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EXPONENTIALLY FITTED INTERPOLATION FORMULAS INVOLVING FIRST AND HIGHER-ORDER DERIVATIVES

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ABSTRACT. We construct exponentially fitted interpolation formulas using the values of the ω-dependent function f as well as its derivatives up to the nth order at a finite number of nodes on a closed interval Ω. The function f is of the form,

$$f(x) = f_1(x)\cos(\omega x) + f_2(x)\sin(\omega x), x \in \Omega,$$

where f_1 and f_2 are smooth enough to be approximated by polynomials on Ω. Some properties of the formulas are newly found. The properties are numerically investigated and reexamined by producing some figures.

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1. Introduction

Phenomena with oscillatory character appear in mechanics and physics. The investigation of such phenomena may imply mathematical operations on the function f with a frequency ω which is of the form,

$$f(x) \quad = \quad f_1(x)\cos(\omega x) + f_2(x)\sin(\omega x), \quad x \in [a, b], \tag{1}$$

where f_1 and f_2 are smooth enough to be approximated by polynomials. The mathematical operations include differentiation and quadrature. The Schrődinger equation provides a good example to explain why the operations are considered on the f defined in (1). In the case of the Schrődinger equation, the solution is partly described by the function in (1). See Chapter 3 of [5] for more details. Since this equation represented the starting point for exponentially fitted techniques, Ixaru[4, 6] derived exponentially fitted formulas for differentiation, quadrature and multistep solvers for ordinary differential equations. Such techniques were further extended to solve boundary value problems [1, 2]. An error analysis for exponentially fitted interpolation formulas was examined [7]. Also,

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extended exponentially fitted interpolation formulas were constructed and investigated at two nodes and at more than two nodes [8]. Now our current research establishes some properties regarding exponentially fitted interpolation formulas involving derivatives up to the nth order at a finite number of nodes.

Technically, when an interpolation formula for a function is constructed at a finite number of nodes, consideration is usually given to the imposition of some conditions on the formula at the given nodes. Such conditions may be provided by the function values or its derivatives up to the nth order at the nodes. According to the conditions selected, a variety of interpolation formulas are generated. We will construct exponentially fitted interpolation formulas which use not only the values of the function f but also of its derivative up to the nth order at $2N+1$ nodes. They will be denoted $\mathcal{I}^n_{2N+1}$. It is instructive to note that in building up these formulas we do not follow the standard procedure to construct interpolation formulas. Specifically, we will not impose the kth derivative conditions on $\mathcal{I}^n_{2N+1}$ such as

$$\frac{d^k}{dx^k}\mathcal{I}^n_{2N+1} \quad = \quad \frac{d^k}{dx^k} f \tag{2}$$

at the nodes where $k = 0, 1, 2, \ldots, n$, so that $\mathcal{I}^n_{2N+1}$ does not necessarily satisfy (2) at the beginning of its construction. However, it will turn out that $\mathcal{I}^n_{2N+1}$ satisfies (2). In fact, the results of (2) can be derived from obtaining some properties of the coefficients of $\mathcal{I}^n_{2N+1}$ at the nodes. Thus, we will focus on producing various results regarding the coefficients of $\mathcal{I}^n_{2N+1}$ at the nodes, in particular that $\mathcal{I}^n_{2N+1}$ satisfies (2).

This article is organized as follows. In Section 2, exponentially fitted interpolation formulas, denoted by $\mathcal{I}^n_{2N+1}$, are introduced. In Section 3, various properties of the coefficients of $\mathcal{I}^n_{2N+1}$ are shown. In Section 4, numerical results are given to illustrate some of the characteristics of $\mathcal{I}^n_{2N+1}$ when $n = 1$ and $N = 2$.

2. Constructing $\mathcal{I}^n_{2N+1}$

For the ω-dependent function f given in (1), we will investigate exponentially fitted interpolation formulas, denoted by $\mathcal{I}^n_{2N+1}$, to use not only the values of the function f but also of its derivatives up to the nth order at $2N+1$ nodes where n and N are positive integers. The $\mathcal{I}^n_{2N+1}$ is defined by

$$\begin{aligned} &f(x_0 + Nht) \\ &\approx \mathcal{I}^n_{2N+1}(t) = \quad \textstyle\sum_{j=0}^{n} h^j \left(\sum_{r=-N}^{N} \alpha_{j,r}(\omega, h, t) f^{(j)}(x_0 + rh) \right) \end{aligned} \tag{3}$$

where x_0 is a real number, $h > 0$ and $-1 \le t \le 1$. The $\alpha_{j,r}(\omega, h, t)$ implies that the coefficient $\alpha_{j,r}$ depends on the values of ω, h and t. For simplicity, we shall write $\alpha_{j,r}$ in place of $\alpha_{j,r}(\omega, h, t)$. In (3), we denote the jth order derivative by

$$f^{(j)} = \frac{d^j f(x)}{dx^j}$$

and assume that $f^{(j)}$ is known at the nodes $x_0 + rh$ where $r = -N, -N+1, -N+2, \ldots, N$. From the formula $\mathcal{I}^n_{2N+1}$, we put

$$\mathcal{T}(f(x), h, \mathcal{V}) \quad = \quad f(x + Nht) - \sum_{j=0}^{n} h^j \left(\sum_{r=-N}^{N} \alpha_{j,r} f^{(j)}(x + rh) \right) \tag{4}$$

where $\mathcal{V}$ is a vector of the coefficients $\alpha_{j,r}$, that is $\mathcal{V} = (\alpha_{0,-N}, \alpha_{0,-N+1}, \ldots, \alpha_{0,N}, \alpha_{1,-N}, \alpha_{1,-N+1}, \ldots, \alpha_{1,N}, \ldots, \alpha_{n,-N}, \alpha_{n,-N+1}, \ldots, \alpha_{n,N})$. To find the values of the coefficients $\alpha_{j,r}$ of $\mathcal{I}^n_{2N+1}$, we may consider

$$\mathcal{T}(x^m \cos(\omega x), h, \mathcal{V}) = 0 \quad \text{and} \quad \mathcal{T}(x^m \sin(\omega x), h, \mathcal{V}) = 0 \tag{5}$$

where $m = 0, 1, 2, \ldots$. But, since $\sin(\omega x)$ and $\cos(\omega x)$ are linear combinations of $\exp(\pm i\omega x)$, we will solve a system of equations satisfying

$$\mathcal{T}(x^m \exp(\pm i\omega x), h, \mathcal{V}) = 0, \quad m = 0, 1, 2, \ldots, (2N+1)(n+1)/2 - 1, \tag{6}$$

where n is an odd number. Thus, the coefficients $\alpha_{j,r}$ of $\mathcal{I}^n_{2N+1}$ will be determined, depending on the values of ω, h and t. At this stage, we do not know that

$$\frac{d^k}{dx^k} \mathcal{I}^n_{2N+1} \quad = \quad \frac{d^k}{dx^k} f, \quad k = 0, 1, 2, \ldots, n, \tag{7}$$

at the nodes, because we do not impose (7) on $\mathcal{I}^n_{2N+1}$ as seen in (6). Therefore, the $\mathcal{I}^n_{2N+1}$ may provide an approximating formula that fits f on $[x_0 - Nh, x_0 + Nh]$ without necessarily matching the f at the given nodes. However, as it will be proved in Corollary 3.9, this $\mathcal{I}^n_{2N+1}$ satisfies (7), and therefore $\mathcal{I}^n_{2N+1}$ represents a genuine interpolation formula. The advantage of the new formula is that, in contrast to formulas in current use, it reproduces the oscillatory behaviour of the interpolated f, or of discrete data with such a behaviour, according to the case. The functions $x^m \exp(\pm i\omega x)$ in (6) will be called reference functions.

In this article we consider interpolation formulas incorporating the values of derivatives up to an odd n. For an even n, we need one more equation to obtain a system with the same number of equations as the number of coefficients, in addition to the reference functions $x^m \exp(\pm i\omega x)$. See [4] for more details about the reference functions to be taken. In the next section, we will see how the values of $\alpha_{j,r}$ are determined.

3. Properties of the coefficients $\alpha_{j,r}$ of $\mathcal{I}^n_{2N+1}$

To determine the values of $\alpha_{j,r}$ of $\mathcal{I}^n_{2N+1}$, let us apply $f(x) = \exp(\pm i\omega x)$ to (4). Then we obtain

$$\begin{aligned} \mathcal{T}(\exp(\mu x), h, \mathcal{V}) \quad &= \quad \exp(\mu x)\psi(\mu h, \mathcal{V}) \\ &\text{and} \\ \mathcal{T}(\exp(-\mu x), h, \mathcal{V}) \quad &= \quad \exp(-\mu x)\psi(-\mu h, \mathcal{V}) \end{aligned} \tag{8}$$

where $\mu = i\omega$ and

$$\psi(u, \mathcal{V}) \quad = \quad \exp(Nut) - \sum_{j=0}^{n} u^j \left(\sum_{r=-N}^{N} \alpha_{j,r} \exp(ru) \right). \tag{9}$$

Using (9), we put

$$\begin{array}{rcl}\Psi^+(Z,\mathcal{V}) & = & \frac{1}{2}(\psi(u,\mathcal{V})+\psi(-u,\mathcal{V})) \\ & \text{and} & \\ \Psi^-(Z,\mathcal{V}) & = & \frac{1}{2u}(\psi(u,\mathcal{V})-\psi(-u,\mathcal{V}))\end{array} \tag{10}$$

where $Z=u^2=(\mu h)^2=-\omega^2h^2$. We consider the case in which $n=2n_1-1$ where n_1 is a positive integer. Denote $\alpha^{\pm}_{j,r}$ by

$$\alpha^+_{j,r}=\alpha_{j,-N-1+r}+\alpha_{j,N+1-r} \text{ and } \alpha^-_{j,r}=\alpha_{j,-N-1+r}-\alpha_{j,N+1-r} \tag{11}$$

where $j=0,1,2,\ldots,n$ and $r=1,2,3,\ldots,N$. Then we can show Lemma 3.1 and Lemma 3.2.

Lemma 3.1. Ψ^+ *is given by*

$$\begin{array}{l}\Psi^+(Z,\mathcal{V}) \\ =\eta_{-1}(N^2Zt^2) \\ \;+\sum_{\beta=1}^{n_1}\Big[-\sum_{r=1}^{N}\alpha^+_{2(\beta-1),r}Z^{\beta-1}\eta_{-1}((N+1-r)^2Z)-\alpha_{2(\beta-1),0}Z^{\beta-1} \\ \qquad\qquad +\sum_{r=1}^{N}\alpha^-_{2\beta-1,r}(N+1-r)Z^{\beta}\eta_0((N+1-r)^2Z)\Big].\end{array} \tag{12}$$

Proof. Use the definition ψ of (9) and see Appendix (or Section 3.4 of [3]) for the definition of η_s where $s=-1,0$. □

Similarly as in Lemma 3.1, we have

Lemma 3.2. Ψ^- *is given by*

$$\begin{array}{l}\Psi^-(Z,\mathcal{V}) \\ =Nt\eta_0(N^2Zt^2) \\ \;+\sum_{\beta=1}^{n_1}\Big[\sum_{r=1}^{N}\alpha^-_{2(\beta-1),r}(N+1-r)Z^{\beta-1}\eta_0((N+1-r)^2Z) \\ \;-\sum_{r=1}^{N}\alpha^+_{2\beta-1,r}Z^{\beta-1}\eta_{-1}((N+1-r)^2Z)-\alpha_{2\beta-1,0}Z^{\beta-1}\Big].\end{array} \tag{13}$$

Proof. The same arguments as in the Proof of Lemma 3.1 are applied. □

Also, using differentiation for the Ixaru's function η_s such as

$$\frac{d}{dZ}\eta_s(Z)=\frac{1}{2}\eta_{s+1}(Z),\ s=\ -1,\ 0,\ 1,\ldots, \tag{14}$$

which is given in Appendix (or Section 3.4 of [3]), we have

Lemma 3.3. *For* $j,k=1,2,3,\ldots,$

$$\begin{array}{l}\frac{d^j}{dZ^j}\left(Z^k\eta_a((N+1-r)^2Z)\right)= \\ \left(\frac{1}{2}\right)^j\Big[\sum_{m=0}^{j-1}\binom{j}{m}2^{j-m}(N+1-r)^{2m}k(k-1)...(k-(j-(m+1))) \\ \qquad\qquad \cdot Z^{k-(j-m)}\eta_{m+a}((N+1-r)^2Z) \\ \qquad +(N+1-r)^{2j}Z^k\eta_{j+a}((N+1-r)^2Z)\Big],\end{array} \tag{15}$$

where $r=1,2,3,\ldots,N$ *and* $a=-1$ *or* 0.

Proof. The conclusion is followed by repeated application of the Product Rule for differentiation with respect to Z. For example, the Product Rule says

$$\begin{aligned}&\tfrac{d}{dZ}\left(Z^k\eta_a((N+1-r)^2Z)\right)\\ &=\tfrac{d}{dZ}\left(Z^k\right)\eta_a((N+1-r)^2Z)+Z^k\tfrac{d}{dZ}\left(\eta_a((N+1-r)^2Z)\right)\\ &=kZ^{k-1}\eta_a((N+1-r)^2Z)+\tfrac{1}{2}(N+1-r)^2Z^k\eta_{a+1}((N+1-r)^2Z).\end{aligned}$$

□

Then exponentially fitted techniques which were explained in [4] suggest that (6) is equivalent to a system of equations,

$$\tfrac{d^j}{dZ^j}\Psi^+(Z,\mathcal{V})=0,\quad \tfrac{d^j}{dZ^j}\Psi^-(Z,\mathcal{V})=0 \tag{16}$$

where $j=0,1,2,\dots,n_1(2N+1)-1$. Note that $(2N+1)(n+1)/2-1=n_1(2N+1)-1$ since $n=2n_1-1$. By applying Lemma 3.1 - 3.3 to (16), the first and second equations of (16) become

$$\begin{aligned}&(Nt)^{2j}\eta_{j-1}(N^2Zt^2)=\\ &\textstyle\sum_{\beta=1}^{n_1}\Big[\sum_{r=1}^{N}\alpha^+_{2(\beta-1),r}\Big[\sum_{k=0}^{\beta-2}\big(j(j-1)\dots(j-(\beta-1-(k+1)))\\ &\cdot 2^{\beta-1-k}(N+1-r)^{2j-2(\beta-1-k)}\tbinom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)-1}((N+1-r)^2Z)\big)\\ &+(N+1-r)^{2j}Z^{\beta-1}\eta_{j-1}((N+1-r)^2Z)\Big]\\ &+\alpha_{2(\beta-1),0}2^j\tfrac{d^j}{dZ^j}Z^{\beta-1}\\ &-\textstyle\sum_{r=1}^{N}\alpha^-_{2\beta-1,r}\Big[\sum_{k=0}^{\beta-1}\big(j(j-1)\dots(j-(\beta-(k+1)))2^{\beta-k}\\ &\cdot(N+1-r)^{2j-2(\beta-k)+1}\tbinom{\beta}{k}Z^k\eta_{j-(\beta-k)}((N+1-r)^2Z)\big)\\ &+(N+1-r)^{2j+1}Z^{\beta}\eta_j((N+1-r)^2Z)\Big]\Big]\end{aligned} \tag{17}$$

and

$$\begin{aligned}&(Nt)^{2j+1}\eta_{j}(N^2Zt^2)=\\ &\textstyle\sum_{\beta=1}^{n_1}\Big[-\sum_{r=1}^{N}\alpha^-_{2(\beta-1),r}\Big[\sum_{k=0}^{\beta-2}\big(j(j-1)\dots(j-(\beta-1-(k+1)))\\ &\cdot 2^{\beta-1-k}(N+1-r)^{2j-2(\beta-1-k)+1}\tbinom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)}((N+1-r)^2Z)\big)\\ &+(N+1-r)^{2j+1}Z^{\beta-1}\eta_{j}((N+1-r)^2Z)\Big]\\ &+\textstyle\sum_{r=1}^{N}\alpha^+_{2\beta-1,r}\Big[\sum_{k=0}^{\beta-2}\big(j(j-1)\dots(j-(\beta-1-(k+1)))2^{\beta-1-k}\\ &\cdot(N+1-r)^{2j-2(\beta-1-k)}\tbinom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)-1}((N+1-r)^2Z)\big)\\ &+(N+1-r)^{2j}Z^{\beta-1}\eta_{j-1}((N+1-r)^2Z)\Big]\\ &+\alpha_{2\beta-1,0}2^j\tfrac{d^j}{dZ^j}Z^{\beta-1}\Big],\end{aligned} \tag{18}$$

respectively. Note that, for $j=0$, (17) and (18) are equivalent to

$$\Psi^+(Z,\mathcal{V})=0 \text{ and } \Psi^-(Z,\mathcal{V})=0$$

where Ψ^+ and Ψ^- are given in (12) and (13). Now we have two systems of linear equations in $\alpha^{\pm}_{j,r}$ and $\alpha_{j,0}$. One is associated with

$$\tfrac{d^j}{dZ^j}\Psi^+(Z,\mathcal{V})=0,\quad j=0,1,2,\dots,n_1(2N+1)-1. \tag{19}$$

The other is associated with

$$\tfrac{d^j}{dZ^j}\Psi^-(Z,\mathcal{V})=0, \quad j=0,1,2,\ldots,n_1(2N+1)-1. \tag{20}$$

Since (19), equivalently (17), is linear in $\alpha^+_{2(\beta-1),r}$, $\alpha_{2(\beta-1),0}$ and $\alpha^-_{2\beta-1,r}$, it is possible to arrange it into a matrix equation,

$$M^{n,+}_{2N+1}X^{n,+}_{2N+1}=Y^{n,+}_{2N+1}, \tag{21}$$

where $X^{n,+}_{2N+1}$ and $Y^{n,+}_{2N+1}$ are both column vectors and $M^{n,+}_{2N+1}$ is a matrix. In detail, $X^{n,+}_{2N+1}$ is a column vector consisting of entries $\alpha^+_{2(\beta-1),r}$, $\alpha_{2(\beta-1),0}$ and $\alpha^-_{2\beta-1,r}$. That is

$$X^{n,+}_{2N+1}=\left[\mathcal{A}^+_0,\alpha_{0,0},\mathcal{A}^-_1,\mathcal{A}^+_2,\alpha_{2,0},\mathcal{A}^-_3,\ldots,\mathcal{A}^+_{2n_1-2},\alpha_{2n_1-2,0},\mathcal{A}^-_{2n_1-1}\right]^T \tag{22}$$

where, for $\beta=1,2,\ldots,n_1$,

$$\mathcal{A}^+_{2\beta-2}=[\alpha^+_{2\beta-2,r}:1\le r\le N] \text{ and } \mathcal{A}^-_{2\beta-1}=[\alpha^-_{2\beta-1,r}:1\le r\le N]. \tag{23}$$

A few words should be said about the notations in (22) and (23). We use the notation $[a_{j,r}:1\le r\le N]$ to denote a vector with N entries $[a_{j,1},a_{j,2},...,a_{j,N}]$. Also, if more than two vectors are placed inside a pair of brackets, we will omit all internal brackets except for the far left and right brackets to make a vector. For example, let us consider a case of $X^{n,+}_{2N+1}$ in (22) with $n=1$ (equivalently, $n_1=1$) and denote it by $X^{1,+}_{2N+1}$. From the definitions of $\mathcal{A}^+_{2\beta-2}$ and $\mathcal{A}^-_{2\beta-1}$ in (23), the $X^{1,+}_{2N+1}$ is expressed by

$$\begin{aligned} X^{1,+}_{2N+1}&=\left[\mathcal{A}^+_0,\alpha_{0,0},\mathcal{A}^-_1\right]^T\\ &=\left[[\alpha^+_{0,r}:1\le r\le N],\alpha_{0,0},[\alpha^-_{1,r}:1\le r\le N]\right]^T\\ &=\Big[[\alpha^+_{0,1},\alpha^+_{0,2},\ldots,\alpha^+_{0,N}],\alpha_{0,0},[\alpha^-_{1,1},\alpha^-_{1,2},\ldots,\alpha^-_{1,N}]\Big]^T. \end{aligned}$$

Then we omit all internal brackets except for the far left and right brackets in the above to make a vector

$$\Big[\alpha^+_{0,1},\alpha^+_{0,2},\ldots,\alpha^+_{0,N},\alpha_{0,0},\alpha^-_{1,1},\alpha^-_{1,2},\ldots,\alpha^-_{1,N}\Big]^T. \tag{24}$$

Thus we regard $X^{1,+}_{2N+1}$ as a vector in (24). Also, we denote $M(u,v)$ and $Y(u)$ by the (u,v) entry of a matrix M and the uth entry of a vector Y, respectively. Now, all entries of $M^{n,+}_{2N+1}$ and $Y^{n,+}_{2N+1}$ in (21) are obtained from (17).

Theorem 3.4. *For $j=0,1,2,\ldots,n_1(2N+1)-1$, $\beta=1,2,3,\ldots,n_1$ and $r=1,2,3,\ldots,N$, the matrix $M^{n,+}_{2N+1}$ and column vector $Y^{n,+}_{2N+1}$ satisfy*

(i): *the $(j+1,(2N+1)(\beta-1)+r)$ entry of $M^{n,+}_{2N+1}$ is given by*

$$\begin{aligned} &M^{n,+}_{2N+1}(j+1,(2N+1)(\beta-1)+r)=\\ &\textstyle\sum_{k=0}^{\beta-2}\Big[j(j-1)\ldots(j-(\beta-1-(k+1)))2^{\beta-1-k}\\ &\cdot(N+1-r)^{2j-2(\beta-1-k)}\binom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)-1}((N+1-r)^2Z)\Big]\\ &+(N+1-r)^{2j}Z^{\beta-1}\eta_{j-1}((N+1-r)^2Z), \end{aligned}$$

(ii): *the* $(j+1,(2N+1)\beta-N)$ *entry of* $M^{n,+}_{2N+1}$ *is given by*

$$M^{n,+}_{2N+1}(j+1,(2N+1)\beta-N)=2^j\tfrac{d^j}{dZ^j}Z^{\beta-1},$$

(iii): *the* $(j+1,(2N+1)\beta-N+r)$ *entry of* $M^{n,+}_{2N+1}$ *is given by*

$$\begin{aligned}&M^{n,+}_{2N+1}(j+1,(2N+1)\beta-N+r)=\\&-\Big[\textstyle\sum_{k=0}^{\beta-1}\big[j(j-1)\dots(j-(\beta-(k+1)))2^{\beta-k}\\&\cdot(N+1-r)^{2j-2(\beta-k)+1}\binom{\beta}{k}Z^k\eta_{j-(\beta-k)}((N+1-r)^2Z)\Big]\\&+(N+1-r)^{2j+1}Z^\beta\eta_j((N+1-r)^2Z)\big],\end{aligned}$$

(iv): *the* $(j+1)$*th entry of* $Y^{n,+}_{2N+1}$ *is given by*

$$Y^{n,+}_{2N+1}(j+1)=(Nt)^{2j}\eta_{j-1}(N^2Zt^2).$$

Thus $X^{n,+}_{2N+1}$ consisting of $\alpha^+_{2(\beta-1),r}$, $\alpha_{2(\beta-1),0}$ and $\alpha^-_{2\beta-1,r}$ can be determined by solving (21) with the results of Theorem 3.4. Similarly as in (21), (20) (equivalently (18)) can be replaced by another matrix equation,

$$M^{n,-}_{2N+1}X^{n,-}_{2N+1}=Y^{n,-}_{2N+1} \tag{25}$$

because it is linear in $\alpha^-_{2(\beta-1),r}$, $\alpha^+_{2\beta-1,r}$ and $\alpha_{2\beta-1,0}$. In (25), $X^{n,-}_{2N+1}$ is a column vector whose entries are $\alpha^-_{2(\beta-1),r}$, $\alpha^+_{2\beta-1,r}$ and $\alpha_{2\beta-1,0}$. It is denoted by

$$X^{n,-}_{2N+1}=\left[\mathcal{A}^-_0,\mathcal{A}^+_1,\alpha_{1,0},\mathcal{A}^-_2,\mathcal{A}^+_3,\alpha_{3,0},\dots,\mathcal{A}^-_{2n_1-2},\mathcal{A}^+_{2n_1-1},\alpha_{2n_1-1,0}\right]^T \tag{26}$$

where, for $\beta=1,2,3,\dots,n_1$,

$$\mathcal{A}^-_{2\beta-2}=[\alpha^-_{2\beta-2,r}:1\le r\le N]\quad\text{and}\quad\mathcal{A}^+_{2\beta-1}=[\alpha^+_{2\beta-1,r}:1\le r\le N].$$

In the following, all entries of $M^{n,-}_{2N+1}$ and $Y^{n,-}_{2N+1}$ in (25) are obtained from (18):

Theorem 3.5. *For* $j=0,1,2,\dots,n_1(2N+1)-1$, $\beta=1,2,3,\dots,n_1$ *and* $r=1,2,3,\dots,N$, *the matrix* $M^{n,-}_{2N+1}$ *and column vector* $Y^{n,-}_{2N+1}$ *satisfy*

(i): *the* $(j+1,(2N+1)(\beta-1)+r)$ *entry of* $M^{n,-}_{2N+1}$ *is given by*

$$\begin{aligned}&M^{n,-}_{2N+1}(j+1,(2N+1)(\beta-1)+r)=\\&-\Big[\textstyle\sum_{k=0}^{\beta-2}\big[j(j-1)\dots(j-(\beta-1-(k+1)))2^{\beta-1-k}\\&\cdot(N+1-r)^{2j-2(\beta-1-k)+1}\binom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)}((N+1-r)^2Z)\Big]\\&+(N+1-r)^{2j+1}Z^{\beta-1}\eta_j((N+1-r)^2Z)\big],\end{aligned}$$

(ii): *the* $(j+1,(2N+1)(\beta-1)+N+r)$ *entry of* $M^{n,-}_{2N+1}$ *is given by*

$$\begin{aligned}&M^{n,-}_{2N+1}(j+1,(2N+1)(\beta-1)+N+r)=\\&\textstyle\sum_{k=0}^{\beta-2}\big[j(j-1)\dots(j-(\beta-1-(k+1)))2^{\beta-1-k}\\&\cdot(N+1-r)^{2j-2(\beta-1-k)}\binom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)-1}((N+1-r)^2Z)\big]\\&+(N+1-r)^{2j}Z^{\beta-1}\eta_{j-1}((N+1-r)^2Z),\end{aligned}$$

(iii): *the* $(j+1,(2N+1)\beta)$ *entry of* $M^{n,-}_{2N+1}$ *is given by*

$$M^{n,-}_{2N+1}(j+1,(2N+1)\beta) = 2^j \tfrac{d^j}{dZ^j} Z^{\beta-1},$$

(iv): *the* $(j+1)th$ *entry of* $Y^{n,-}_{2N+1}$ *is given by*

$$Y^{n,-}_{2N+1}(j+1) = (Nt)^{2j+1}\eta_j(N^2Zt^2).$$

Thus, $X^{n,-}_{2N+1}$ whose entries are $\alpha^-_{2(\beta-1),r}$, $\alpha^+_{2\beta-1,r}$ and $\alpha_{2\beta-1,0}$ is also determined by solving (25) with the results of Theorem 3.5. By the way, our main task in this article is to show that the properties of $\mathcal{I}^n_{2N+1}$ such as

$$\tfrac{d^k}{dx^k}\mathcal{I}^n_{2N+1}(t) \quad = \quad \tfrac{d^k}{dx^k} f(x), \quad k = 0,1,2,\ldots,n,$$

are satisfied at the nodes where $x = x_0 + Nht$. It is necessary to obtain first and higher-order derivatives of the coefficients of $\mathcal{I}^n_{2N+1}$ up to the nth order at the nodes. This can be achieved by calculating first and higher-order derivatives of

$$Y^{n,+}_{2N+1} \quad \text{and} \quad Y^{n,-}_{2N+1} \tag{27}$$

with respect to t at $t = \pm\frac{N+1-r}{N}$ or 0 where $r = 1,2,3,\ldots,N$. Therefore, we first consider differentiating $Y^{n,+}_{2N+1}$ with respect to t.

Lemma 3.6. *For* $\beta = 1,2,3,\ldots,n_1$ *and* $r = 1,2,3,\ldots,N$,

(i):

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(\,\cdot\,)\right]_{t=\pm\frac{N+1-r}{N}} = N^{2(\beta-1)} M^{n,+}_{2N+1}(\,\cdot\,,(2N+1)(\beta-1)+r),$$

(ii):

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(\,\cdot\,)\right]_{t=0} \quad = \quad N^{2(\beta-1)} M^{n,+}_{2N+1}(\,\cdot\,,(2N+1)\beta-N),$$

(iii):

$$\left[\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,+}_{2N+1}(\,\cdot\,)\right]_{t=-\frac{N+1-r}{N}} = N^{2\beta-1} M^{n,+}_{2N+1}(\,\cdot\,,(2N+1)\beta-N+r),$$

(iv):

$$\left[\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,+}_{2N+1}(\,\cdot\,)\right]_{t=\frac{N+1-r}{N}} \quad = \quad -\left[\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,+}_{2N+1}(\,\cdot\,)\right]_{t=-\frac{N+1-r}{N}},$$

(v):

$$\left[\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,+}_{2N+1}(\,\cdot\,)\right]_{t=0} \quad = \quad \mathbb{O}.$$

Proof. Using (iv) of Theorem 3.4, (14) and

$$\eta_1(N^2Zt^2) = \tfrac{\eta_{-1}(N^2Zt^2)-\eta_0(N^2Zt^2)}{N^2Zt^2}, \tag{28}$$

we have

$$\begin{gathered}\tfrac{d^{2\beta-2}}{dt^{2\beta-2}} Y^{n,+}_{2N+1}(1) = N^{2\beta-2} Z^{\beta-1}\eta_{-1}(N^2Zt^2)\\ \text{and}\\ \tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,+}_{2N+1}(1) = N^{2\beta-1}(Nt)Z^{\beta}\eta_0(N^2Zt^2)\end{gathered} \tag{29}$$

where $\beta = 1, 2, 3, \ldots, n_1$. From the Appendix, we know that

$$\eta_j(N^2Zt^2) = \frac{\eta_{j-2}(N^2Zt^2) - (2j-1)\eta_{j-1}(N^2Zt^2)}{N^2Zt^2}, j = 1, 2, 3, \ldots. \tag{30}$$

In fact, (28) comes from $j = 1$ in (30). Again, using (iv) of Theorem 3.4 and (14),

$$\begin{aligned} &\tfrac{d}{dt} Y^{n,+}_{2N+1}(j+1) = \tfrac{d}{dt}(Nt)^{2j}\eta_{j-1}(N^2Zt^2) \\ &= N\left[2j(Nt)^{2j-1}\eta_{j-1}(N^2Zt^2) + (Nt)^{2j+1}Z\eta_j(N^2Zt^2)\right]. \end{aligned} \tag{31}$$

Then, for $j \geq 1$,

$$\begin{aligned} &\tfrac{d^2}{dt^2} Y^{n,+}_{2N+1}(j+1) = \tfrac{d}{dt}\left(\tfrac{d}{dt} Y^{n,+}_{2N+1}(j+1)\right) \\ &= N\tfrac{d}{dt}\left[(Nt)^{2j-1}\left(\eta_{j-2}(N^2Zt^2) + \eta_{j-1}(N^2Zt^2)\right)\right] \quad \text{(by (30))} \\ &= N^2\left[(2j-1)(Nt)^{2j-2}\left(\eta_{j-2}(N^2Zt^2) + \eta_{j-1}(N^2Zt^2)\right)\right. \\ &\left.+(Nt)^{2j}\left(Z\eta_{j-1}(N^2Zt^2) + Z\eta_j(N^2Zt^2)\right)\right] \\ &= N^2\left[2j(Nt)^{2j-2}\eta_{j-2}(N^2Zt^2) + (Nt)^{2j}Z\eta_{j-1}(N^2Zt^2)\right] \text{ (by (30))}. \end{aligned} \tag{32}$$

Similarly as in (32), for $j \geq 1$,

$$\begin{aligned} &\tfrac{d^3}{dt^3} Y^{n,+}_{2N+1}(j+1) = \tfrac{d}{dt}\left(\tfrac{d^2}{dt^2} Y^{n,+}_{2N+1}(j+1)\right) \\ &= N^3\left[j(j-1)2^2(Nt)^{2j-3}\eta_{j-2}(N^2Zt^2)\right. \\ &\left.+(j)2(Nt)^{2j-1}2Z\eta_{j-1}(N^2Zt^2) + (Nt)^{2j+1}Z^2\eta_j(N^2Zt^2)\right]. \end{aligned} \tag{33}$$

In general, for $1 \leq \beta \leq n_1$ and $j \geq 1$,

$$\begin{aligned} &\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(j+1) = \\ &N^{2(\beta-1)}\left[\sum_{k=0}^{\beta-2} j(j-1)(j-2)\ldots(j-(\beta-1-(k+1)))\right. \\ &\cdot 2^{\beta-1-k}(Nt)^{2(j-(\beta-1-k))}\tbinom{\beta-1}{k}Z^k\eta_{j-(\beta-1-k)-1}(N^2Zt^2) \\ &\left.+(Nt)^{2j}Z^{\beta-1}\eta_{j-1}(N^2Zt^2)\right] \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,+}_{2N+1}(j+1) = \\ &N^{2\beta-1}\left[\sum_{k=0}^{\beta-1} j(j-1)(j-2)\ldots(j-(\beta-(k+1)))\right. \\ &\left.\cdot 2^{\beta-k}(Nt)^{2(j-(\beta-k))+1}\tbinom{\beta}{k}Z^k\eta_{j-(\beta-k)}(N^2Zt^2) + (Nt)^{2j+1}Z^\beta\eta_j(N^2Zt^2)\right] \end{aligned} \tag{35}$$

where $\eta_s \equiv 0$ for $s = -2, -3, -4, \ldots$. Using the first equation of (29) and (34), (i) is obtained from (i) of Theorem 3.4. Likewise, using the second equation of (29) and (35), (iii) is obtained from (iii) of Theorem 3.4. Also, (iv) is shown by (iii) and (35). To get the results of (ii), (34) is rearranged by

$$\begin{aligned} &\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(j+1) = N^{2(\beta-1)}\left[\sum_{m=2}^{\beta} j(j-1)(j-2)\ldots(j-(m-2))\right. \\ &\cdot 2^{m-1}(Nt)^{2(j-(m-1))}\tbinom{\beta-1}{\beta-m}Z^{\beta-m}\eta_{j-m}(N^2Zt^2) \\ &\left.+(Nt)^{2j}Z^{\beta-1}\eta_{j-1}(N^2Zt^2)\right]. \end{aligned} \tag{36}$$

The right hand side of (36) contains β terms and each term contains a factor Nt with an exponent. If the exponent of Nt is negative (equivalently if $m > j+1$), the related term vanishes. Therefore, the right hand side of (36) is continuous at $t = 0$. From this fact, we can obtain

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(j+1)\right]_{t=0} \tag{37}$$

by

$$\lim_{t\to 0}\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(j+1)\right].$$

Thus, for $1 \le j \le \beta - 1$, (36) says that

$$\begin{aligned} (37) &= N^{2(\beta-1)} j(j-1)(j-2)\dots(1)2^j \tbinom{\beta-1}{\beta-1-j} Z^{\beta-(j+1)} \\ &= N^{2(\beta-1)} 2^j (\beta-1)(\beta-2)(\beta-3)\dots(\beta-j) Z^{\beta-(j+1)} \end{aligned}$$

because only one term survives in the right hand side of (36). In general, we have

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(j+1)\right]_{t=0} =$$

$$\begin{cases} N^{2(\beta-1)} Z^{\beta-1} & \text{if } j = 0, \\ N^{2(\beta-1)} 2^j (\beta-1)(\beta-2)\dots(\beta-j) Z^{\beta-(j+1)} & \text{if } 1 \le j \le \beta-1, \\ 0 & \text{if } j \ge \beta. \end{cases} \tag{38}$$

Note that the right hand side of (38) is expressed by

$$N^{2(\beta-1)} 2^j \frac{d^j}{dZ^j} Z^{\beta-1}$$

where $j = 0, 1, 2, \dots$. Using (ii) of Theorem 3.4,

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,+}_{2N+1}(j+1)\right]_{t=0} = N^{2(\beta-1)} M^{n,+}_{2N+1}(j+1, (2N+1)\beta - N).$$

Thus, (ii) is proved. Similarly as done by (34) to prove (ii), (v) is proved by (35). □

Similarly as in Lemma 3.6, $M^{n,-}_{2N+1}$ and $Y^{n,-}_{2N+1}$ are closely related as follows.

Lemma 3.7. *For* $\beta = 1, 2, 3, \dots, n_1$ *and* $r = 1, 2, 3, \dots, N$,

(i):

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,-}_{2N+1}(\,\cdot\,)\right]_{t=-\frac{N+1-r}{N}} = N^{2(\beta-1)} M^{n,-}_{2N+1}(\,\cdot\,, (2N+1)(\beta-1)+r),$$

(ii):

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,-}_{2N+1}(\,\cdot\,)\right]_{t=\frac{N+1-r}{N}} = -\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,-}_{2N+1}(\,\cdot\,)\right]_{t=-\frac{N+1-r}{N}},$$

(iii):

$$\left[\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,-}_{2N+1}(\,\cdot\,)\right]_{t=0} = \mathbb{O},$$

(iv):

$$\left[\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,-}_{2N+1}(\,\cdot\,)\right]_{t=\pm\frac{N+1-r}{N}} = N^{2\beta-1} M^{n,-}_{2N+1}(\,\cdot\,, (2N+1)(\beta-1)+N+r),$$

(v):

$$\left[\frac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,-}_{2N+1}(\,\cdot\,)\right]_{t=0} \quad = \quad N^{2\beta-1} M^{n,-}_{2N+1}(\,\cdot\,,(2N+1)\beta).$$

Proof. The proof proceeds similarly as in the proof of Lemma 3.6 but by using Theorem 3.5 instead of Theorem 3.4. In fact, as done with (34) and (35) in Lemma 3.6, this can be done by

$$\begin{aligned}&\tfrac{d^{2(\beta-1)}}{dt^{2(\beta-1)}} Y^{n,-}_{2N+1}(j+1) = \\ &N^{2(\beta-1)}\Big[\textstyle\sum_{k=0}^{\beta-2} j(j-1)(j-2)\dots(j-(\beta-1-(k+1))) \\ &\cdot 2^{\beta-1-k}(Nt)^{2(j-(\beta-1-k))+1}\binom{\beta-1}{k} Z^k \eta_{j-(\beta-1-k)}(N^2Zt^2) \\ &+(Nt)^{2j+1} Z^{\beta-1}\eta_j(N^2Zt^2)\Big]\end{aligned}$$

and

$$\begin{aligned}&\tfrac{d^{2\beta-1}}{dt^{2\beta-1}} Y^{n,-}_{2N+1}(j+1) = \\ &N^{2\beta-1}\Big[\textstyle\sum_{k=0}^{\beta-2} j(j-1)(j-2)\dots(j-(\beta-1-(k+1))) \\ &\cdot 2^{\beta-1-k}(Nt)^{2(j-(\beta-1-k))}\binom{\beta-1}{k} Z^k \eta_{j-(\beta-1-k)-1}(N^2Zt^2) \\ &+(Nt)^{2j} Z^{\beta-1}\eta_{j-1}(N^2Zt^2)\Big].\end{aligned}$$

□

Now we are ready for obtaining first and higher-order derivatives of the coefficients $\alpha_{j,r}$ of $\mathcal{I}^n_{2N+1}$ as follows.

Theorem 3.8. *For $m,j=0,1,2,\dots,n$ and $q,r=-N,\,-N+1,\,-N+2,\,\dots,N$,*

$$\left[\frac{d^m}{dt^m}\alpha_{j,r}\right]_{t=\frac{q}{N}} = \begin{cases} N^m, & \text{if } j=m \text{ and } r=q, \\ 0, & \text{otherwise.}\end{cases}$$

Proof. To solve the two linear systems,

$$M^{n,+}_{2N+1}X^{n,+}_{2N+1} = Y^{n,+}_{2N+1} \text{ and } M^{n,-}_{2N+1}X^{n,-}_{2N+1} = Y^{n,-}_{2N+1}, \tag{39}$$

apply the Cramer's Rule to the linear systems, respectively, by using Theorem 3.4 and 3.5. Then the mth derivatives of $X^{n,+}_{2N+1}$ and $X^{n,-}_{2N+1}$, equivalently the mth derivatives of $\alpha^{\pm}_{j,k}$ and $\alpha_{j,0}$, with respect to t at $t=q/N$ are derived from Lemma 3.6 and 3.7 where $m,j=0,1,2,\dots,n$, $k=1,2,3,\dots,N$, and $q=-N,-N+1,\dots,N$. It is known that the determinant of a square matrix M is equal to 0 if two columns (or rows) of the matrix M are equal. This determinant property is used to derive the mth derivatives of $\alpha^{\pm}_{j,k}$ and $\alpha_{j,0}$ when the Cramer's Rule is applied. Finally, use (11) to get the mth derivative of $\alpha_{j,r}$ where $j=0,1,2,\dots,n$, $r=-N,-N+1,\dots,N$ and $r\neq 0$. For easy understanding, let us consider the case of $n=1$ and $N=1$ in (39). From Theorem 3.4 and (22), the first equation of (39), $M^{1,+}_3 X^{1,+}_3 = Y^{1,+}_3$, becomes

$$\begin{bmatrix} \eta_{-1}(Z) & 1 & -Z\eta_0(Z) \\ \eta_0(Z) & 0 & -(1\cdot 2\eta_0(Z)+Z\eta_1(Z)) \\ \eta_1(Z) & 0 & -(2\cdot 2\eta_1(Z)+Z\eta_2(Z)) \end{bmatrix} \begin{bmatrix} \alpha^+_{0,1} \\ \alpha_{0,0} \\ \alpha^-_{1,1} \end{bmatrix} = \begin{bmatrix} \eta_{-1}(Zt^2) \\ t^2\eta_0(Zt^2) \\ t^4\eta_1(Zt^2) \end{bmatrix}. \tag{40}$$

Lemma 3.6 says that

$$\begin{aligned} &(i) && \Big[Y_3^{1,+}(\,\cdot\,)\Big]_{t=\pm 1} = M_3^{1,+}(\,\cdot\,,1),\ (ii)\ \Big[Y_3^{1,+}(\,\cdot\,)\Big]_{t=0} = M_3^{1,+}(\,\cdot\,,2),\\ &(iii) && \Big[\tfrac{d}{dt}Y_3^{1,+}(\,\cdot\,)\Big]_{t=-1} = M_3^{1,+}(\,\cdot\,,3),\\ &(iv) && \Big[\tfrac{d}{dt}Y_3^{1,+}(\,\cdot\,)\Big]_{t=1} = -M_3^{1,+}(\,\cdot\,,3),\ (v)\ \Big[\tfrac{d}{dt}Y_3^{1,+}(\,\cdot\,)\Big]_{t=0} = \mathbb{O}. \end{aligned} \tag{41}$$

Then, by applying the Cramer's Rule to (40), we have

$$\begin{array}{lll} (i) & \alpha_{0,1}^{+} = 1 \text{ at } t=-1,1, & \\ & \alpha_{0,1}^{+} = 0 \text{ at } t=0, & \frac{d}{dt}\alpha_{0,1}^{+} = 0 \text{ at } t=-1,0,1,\\ (ii) & \alpha_{0,0} = 1 \text{ at } t=0, & \\ & \alpha_{0,0} = 0 \text{ at } t=-1,1, & \frac{d}{dt}\alpha_{0,0} = 0 \text{ at } t=-1,0,1,\\ (iii) & \frac{d}{dt}\alpha_{1,1}^{-} = 1 \text{ at } t=-1, & \frac{d}{dt}\alpha_{1,1}^{-} = -1 \text{ at } t=1,\\ & \alpha_{1,1}^{-} = 0 \text{ at } t=-1,0,1, & \frac{d}{dt}\alpha_{1,1}^{-} = 0 \text{ at } t=0. \end{array} \tag{42}$$

From Theorem 3.5 and (26), the second equation of (39), $M_3^{1,-}X_3^{1,-} = Y_3^{1,-}$, becomes

$$\begin{bmatrix} -\eta_0(Z) & \eta_{-1}(Z) & 1\\ -\eta_1(Z) & \eta_0(Z) & 0\\ -\eta_2(Z) & \eta_1(Z) & 0 \end{bmatrix} \begin{bmatrix} \alpha_{0,1}^{-}\\ \alpha_{1,1}^{+}\\ \alpha_{1,0} \end{bmatrix} = \begin{bmatrix} t\eta_0(Zt^2)\\ t^3\eta_1(Zt^2)\\ t^5\eta_2(Zt^2) \end{bmatrix}. \tag{43}$$

Lemma 3.7 says that

$$\begin{aligned} &(i) && \Big[Y_3^{1,-}(\,\cdot\,)\Big]_{t=-1} = M_3^{1,-}(\,\cdot\,,1),\ (ii)\ \Big[Y_3^{1,-}(\,\cdot\,)\Big]_{t=1} = -M_3^{1,-}(\,\cdot\,,1),\\ &(iii) && \Big[Y_3^{1,-}(\,\cdot\,)\Big]_{t=0} = \mathbb{O},\quad (iv)\ \Big[\tfrac{d}{dt}Y_3^{1,-}(\,\cdot\,)\Big]_{t=\pm 1} = M_3^{1,-}(\,\cdot\,,2),\\ &(v) && \Big[\tfrac{d}{dt}Y_3^{1,-}(\,\cdot\,)\Big]_{t=0} = M_3^{1,-}(\,\cdot\,,3). \end{aligned} \tag{44}$$

Then, by applying the Cramer's Rule to (43), we have

$$\begin{array}{lll} (i) & \alpha_{0,1}^{-} = 1 \text{ at } t=-1, & \alpha_{0,1}^{-} = -1 \text{ at } t=1,\\ & \alpha_{0,1}^{-} = 0 \text{ at } t=0, & \frac{d}{dt}\alpha_{0,1}^{-} = 0 \text{ at } t=-1,0,1,\\ (ii) & \frac{d}{dt}\alpha_{1,1}^{+} = 1 \text{ at } t=-1,1, & \\ & \alpha_{1,1}^{+} = 0 \text{ at } t=-1,0,1, & \frac{d}{dt}\alpha_{1,1}^{+} = 0 \text{ at } t=0,\\ (iii) & \frac{d}{dt}\alpha_{1,0} = 1 \text{ at } t=0, & \\ & \alpha_{1,0} = 0 \text{ at } t=-1,0,1, & \frac{d}{dt}\alpha_{1,0} = 0 \text{ at } t=-1,1. \end{array} \tag{45}$$

Using (11), we get

$$\begin{aligned} &\alpha_{0,-1} = (\alpha_{0,1}^{+} + \alpha_{0,1}^{-})/2, \quad \alpha_{0,1} = (\alpha_{0,1}^{+} - \alpha_{0,1}^{-})/2,\\ &\alpha_{1,-1} = (\alpha_{1,1}^{+} + \alpha_{1,1}^{-})/2, \quad \alpha_{1,1} = (\alpha_{1,1}^{+} - \alpha_{1,1}^{-})/2. \end{aligned} \tag{46}$$

Thus, (42) and (45) give

$$\begin{array}{lll} (i) & \alpha_{0,-1} = 1 \text{ at } t=-1, & \alpha_{0,-1} = 0 \text{ at } t=0,1,\\ & \alpha_{0,1} = 1 \text{ at } t=1, & \alpha_{0,1} = 0 \text{ at } t=-1,0,\\ (ii) & \alpha_{1,-1} = 0 \text{ at } t=-1,0,1, & \alpha_{1,1} = 0 \text{ at } t=-1,0,1 \end{array} \tag{47}$$

and

$$\begin{array}{llll} (i) & \frac{d}{dt}\alpha_{0,-1} = 0 \text{ at } t=-1,0,1, & \frac{d}{dt}\alpha_{0,1} = 0 \text{ at } t=-1,0,1, & \\ (ii) & \frac{d}{dt}\alpha_{1,-1} = 1 \text{ at } t=-1, & \frac{d}{dt}\alpha_{1,-1} = 0 \text{ at } t=0,1, & (48) \\ & \frac{d}{dt}\alpha_{1,1} = 1 \text{ at } t=1, & \frac{d}{dt}\alpha_{1,1} = 0 \text{ at } t=-1,0. & \end{array}$$

From (ii) of (42), (iii) of (45) and (47) and (48), we conclude that, for $m, j = 0, 1$ and $q, r = -1, 0, 1$,

$$\left[\frac{d^m}{dt^m}\alpha_{j,r}\right]_{t=q} = \begin{cases} 1, & \text{if } j=m \text{ and } r=q, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, other cases are proved. □

Corollary 3.9. *For $m = 0, 1, 2, \ldots, n$ and $q = -N, -N+1, -N+2, \ldots, N$,*

$$\left[\tfrac{d^m}{dx^m}\mathcal{I}^n_{2N+1}(t)\right]_{x=x_0+qh} = \left[\tfrac{d^m f(x)}{dx^m}\right]_{x=x_0+qh} \tag{49}$$

where $x = x_0 + Nht$.

Proof. Theorem 3.8 says that

$$\begin{aligned} &\left[\tfrac{d^m}{dx^m}\mathcal{I}^n_{2N+1}(t)\right]_{x=x_0+qh} \\ &= \left[\tfrac{d^m}{dt^m}\mathcal{I}^n_{2N+1}(t)\right]_{t=\frac{q}{N}} \cdot \tfrac{1}{(Nh)^m} \\ &= \left[\textstyle\sum_{j=0}^n h^j \left(\sum_{r=-N}^N \left[\tfrac{d^m}{dt^m}\alpha_{j,r}\right]_{t=\frac{q}{N}} f^{(j)}(x_0+rh)\right)\right] \cdot \tfrac{1}{(Nh)^m} \\ &= h^m \left(\textstyle\sum_{r=-N}^N \left[\tfrac{d^m}{dt^m}\alpha_{m,r}\right]_{t=\frac{q}{N}} f^{(m)}(x_0+rh)\right) \cdot \tfrac{1}{(Nh)^m} \\ &= h^m \left(\left[\tfrac{d^m}{dt^m}\alpha_{m,q}\right]_{t=\frac{q}{N}} f^{(m)}(x_0+qh)\right) \cdot \tfrac{1}{(Nh)^m} \\ &= h^m \left(N^m f^{(m)}(x_0+qh)\right) \cdot \tfrac{1}{(Nh)^m} \\ &= f^{(m)}(x_0+qh) = \left[\tfrac{d^m f(x)}{dx^m}\right]_{x=x_0+qh}. \end{aligned}$$

□

Corollary 3.9 states that the mth derivative of $\mathcal{I}^n_{2N+1}$ with respect to x is equal to the mth derivative of f with respect to x at the nodes where $m = 0, 1, 2, \ldots, n$.

4. Numerical results

To illustrate numerical results, let us choose an example function f on the domain $[a, b] = [-1, 1]$,

$$f(x) = (1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5)\sin(\omega x + 1), \quad \omega = 35, \tag{50}$$

which is of the form (1) with $f_1(x) = \sin(1)(1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5)$ and $f_2(x) = \cos(1)(1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5)$. For this function, we will investigate $\mathcal{I}^n_{2N+1}$ in (3) with the case of $N = 2$ and $n = 1$ when $x_0 = 0$ and $h = 1/2$. Let us denote it by $\mathcal{I}^1_5$. Thus, $\mathcal{I}^1_5$ is expressed by

$$\mathcal{I}^1_5(t) = \textstyle\sum_{r=-2}^2 \alpha_{0,r} f(r/2) + h \sum_{r=-2}^2 \alpha_{1,r} \left[\tfrac{df(x)}{dx}\right]_{x=r/2}. \tag{51}$$

FIGURE 1. Comparison between $f(x)$ and EF on $[-1, 1]$.

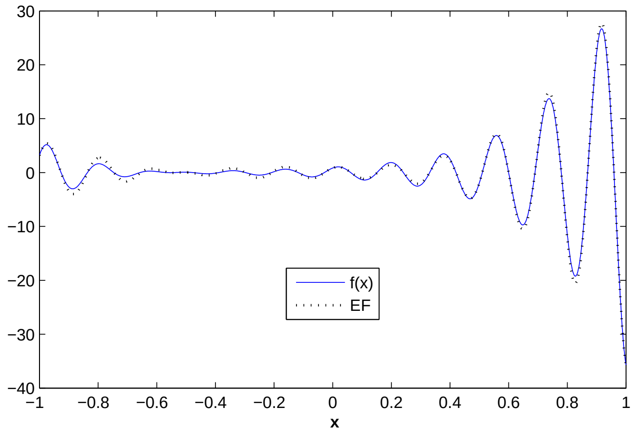
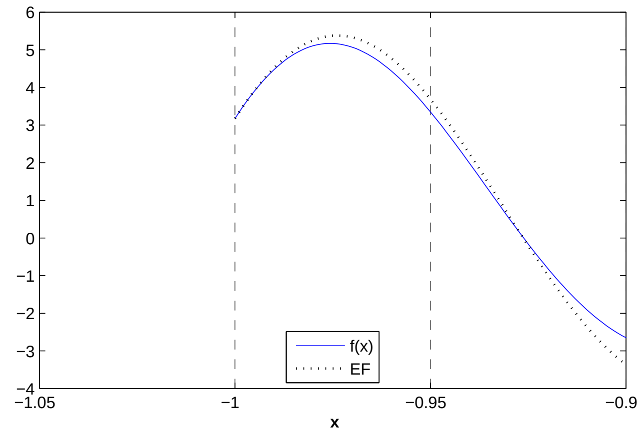


FIGURE 2. Comparison between $f(x)$ and EF near $x = -1$.



Note that $\mathcal{I}_5^1$ uses not only pointwise values of the function f but also of its first derivative at five nodes (equivalently, at $x = -1, -1/2, 0, 1/2, 1)$ on $[a, b] = [-1, 1]$. As explained in Section 2, $\mathcal{I}_5^1$ is constructed to be exact for $f(x) = x^m \exp(\pm i\omega x)$ where $m = 0, 1, 2, 3, 4$. Therefore, some discrepancies between the example function f and $\mathcal{I}_5^1$ may happen because f_1 and f_2 include $constant \times x^5$. The example function f is compared with $\mathcal{I}_5^1$ on the domain $[-1, 1]$ in Figure 1 where the solid and dotted lines are represented by f and $\mathcal{I}_5^1$(=EF), respectively. Obviously, Figure 1 shows some discrepancies between the two lines. Now, let us magnify the two lines in Figure 1 near $x = -1, -1/2$. Thus, Figure 2 and 3 are provided to compare the two lines near $x = -1$ and $-1/2$, respectively. In details, Fig. 2 shows that $\mathcal{I}_5^1$ has the same value as f at $x = -1$ and it does the same slope as f at the same node. The same argument as in Figure 2 is applied for Figure 3 at $x = -1/2$. In fact, it is numerically confirmed that $\mathcal{I}_5^1$ passes through f at the remaining nodes, $x = 0, 1/2, 1$, and it has the same slope as f at the same nodes. In other words, the behaviors of $\mathcal{I}_5^1$ at the nodes are in full agreement with the theoretical results given by (49). All computational results in the figures are obtained from Matlab[9].

In this article, we concentrated on obtaining theoretical results given by (49) regarding $\mathcal{I}_{2N+1}^n$ at the nodes. Finally, we may suggest another investigation to construct exponentially fitted interpolation formulas at unequally spaced nodes. Maybe our study will afford a good foundation for the investigation.

FIGURE 3. Comparison between $f(x)$ and EF near $x = -1/2$.

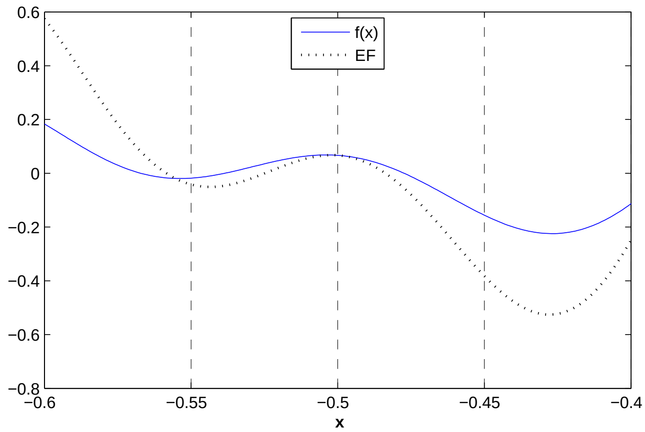


Figure Captions.

(1): The notation ‘EF’ in Figure $1-3$ is used to denote $\mathcal{I}_5^1$.
(2): In Figure 1-3, $\omega = 35$ and
$f(x) = (1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5)\sin(\omega x + 1)$.

Figure 1: Comparison between $f(x)$ and EF on $[-1, 1]$.
Figure 2: Comparison between $f(x)$ and EF near $x = -1$.
Figure 3: Comparison between $f(x)$ and EF near $x = -1/2$.

Appendix

1. Define functions η_s by

(i):

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0, \end{cases}$$

(ii):

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0, \end{cases}$$

(iii): for $Z \neq 0$ let

$$\eta_s(Z) = (\eta_{s-2}(Z) - (2s-1)\eta_{s-1}(Z))/Z \quad (s = 1, 2, 3, \ldots),$$

for $Z = 0$ let

$$\eta_s(0) = \frac{2^s s!}{(2s+1)!} \quad (s = 1, 2, 3, \ldots).$$

2. The functions defined above satisfy the following two properties.
(i) Power series:

$$\eta_s(Z) = 2^s \sum_{q=0}^{\infty} g_{sq} Z^q/(2q + 2s + 1)!$$

with

$$g_{sq} = \begin{cases} 1 & \text{if } s = 0 \\ (q+1)(q+2)\ldots(q+s) & \text{if } s = 1, 2, 3, \ldots. \end{cases}$$

(ii) Differentiation with respect to Z:

$$\frac{d}{dZ}\eta_s(Z) = \frac{1}{2}\eta_{s+1}(Z), \quad s = -1, 0, 1, \ldots.$$

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