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REMARKS ON NEIGHBORHOODS OF INDEPENDENT SETS AND (a, b, k)-CRITICAL GRAPHS†

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ABSTRACT. Let a and b be two even integers with $2 \leq a < b$, and let k be a nonnegative integer. Let G be a graph of order n with $n \geq \frac{(a+b-1)(a+b-2)+bk-2}{b}$. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, it is proved that G is an (a, b, k)-critical graph if

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 2}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b + bk - 3}{a+b-1}.$$

Furthermore, it is shown that the result in this paper is best possible in some sense.

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1. Introduction

The graphs considered in this article will be finite undirected graphs which have neither multiple edges nor loops. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For each $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. The minimum degree of G is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we write $N_G(S) = \cup_{x \in S} N_G(x)$. We denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. If $G[S]$ has no edges, then

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we call S independent. For two disjoint vertex subsets S and T of G, we use $e_G(S,T)$ to denote the number of edges from S to T.

Let a, b and k be nonnegative integers with $1 \leq a \leq b$. An $[a,b]$-factor of G is defined to be a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. If $a = b = r$, then an $[a,b]$-factor of G is called an r-factor of G. A graph G is called an (a,b,k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a,b]$-factor. If $a = b = r$, then an (a,b,k)-critical graph is simply called an (r,k)-critical graph. In particular, a $(1,k)$-critical graph is simply called a k-critical graph.

Many authors have investigated graph factors [1–7]. Liu and Yu [8] gave the characterization of (r,k)-critical graphs. Li [9] showed a degree condition for graphs to be (a,b,k)-critical graphs. Zhou [10–12] obtained some results on (a,b,k)-critical graphs. Liu and Liu [13] showed a neighborhood condition for the existence of (a,b,k)-critical graphs. Liu and Wang [14] obtained a necessary and sufficient condition for a graph to be an (a,b,k)-critical graph. The following result on k-factors and (a,b,k)-critical graphs are known.

Theorem 1.1 (Woodall [15]). *Let $k \geq 2$ be an integer and G a graph of order n with $n \geq 4k-6$. If k is odd, then n is even and G is connected. Let G satisfy*

$$|N_G(X)| \geq \frac{|X| + (k-1)n - 1}{2k-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) \geq \frac{k-1}{2k-1}(n+2).$$

Then G has a k-factor.

Theorem 1.2 (Zhou and Xu [11]). *Let a, b and k be nonnegative integers with $1 \leq a < b$, and let G be a graph of order n with $n \geq \frac{(a+b)(a+b-2)}{b} + k$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 1}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b + bk - 2}{a+b-1}.$$

Then G is an (a,b,k)-critical graph.

Zhou and Xu [11] also showed that the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$ in Theorem 1.2 cannot be replaced by $|N_G(X)| \geq \frac{(a-1)n+|X|+bk-1}{a+b-1}$. For the proof of the optimality (in this sense), they considered the case when $\frac{(a+b-1)^2}{2(a-1)}$ is an integer. Then they constructed a non (a,b,k)-critical graph G with $|N_G(X)| \geq \frac{(a-1)n+|X|+bk-1}{a+b-1}$. It is easy to see that in this case, either a is odd and b is even, or a is even and b is odd. Thus, the question is:

Is the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$ optimal in the other cases? (i.e., a and b have same parity).

In this paper, we study this question when the integers a and b are both even. In this case, we improve our previous result and obtain the following theorem. Furthermore, we use some new techniques in the proof of the main result.

Theorem 1.3. *Let a and b be two even integers with $2 \le a < b$ and k be a non-negative integer, and let G be a graph of order n with $n \ge \frac{(a+b-1)(a+b-2)+bk-2}{b}$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 2}{a+b-1} \tag{1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b + bk - 3}{a+b-1}. \tag{2}$$

Then G is an (a, b, k)-critical graph.

If $k = 0$ in Theorem 1.3, then we obtain the following corollary.

Corollary 1.4. *Let a and b be two even integers with $2 \le a < b$, and let G be a graph of order n with $n \ge \frac{(a+b-1)(a+b-2)-2}{b}$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| - 2}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b - 3}{a+b-1}.$$

Then G has an $[a, b]$-factor.

2. The Proof of Theorem 1.3

Let a and b be two positive integers with $a < b$, and let G be a graph. For any $S \subseteq V(G)$, define

$$d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$$

and

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T|,$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le a - 1\}$. In the following, we define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

Obviously, $0 \le h \le a - 1$.

The following lemmas are applied in the proof of Theorem 1.3.

Lemma 2.1 (Liu and Wang [14]). *Let a, b and k be nonnegative integers with $1 \le a < b$, and let G be a graph of order $n \ge a+k+1$. Then G is (a,b,k)-critical if and only if for any $S \subseteq V(G)$ with $|S| \ge k$*

$$\delta_G(S,T) \ge bk,$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le a-1\}$.

Lemma 2.2 (Zhou, Xu and Wu [12]). *Let a and b be two even integers with $2 \le a < b$, and let k be a nonnegative integer. Let G be a graph of order n. If $\delta_G(S,T) \le bk-1$ for some $S \subseteq V(G)$, then $|S| \le \frac{(a-h)n+bk-2}{a+b-h}$.*

In the following, we prove Theorem 1.3.

Proof. Suppose that G satisfies the hypothesis of Theorem 1.3, but is not an (a,b,k)-critical graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ with $|S| \ge k$ such that

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \le bk-1, \tag{3}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le a-1\}$. Obviously, $T \ne \emptyset$ by (3). Let h be as in the previous, and $0 \le h \le a-1$. We choose $x_1 \in T$ such that $d_{G-S}(x_1) = h$. Then the following inequalities hold.

$$\delta(G) \le d_G(x_1) \le d_{G-S}(x_1) + |S| = h + |S|,$$

which implies

$$|S| \ge \delta(G) - h. \tag{4}$$

In view of (2) and (4), we obtain

$$|S| \ge \delta(G) - h > \frac{(a-1)n+a+b+bk-3}{a+b-1} - h. \tag{5}$$

We shall consider various cases by the value of h and derive a contradiction in each case.

Case 1. $h = 0$.

Set $X = \{x \in T : d_{G-S}(x) = 0\}$. Clearly, $X \ne \emptyset$ and X is independent. Thus, by (1) we have

$$\frac{(a-1)n+|X|+bk-2}{a+b-1} < |N_G(X)| \le |S|. \tag{6}$$

Subcase 1.1. $|S| + |T| \le n-1$.

According to (3), $|S| + |T| \le n-1$ and $a \ge 2$, we obtain

$$\begin{aligned} bk-1 &\ge \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &\ge b|S| + |T| - |X| - a|T| \\ &= b|S| - (a-1)|T| - |X| \\ &\ge b|S| - (a-1)(n-1-|S|) - |X| \\ &= (a+b-1)|S| - |X| - (a-1)n + a - 1 \\ &\ge (a+b-1)|S| - |X| - (a-1)n + 1, \end{aligned}$$

which implies

$$|S| \leq \frac{(a-1)n + |X| + bk - 2}{a+b-1},$$

which contradicts (6).

Subcase 1.2. $|S| + |T| = n$.

Using (3), (6) and $|S| + |T| = n$, we obtain

$$\begin{aligned} bk - 1 &\geq \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &\geq b|S| + |T| - |X| - a|T| \\ &= b|S| - (a-1)|T| - |X| \\ &= b|S| - (a-1)(n - |S|) - |X| \\ &= (a+b-1)|S| - (a-1)n - |X| \\ &\geq (a+b-1) \cdot \frac{(a-1)n + |X| + bk - 1}{a+b-1} - (a-1)n - |X| \\ &= bk - 1, \end{aligned}$$

which implies

$$d_{G-S}(T) = |T| - |X| \tag{7}$$

and

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| = bk - 1. \tag{8}$$

Claim 1. $d_{G-S}(T)$ is even.

Proof. Obviously, $X \subseteq T$. If $|X| = |T|$, then from (7) we have $d_{G-S}(T) = 0$. In the following we assume that $|X| < |T|$. In terms of (7) and the definition of X, we have $d_{G-S}(v) = 1$ for any $v \in T \setminus X$. Combining this with $|S| + |T| = n$, we obtain $d_{G[T\setminus X]}(v) = 1$ for any $v \in T \setminus X$, and so $G[T \setminus X]$ is a perfect matching. Hence, $|T| - |X|$ is even. In view of (7), $d_{G-S}(T)$ is even. This completes the proof of Claim 1. □

According to Claim 1 and $a - b \equiv 0 \pmod 2$, we have $\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T|$ is even. Which contradicts (8).

Case 2. $1 \leq h \leq a - 1$.

According to (3) and Lemma 2.2, we have

$$|S| \leq \frac{(a-h)n + bk - 2}{a+b-h}. \tag{9}$$

Using (5) and (9), we obtain

$$\frac{(a-h)n + bk - 2}{a+b-h} > \frac{(a-1)n + a + b + bk - 3}{a+b-1} - h. \tag{10}$$

If $h = 1$, then by (10) we have $\frac{(a-1)n+bk-2}{a+b-1} > \frac{(a-1)n+a+b+bk-3}{a+b-1} - 1 = \frac{(a-1)n+bk-2}{a+b-1}$. That is a contradiction. In the following, we assume that $2 \leq h \leq a - 1$.

If the left-hand and right-hand sides of (10) are denoted by A and B respectively, then (10) says that

$$A - B > 0. \tag{11}$$

Multiplying (11) by $(a+b-1)(a+b-h)$ and rearranging, we obtain

$$\begin{aligned} 0 &< (a+b-1)(a+b-h)(A-B) \\ &= (a+b-1)(a+b-h)(\frac{(a-h)n+bk-2}{a+b-h} \\ &\quad -\frac{(a-1)n+a+b+bk-3}{a+b-1}+h) \\ &= -(h-1)(bn-(a+b-1)(a+b-h)-bk+2). \end{aligned}$$

Combining this with $2 \le h \le a-1$, we have

$$n < \frac{(a+b-1)(a+b-h)+bk-2}{b} \le \frac{(a+b-1)(a+b-2)+bk-2}{b},$$

which contradicts that $n \ge \frac{(a+b-1)(a+b-2)+bk-2}{b}$.

From the contradictions we deduce that G is an (a,b,k)-critical graph. This Completes the proof of Theorem 1.3. □

Remark 2.1. Let $b > a \ge 2$ be two even integers such that $\frac{(a+b-1)(a+b-2)}{2(a-1)}$ is an integer, and let k be a nonnegative integer. We write $n = \frac{(a+b-1)(a+b-2)}{a-1}+k$. It is easy to see that n is an integer. In the following, let us show that the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-2}{a+b-1}$ in Theorem 1.3 can not be replaced by $|N_G(X)| \ge \frac{(a-1)n+|X|+bk-2}{a+b-1}$. We can show this by constructing a graph $G = K_{a+b+k-1} \bigvee ((a+b+1)K_1 \cup (\frac{(a+b-1)(a+b-2)}{2(a-1)}-(a+b))K_2)$. Let $X = V((a+b+1)K_1)$. Then $\delta(G) = a+b+k-1 > \frac{(a-1)n+a+b+bk-3}{a+b-1}$ and $|N_G(X)| = a+b+k-1 = \frac{(a-1)n+|X|+bk-2}{a+b-1}$, and it is easy to see from this that $|N_G(X)| \ge \frac{(a-1)n+|X|+bk-2}{a+b-1}$ for every non-empty independent subset X of $V(G)$. Let $S = V(K_{a+b+k-1}) \subseteq V(G)$, $T = V((a+b+1)K_1 \cup (\frac{(a+b-1)(a+b-2)}{2(a-1)}-(a+b))K_2) \subseteq V(G)$. Then $|S| = a+b+k-1$, $|T| = \frac{(a+b-1)(a+b-2)}{(a-1)}-(a+b)+1$, and $d_{G-S}(T) = \frac{(a+b-1)(a+b-2)}{(a-1)}-2(a+b)$. Thus, we get

$$\begin{aligned} \delta_G(S,T) &= b|S|+d_{G-S}(T)-a|T| \\ &= b(a+b+k-1)+\frac{(a+b-1)(a+b-2)}{(a-1)}-2(a+b) \\ &\quad -a(\frac{(a+b-1)(a+b-2)}{(a-1)}-(a+b)+1) \\ &= bk-2 < bk. \end{aligned}$$

According to Lemma 2.1, G is not an (a,b,k)-critical graph. In the above sense, the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-2}{a+b-1}$ in Theorem 1.3 is best possible.

Remark 2.2. Zhou and Xu [11] proved Theorem 1.2, and showed that the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$ is sharp when either a is odd and b is even, or a is even and b is odd. In this paper, we improve the condition by

$|N_G(X)| > \frac{(a-1)n+|X|+bk-2}{a+b-1}$ when a and b are both even, and show the condition in this case is sharp. Thus, we present the following problem:

Open Problem. Let a, b and k be three nonnegative integers such that $1 \leq a < b$, a and b are both odd. Suppose that n is sufficiently large for a, b and k, $\delta(G) > \frac{(a-1)n+a+b+bk-3}{a+b-1}$, and $|N_G(X)| > \frac{(a-1)n+|X|+bk-2}{a+b-1}$. Then, whether a graph G of order n is (a, b, k)-critical or not?

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