x) + \frac{1}{x}U'(x) + U(x) - \frac{5}{4} - \frac{x^2}{16}\right] = 0,$$

or equivalently,

$$U''(x) = u_0(x) - p\left[u_0(x) + \frac{1}{x}U'(x) + U(x) - \frac{5}{4} - \frac{x^2}{16}\right] \tag{45}$$

Applying the inverse operator $L^{-1} = \int_0^x \int_0^t (\cdot) d\tau dt$, to both side of Eq. (45) we have

$$\begin{aligned} U(x) = U(0) + &\int_0^x \int_0^t u_0(\tau) d\tau dt \\ &- p \int_0^x \int_0^t \left[u_0(\tau) + \frac{1}{\tau} U'(\tau) + U(\tau) - \frac{5}{4} - \frac{\tau^2}{16} \right] d\tau dt \end{aligned} \tag{46}$$

Suppose the solution $U(x)$ of Eq. (46) is represented as

$$U(x) = U_0(x) + pU_1(x) + p^2 U_2(x) + \cdots, \tag{47}$$

where $U_i(x)$ are functions which should be determined. Substituting Eq. (47) into Eq. (46) and comparing coefficients of terms with identical powers of p, we obtain

$$\begin{aligned} p^0 &: U_0(x) = U(0) + \int_0^x \int_0^t u_0(\tau) d\tau dt \\ p^1 &: U_1(x) = -\int_0^x \int_0^t \left[u_0(\tau) + \frac{1}{\tau} U_0'(\tau) + U_0(\tau) - \frac{5}{4} - \frac{\tau^2}{16} \right] d\tau dt \\ p^n &: U_n(x) = -\int_0^x \int_0^t \left[\frac{1}{\tau} U_{n-1}'(\tau) + U_{n-1}(\tau) \right] d\tau dt \quad (n \geq 2). \end{aligned}$$

Let $u_0(x) = \sum_{n=0}^{\infty} a_n x^n$, $U'(0) = u'(0) = 0$ and $U(0) = \alpha$, where α is a constant to be determined. Then we have

$$U_0(x) = \alpha + \int_0^x \int_0^t \sum_{n=0}^{\infty} a_n \tau^n d\tau dt = \alpha + \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)} x^{n+2}, \tag{48}$$

$$\begin{aligned} U_1(x) &= -\int_0^x \int_0^t \left[\sum_{n=0}^{\infty} a_n \tau^n + \frac{1}{\tau} \sum_{n=0}^{\infty} \frac{a_n}{n+1} \tau^{n+1} + \alpha + \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)} \tau^{n+2} - \frac{5}{4} - \frac{\tau^2}{16} \right] d\tau dt \\ &= -\sum_{n=0}^{\infty} \frac{a_n}{(n+1)^2} x^{n+2} - \frac{1}{2} \alpha x^2 - \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)(n+3)(n+4)} x^{n+4} + \frac{5}{8} x^2 + \frac{1}{192} x^4 \\ &= \left(\frac{5}{8} - \frac{1}{2}\alpha - a_0 \right) x^2 - \frac{a_1}{4} x^3 + \left(\frac{1}{192} - \frac{a_2}{9} - \frac{a_0}{24} \right) x^4 \\ &\qquad - \sum_{n=0}^{\infty} \frac{1}{n+4} \left(\frac{a_{n+3}}{n+4} + \frac{a_n}{(n+2)(n+3)(n+5)} \right) x^{n+5}. \end{aligned} \tag{49}$$

Letting $U_1(x) = 0$, one obtains

$$\begin{aligned} &\frac{5}{8} - \frac{1}{2}\alpha - a_0 = 0, \quad \frac{a_1}{4} = 0 \\ &\frac{1}{192} - \frac{a_2}{9} - \frac{a_0}{24} = 0 \\ &\frac{1}{k+1} \left(\frac{a_k}{k+1} + \frac{a_{k-2}}{(k-1)k(k+2)} \right) = 0 \quad (k \geq 3). \end{aligned} \tag{50}$$

From Eq. (50),

$$\begin{aligned} a_0 &= \frac{5}{8} - \frac{1}{2}\alpha \\ a_1 &= 0 \\ a_2 &= \frac{3}{64} - \frac{3}{8}a_0 \\ a_k &= -\frac{a_{k-2}(k+1)}{(k-1)k(k+2)} \quad (k \geq 3). \end{aligned} \tag{51}$$

From Eqs. (48) and (51),

$$\begin{aligned} u(x) = U_0(x) &= \alpha + \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)} x^{n+2} \\ &= \alpha + \frac{a_0}{1\cdot 2}x^2 + \frac{a_1}{2\cdot 3}x^{+} \frac{a_2}{3\cdot 4}x^3 + \cdots \end{aligned} \tag{52}$$

To find an approximate value of α, we apply the boundary condition $u(1) = \frac{17}{16}$ which is never used in the above computational process. For this paper, the α is approximated by solving the following equation

$$u_{15}(1) = \frac{17}{16}, \tag{53}$$

where $u_{15}(x) = \alpha + \sum_{n=0}^{15} \frac{a_n}{(n+1)(n+2)} x^{n+2}$. Solving Eq. (53) with Mathematica, we can obtain an approximate value of $\alpha = 1$. Substituting this approximate value of α into Eq. (52), one obtains

$$\begin{aligned} \widehat{u}(x) = U_0(x) &= 1 + \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)} x^{n+2} \\ &= 1 + \frac{a_0}{1\cdot 2}x^2 + \frac{a_1}{2\cdot 3}x^{+} \frac{a_2}{3\cdot 4}x^3 + \cdots \\ &= 1 + \frac{a_0}{2}x^2 = 1 + \frac{1}{16}x^2, \end{aligned} \tag{54}$$

which is the exact solution of BVP (38).

In Tables 3 and 4, we compare the numerical results obtained by VIM, NHPM and FDM with those of the exact solution, where FDM(n) means an approximate solution obtained by the finite difference method with the mesh size $h = \frac{1}{n}$. As can be seen in Table 4, NHPM is more accurate than VIM and FDM, and our technique of VIM is more accurate than the VIM in Lu [16].

4. Conclusion

In this paper, we proposed a new technique of the VIM and NHPM for solving two-point boundary value problems. Our technique of VIM performed better than Lu's technique in [16]. Our technique of NHPM also performed very well as compared with FDM (See Tables 1 to 4). Therefore, we conclude that our technique of VIM and NHPM are very effective, reliable and powerful methods

Table 3. Numerical results for Example 3.2

x	Exact Solution	VIM	VIM(Lu [16])	NHPM	FDM(80)	FDM(160)
0.0	1	1.06742	0.8646	1	1.00003	1.00001
0.1	1.00063	1.06742	0.8665	1.00063	1.00064	1.00063
0.2	1.0025	1.06742	0.8723	1.0025	1.00251	1.0025
0.3	1.00563	1.06739	0.8820	1.00563	1.00563	1.00563
0.4	1.01	1.0673	0.8956	1.01	1.01001	1.01
0.5	1.01563	1.06713	0.9131	1.01563	1.01563	1.01563
0.6	1.0225	1.06681	0.9346	1.0225	1.0225	1.0225
0.7	1.03063	1.06627	0.9603	1.03063	1.03063	1.03063
0.8	1.04	1.06745	0.9901	1.04	1.04	1.04
0.9	1.05063	1.06423	1.0241	1.05063	1.05063	1.05063

Table 4. Absolute errors of numerical methods for Example 3.2 compared to the exact solution

x	VIM	VIM(Lu [16])	NHPM	FDM(80)	FDM(160)
0.0	0.0674236	0.1354	0.0	3.24313×10^{-5}	8.95455×10^{-6}
0.1	0.0667981	0.13413	0.0	7.64539×10^{-6}	1.83782×10^{-6}
0.2	0.0649161	0.1302	0.0	9.16605×10^{-6}	2.29183×10^{-6}
0.3	0.0617607	0.12363	0.0	2.05549×10^{-6}	3.23593×10^{-6}
0.4	0.0573034	0.1144	0.0	5.49701×10^{-6}	1.37439×10^{-6}
0.5	0.0515036	0.10253	0.0	7.60917×10^{-7}	3.94013×10^{-6}
0.6	0.0443081	0.0879	0.0	3.17265×10^{-6}	7.93232×10^{-7}
0.7	0.03565	0.07033	0.0	2.75834×10^{-6}	4.43954×10^{-6}
0.8	0.0254474	0.0499	0.0	1.41426×10^{-6}	3.53594×10^{-7}
0.9	0.0136025	0.02653	0.0	4.32888×10^{-6}	4.83221×10^{-6}

for solving two-point boundary value problems. All numerical computations in this paper were performed using Mathematica.

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