J. Appl. Math. & Informatics Vol. **31***(2013), No. 5 - 6, pp. 651 - 659*
http://dx.doi.org/10.14317/jami.2013.651

CUBIC s-REGULAR GRAPHS OF ORDER $12p$, $36p$, $44p$, $52p$, $66p$, $68p$ AND $76p$†

JU-MOK OH

ABSTRACT. A graph is *s-regular* if its automorphism group acts regularly on the set of its s-arcs. In this paper, the cubic s-regular graphs of order $12p$, $36p$, $44p$, $52p$, $66p$, $68p$ and $76p$ are classified for each $s \geq 1$ and each prime p. The number of cubic s-regular graphs of order $12p$, $36p$, $44p$, $52p$, $66p$, $68p$ and $76p$ is 4, 3, 7, 8, 1, 4 and 1, respectively. As a partial result, we determine all cubic s-regular graphs of order $70p$ except for $p = 31, 41$.

AMS Mathematics Subject Classification : 05C25, 20B25.
Key words and phrases : cubic s-regular graph.

1. Introduction

Throughout this paper, graphs are finite, simple, undirected and connected. For a graph X, let $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ denote the vertex set, the edge set and the full automorphism group of X, respectively. An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. A 0-arc-transitive graph is called *vertex-transitive*, and a 1-arc-transitive graph is called *arc-transitive* or *symmetric*. A subgroup of the automorphism group of a graph X is said to be *s-regular* if it acts regularly on the set of s-arcs of X. In particular, if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of X then X is said to be *s-regular*. Thus, if a graph X is s-regular then $\mathrm{Aut}(X)$ is transitive on the set of s-arcs and the only automorphism fixing an s-arc is the identity automorphism of X.

Tutte [18] showed that every finite cubic symmetric graph is s-regular for some $s \geq 1$, and that this s is at most five. It follows that a connected cubic symmetric graph of order n is s-regular if and only if the order of its automorphism group

Received May 20, 2012. Revised April 18, 2013. Accepted May 25, 2013.
†This work was supported by the Korea Research Foundation Grant funded by the Korean Government(KRF-2009-353-C00040).

is $n \cdot 3 \cdot 2^{s-1}$. Conder and Dobcsányi [3, 4] classified the cubic s-regular graphs up to order 2048. Cheng and Oxley [2] classified the cubic s-regular graphs of order $2p$ (in fact, they classified the symmetric graphs of order $2p$ with any valency). Feng et al. [7, 8, 9, 10, 11, 12] classified the cubic s-regular graphs with order $2p^2$, $2p^3$, $4p^i$, $6p^i$, $8p^i$ and $10p^i$ for any prime p and each $i = 1, 2$. Using those results, Feng and Zhou [13] completed the classification of cubic s-regular graphs of order $2pq$ for any primes p and q. The author [14, 15] classified the cubic s-regular graphs of order $16p$ and $18p$ for any prime p.

2. Main results

Lemma 2.1. *Let p, q and r be primes.*

(1) *If a and b be non-negative integers, then every group of order $p^a q^b$ is solvable* [17, Theorem 8.5.3].
(2) *Every group of order pqr is solvable* [17, Theorem 5.4.1].
(3) *Every finite group of odd order is solvable* [6, Feit-Thompson Theorem].

Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. Denote by $\underline{X}$ the quotient graph corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in $\underline{X}$ whenever there is an edge between those orbits in X.

Lemma 2.2 (Theorem 9, [16]). *Let X be a connected symmetric graph of prime valency and G an s-arc-transitive subgroup of* $\mathrm{Aut}(X)$ *for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-arc-transitive subgroup of* $\mathrm{Aut}(\underline{X})$ *where $\underline{X}$ is the quotient graph of X corresponding to the orbits of N.*

Because we only deal with simple connected graphs, the smallest cubic symmetric graph is the complete graph K_4 of order 4. The first few orders of cubic symmetric graphs are 4, 6, 8, 10, 14, 16, 18, 20, 24, 26, 28, 30, 32, 38, 40, 42, 48, 50, 54, 56, 56, 56, 60, 62, 64, 72, 74, 78 and 80 (see [4]). Hence there do not exist cubic symmetric graphs of order 12, 22, 34, 36, 44, 46, 52, 58, 66, 68, 70 and 76. This fact is used to prove Theorem 2.4. In view of Lemma 2.2, one might see that there exist only finitely many cubic symmetric graphs of order $12p$, $22p$, $34p$, $36p$, $44p$, $46p$, $52p$, $58p$, $66p$, $68p$, $70p$ and $76p$. Since the cases for $22p$, $34p$, $46p$ and $58p$ were already treated by Feng and Zhou [13] (these cases can be also done by a similar method to that described in the proof Theorem 2.4), we only consider the remaining cases in this paper.

By [3, 4], we have the following lemma.

Lemma 2.3. *Let p be a prime. Let X be a cubic symmetric graph.*

(1) *If X has order $12p$ and $p \leq 71$, then X is isomorphic to one of the graphs in Table 1.*
(2) *If X has order $36p$ and $p \leq 53$, then X is isomorphic to one of the graphs in Table 2.*

(3) *If X has order $44p$ and $p \leq 43$, then X is isomorphic to one of the graphs in Table* 3.
(4) *If X has order $52p$ and $p \leq 37$, then X is isomorphic to one of the graphs in Table* 4.
(5) *If X has order $66p$ and $p \leq 31$, then X is isomorphic to the graph in Table* 5.
(6) *If X has order $68p$ and $p \leq 29$, then X is isomorphic to one of the graphs in Table* 6.
(7) *If X has order $70p$ and $p \leq 29$, then X is isomorphic to one of the graphs in Table* 7.
(8) *If X has order $76p$ and $p \leq 23$, then X is isomorphic to the graph in Table* 8.

TABLE 1. Cubic symmetric graphs of order $12p$ with $p \leq 71$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{24}	$12 \cdot 2 = 24$	2	6	4	*Yes*
F_{60}	$12 \cdot 5 = 60$	2	9	5	*No*
F_{84}	$12 \cdot 7 = 84$	2	7	7	*No*
F_{204}	$12 \cdot 17 = 204$	4	12	9	*Yes*

TABLE 2. Cubic symmetric graphs of order $36p$ with $p \leq 53$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{72}	$36 \cdot 2 = 72$	2	6	8	*Yes*
F_{108}	$36 \cdot 3 = 108$	2	9	7	*No*
F_{468}	$36 \cdot 13 = 468$	5	12	13	*Yes*

TABLE 3. Cubic symmetric graphs of order $44p$ with $p \leq 43$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{220A}	$44 \cdot 5$	2	10	9	*Yes*
F_{220B}	$44 \cdot 5$	2	10	9	*No*
F_{220C}	$44 \cdot 5$	3	10	10	*Yes*
F_{1012A}	$44 \cdot 23$	2	11	11	*No*
F_{1012B}	$44 \cdot 23$	3	16	12	*Yes*
F_{1012C}	$44 \cdot 23$	3	11	11	*No*
F_{1012D}	$44 \cdot 23$	3	11	11	*No*

The following is the main result of this paper.

Theorem 2.4. *Let p be a prime. Let X be a cubic symmetric graph.*

Table 4. Cubic symmetric graphs of order $52p$ with $p \leq 37$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{104}	$52 \cdot 2$	1	6	9	*Yes*
F_{364A}	$52 \cdot 7$	2	7	12	*No*
F_{364B}	$52 \cdot 7$	2	7	11	*No*
F_{364C}	$52 \cdot 7$	2	12	10	*Yes*
F_{364D}	$52 \cdot 7$	2	12	9	*No*
F_{364E}	$52 \cdot 7$	2	12	9	*Yes*
F_{364F}	$52 \cdot 7$	2	7	13	*No*
F_{364G}	$52 \cdot 7$	3	12	12	*Yes*

Table 5. Cubic symmetric graphs of order $66p$ with $p \leq 31$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{726}	$66 \cdot 11$	2	6	22	*Yes*

Table 6. Cubic symmetric graphs of order $68p$ with $p \leq 29$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{204}	$68 \cdot 3$	4	12	9	*Yes*
F_{240A}	$68 \cdot 5$	2	9	10	*No*
F_{240B}	$68 \cdot 5$	2	10	11	*Yes*
F_{240C}	$68 \cdot 5$	2	8	10	*Yes*

Table 7. Cubic symmetric graphs of order $70p$ with $p \leq 29$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{350}	$70 \cdot 5$	1	6	17	*Yes*
F_{2030A}	$70 \cdot 29$	2	7	16	*No*
F_{2030B}	$70 \cdot 29$	2	10	13	*No*
F_{2030C}	$70 \cdot 29$	3	15	15	*No*

(1) *If X has order $12p$, then X is isomorphic to the 2-regular graphs F_{24}, F_{60}, F_{84} or the 4-regular graph F_{204}.*
(2) *If X has order $36p$, then X is isomorphic to the 2-regular graphs F_{72}, F_{108} or the 5-regular graph F_{468}.*
(3) *If X has order $44p$, then X is isomorphic to the 2-regular graphs F_{220A}, F_{220B}, F_{1012A} or the 3-regular graphs F_{220C}, F_{1012B}, F_{1012C}, F_{1012D}.*
(4) *If X has order $52p$, then X is isomorphic to the 1-regular graph F_{104}, the 2-regular graphs F_{364A}, F_{364B},F_{364C},F_{364D},F_{364E},F_{364F} or the 3-regular graph F_{364G}.*
(5) *If X has order $66p$, then X is isomorphic to the 2-regular graph F_{726}.*

TABLE 8. Cubic symmetric graphs of order $76p$ with $p \leq 23$

Graph	*Order*	*s-regular*	*Girth*	*Diameter*	*Bipartite?*
F_{152}	$76 \cdot 2$	1	6	11	*Yes*

(6) *If X has order $68p$, then X is isomorphic to the 2-regular graphs F_{240A}, F_{240B}, F_{240C} or the 4-regular graph F_{204}.*
(7) *If X has order $70p$ with $p \neq 31, 41$, then X is isomorphic to the 1-regular graph F_{350}, the 2-regular graphs F_{2030A}, F_{2030B} or the 3-regular graph F_{2030C}.*
(8) *If X has order $76p$, then X is isomorphic to the 1-regular graph F_{152}.*

Remark 2.1. If there exists a cubic symmetric graph X of order $70 \cdot 31 = 2170$ and $70 \cdot 41 = 2870$, respectively, then by following the proof of Theorem 2.4(7), one can see that X is 5-regular and 3-arc-transitive whose automorphism group contains $\mathrm{PSL}_2(31)$ and $\mathrm{PSL}_2(41)$ as a minimal normal subgroup. The author could not determine their existence.

Proof. By Lemma 2.3, it suffices to prove that there does not exist a cubic symmetric graph of order $12p$, $36p$, $44p$, $52p$, $66p$, $68p$, $70p$ and $76p$ if $p > 71$, 53, 43, 37, 31, 29, $(p > 31, p \neq 41)$ and 23, respectively. Throughout the proof we let $A := \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A and $N_A(P)$ by the normalizer of P in A. Then by Sylow theorem the number of Sylow p-subgroups of A is $np + 1 = |A : N_A(P)|$ for some non-negative integer n. By Tutte [18] X is at most 5-regular, and hence $|A|$ is a divisor of $3 \cdot 2^4 \cdot |V(X)| = 48 \cdot |V(X)|$.
(1) Suppose that there exists a cubic symmetric graph X of order $12p$ with $p > 71$. If P is normal in A, by Lemma 2.2 the quotient graph of X corresponding to the orbits of P is a cubic symmetric graph of order 12, which is impossible by [4]. Thus P is not normal in A. Since $|A|$ is a divisor of $48 \cdot 12p$, $np + 1$ is a divisor of $48 \cdot 12 = 2^6 \cdot 3^2$. Furthermore, since $np + 1 \geq 74$, we have $(n, p) = (1, 191)$ and $np + 1 = 2^6 \cdot 3$. This implies that X is 5-regular. Let M be a minimal normal subgroup of A and let $\underline{X}$ be the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian then $\underline{X}$ is 5-regular with order $4 \cdot 191$ or $6 \cdot 191$, which is impossible by [3]. Thus $M = T_1 \times T_2 \times \cdots \times T_t$ where T_i $(1 \leq i \leq t)$ are isomorphic non-abelian simple groups. By Lemma 2.1, $|T_i|$ has at least three prime factors and even order. Notice that $|A|$ is a divisor of $2^6 \cdot 3^2 \cdot 191$. Then $t = 1$ and M is a non-abelian simple group. Thus M has order $2^{\ell_1} \cdot 3^{\ell_2} \cdot 191$ for some $1 \leq \ell_1 \leq 6$ and $1 \leq \ell_2 \leq 2$. However, there is no simple group with such orders (see [5]).
(2) Suppose that there exists a cubic symmetric graph X of order $36p$ with $p > 53$. Since there is no cubic symmetric graph of order 36, P is not normal in A. Since $|A|$ is a divisor of $48 \cdot 36p$, $np + 1 \mid 48 \cdot 36$. Furthermore, since $np+1 \geq 60$, we have $(n, p) = (1, 71), (1, 107), (11, 157), (1, 191), (1, 431), (1, 863)$ and $np+1 = 2^3 \cdot 3^2$, $2^2 \cdot 3^3$, $2^6 \cdot 3^3$, $2^6 \cdot 3$, $2^4 \cdot 3^3$, $2^5 \cdot 3^3$. This implies that if $p \neq 107$,

then X is at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian, then $\underline{X}$ is symmetric with order $4p$, $12p$ or $18p$ where $p = 71$, 107, 157, 191, 431, 863 (in particular, if $p \neq 107$, by Lemma 2.2 $\underline{X}$ is at least 2-arc-transitive), which is impossible by [9, Theorem 6.2], (1) and [15, Theorem 2.4]. Thus $M = T_1 \times T_2 \times \cdots \times T_t$ where T_i $(1 \leq i \leq t)$ are isomorphic non-abelian simple groups. By Lemma 2.1, $|T_i|$ has at least three prime factors and even order. Since $|A|$ is a divisor of $2^6 \cdot 3^3 p$, $t = 1$ and M is a non-abelian simple group. Thus M has order $2^{\ell_1} \cdot 3^{\ell_2} p$ where $1 \leq \ell_1 \leq 6$, $1 \leq \ell_2 \leq 3$ and $p = 71, 107, 157, 191, 431, 863$. However, there is no simple group with such orders (see [5]).

(3) Suppose that there exists a cubic symmetric graph X of order $44p$ with $p > 43$. Since there is no cubic symmetric graph of order 44, P is not normal in A. Since $np + 1 \geq 48$ and $np + 1 \mid 2^6 \cdot 3 \cdot 11$, we have $(n, p) = (1, 47)$, $(1, 131)$, $(1, 191)$, $(5, 211)$, $(1, 263)$, $(1, 2111)$ and $np + 1 = 2^4 \cdot 3, 2^2 \cdot 3 \cdot 11, 2^6 \cdot 3, 2^5 \cdot 3 \cdot 11$, $2^3 \cdot 3 \cdot 11$, $2^6 \cdot 3 \cdot 11$. This implies that if $p \neq 131$, X is at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian, then $\underline{X}$ is symmetric with order $4p$ or $22p$ where $p = 47, 131, 191, 211, 263,$ 2111 (in particular, if $p \neq 131$, $\underline{X}$ is at least 2-arc-transitive), which is impossible by [9, Theorem 6.2] and [13]. Thus, like in the proof of (2), one can see that M is a non-abelian simple group and has order $2^\ell \cdot 3 \cdot 11$, $2^\ell \cdot 11p$, $2^\ell \cdot 11p$ or $2^\ell \cdot 3 \cdot 11p$ where $1 \leq \ell \leq 6$ and $p = 47$, 131, 191, 211, 263, 2111. However, there is no simple group with such orders (see [5]).

(4) Suppose that there exists a cubic symmetric graph X of order $52p$ with $p \geq 41$. Since there is no cubic symmetric graph of order 52, P is not normal in A. Since $np + 1 \geq 42$ and $np + 1 \mid 2^6 \cdot 3 \cdot 13$, we have $(n, p) = (29, 43)$, $(1, 47)$, $(5, 83)$, $(7, 89)$, $(1, 103)$, $(1, 191)$, $(3, 277)$, $(1, 311)$, $(5, 499)$ and $np+1 = 2^5 \cdot 3 \cdot 13$, $2^4 \cdot 3$, $2^5 \cdot 13$, $2^4 \cdot 3 \cdot 13$, $2^3 \cdot 13$, $2^6 \cdot 3$, $2^6 \cdot 13$, $2^3 \cdot 3 \cdot 13$, $2^6 \cdot 3 \cdot 13$. This implies that X is at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian, then $\underline{X}$ is 2-arc-transitive with order $4p$ or $26p$ where $p = 43$, 47, 83, 87, 103, 191, 277, 311, 499, which is impossible by [9, Theorem 6.2] and [13]. Thus, like in the proof of (2), M is a non-abelian simple group and has order $2^\ell \cdot 3 \cdot 13$, $2^\ell \cdot 3p$, $2^\ell \cdot 13p$ or $2^\ell \cdot 3 \cdot 13p$ where $1 \leq \ell \leq 6$ and $p = 43$, 47, 83, 87, 103, 191, 277, 311, 499. However, there is no simple group with such orders (see [5]).

(5) Suppose that there exists a cubic symmetric graph X of order $66p$ with $p \geq 37$. Since there is no cubic symmetric graph of order 66, P is not normal in A. Since $np+1 \geq 38$ and $np+1 \mid 2^5 \cdot 3^2 \cdot 11$, we have $(n, p) = (7, 41)$, $(1, 43)$, $(1, 47)$, $(1, 71)$, $(5, 79)$, $(7, 113)$, $(1, 131)$, $(1, 197)$, $(5, 211)$, $(1, 263)$, (1583), $(1, 3167)$ and $np + 1 = 2^5 \cdot 3^2$, $2^2 \cdot 11$, $2^4 \cdot 3$, $2^3 \cdot 3^2$, $2^2 \cdot 3^2 \cdot 11$, $2^3 \cdot 3^2 \cdot 11$, $2^2 \cdot 3 \cdot 11$, $2 \cdot 3^2 \cdot 11$, $2^5 \cdot 3 \cdot 11$, $2^3 \cdot 3 \cdot 11$, $2^4 \cdot 3^2 \cdot 11$, $2^5 \cdot 3^2 \cdot 11$. This implies that if $p \neq 197$, X is

at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian, then $\underline{X}$ is symmetric with order $6p$ or $22p$ where $p = 41$, 43, 47, 71, 79, 113, 131, 197, 211, 263, 1583, 3167 (in particular, if $p \neq 197$, then $\underline{X}$ is 2-arc-transitive), which is impossible by [9, Theorem 5.2] and [13]. Thus, like in the proof of (2), M is a non-abelian simple group and has order $2^{\ell_1} \cdot 3^{\ell_2} \cdot 11$, $2^{\ell_1} \cdot 3^{\ell_2} p$, $2^{\ell_1} \cdot 11p$ or $2^{\ell_1} \cdot 3^{\ell_3} \cdot 11p$ where $1 \leq \ell_1 \leq 5$, $1 \leq \ell_2 \leq 2$ and $p = 41$, 43, 47, 71, 79, 113, 131, 197, 211, 263, 1583, 3167. However, there is no simple group with such orders (see [5]).

(6) Suppose that there exists a cubic symmetric graph X of order $68p$ with $p \geq 31$. Since there is no cubic symmetric graph of order 68, P is not normal in A. Since $np + 1 \geq 32$ and $np + 1 \mid 2^6 \cdot 3 \cdot 17$, we have $(n, p) = (1, 31)$, $(11, 37)$, $(1, 47)$, $(1, 67)$, $(1, 101)$, $(5, 163)$, $(3, 181)$, $(1, 191)$, $(7, 233)$, $(13, 251)$, $(1, 271)$, $(1, 1087)$ and $np + 1 = 2^5$, $2^3 \cdot 3 \cdot 17$, $2^4 \cdot 3$, $2^2 \cdot 17$, $2 \cdot 3 \cdot 17$, $2^4 \cdot 3 \cdot 17$, $2^5 \cdot 17$, $2^6 \cdot 3$, $2^5 \cdot 3 \cdot 17$, $2^6 \cdot 3 \cdot 17$, $2^4 \cdot 17$, $2^6 \cdot 17$. This implies that if $p \neq 67$, 101, X is at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian, then $\underline{X}$ is symmetric with order $4p$ or $34p$ where $p = 31$, 37, 47, 67, 101, 163, 181, 191, 233, 251, 271, 1087 (in particular, if $p \neq 67$, 101, then $\underline{X}$ is 2-arc-transitive), which is impossible by [9, Theorem 6.2] and [13]. Thus, like in the proof of (2), M is a non-abelian simple group and has order $2^\ell \cdot 3 \cdot 17$, $2^\ell \cdot 3p$, $2^\ell \cdot 17p$ or $2^\ell \cdot 3 \cdot 17p$ where $1 \leq \ell \leq 6$ and $p = 31$, 37, 47, 67, 101, 163, 181, 191, 233, 251, 271, 1087. However, there is no simple group with such orders (see [5]).

(7) Suppose that there exists a cubic symmetric graph X of order $70p$ with $p > 31$ and $p \neq 41$. Since there is no cubic symmetric graph of order 70, P is not normal in A. Since $np+1 \geq 38$ and $np+1 \mid 2^5 \cdot 3 \cdot 5 \cdot 7$, we have $(n, p) = (3, 37)$, $(13, 43)$, $(1, 47)$, $(3, 53)$, $(1, 59)$, $(11, 61)$, $(5, 67)$, $(23, 73)$, $(1, 79)$, $(1, 83)$, $(1, 139)$, $(1, 167)$, $(1, 223)$, $(1, 239)$, $(3, 373)$, $(1, 419)$, $(1, 479)$, $(1, 839)$, $(1, 3359)$ and $np+1 = 2^4 \cdot 7$, $2^4 \cdot 5 \cdot 7$, $2^4 \cdot 3$, $2^5 \cdot 5$, $2^2 \cdot 3 \cdot 5$, $2^5 \cdot 3 \cdot 7$, $2^4 \cdot 3 \cdot 7$, $2^4 \cdot 3 \cdot 5 \cdot 7$, $2^4 \cdot 5$, $2^2 \cdot 3 \cdot 7$, $2^2 \cdot 5 \cdot 7$, $2^3 \cdot 3 \cdot 7$, $2^5 \cdot 7$, $2^4 \cdot 3 \cdot 5$, $2^5 \cdot 5 \cdot 7$, $2^2 \cdot 3 \cdot 5 \cdot 7$, $2^5 \cdot 3 \cdot 5$, $2^3 \cdot 3 \cdot 5 \cdot 7$, $2^5 \cdot 3 \cdot 5 \cdot 7$. This implies that X is at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian, then $\underline{X}$ is 2-arc-transitive with order $10p$ or $14p$ where $p = 37$, 43, 47, 53, 59, 61, 67, 73, 79, 83, 139, 167, 223, 239, 373, 419, 479, 839, 3359, which is impossible by [8, Theorem 5.1] and [13]. Thus, like in the proof of (2), M is a non-abelian simple group and has order $2^\ell \cdot 3 \cdot 5$, $2^\ell \cdot 3 \cdot 7$, $2^\ell \cdot 3p$, $2^\ell \cdot 5 \cdot 7$, $2^\ell \cdot 5p$, $2^\ell \cdot 7p$, $2^\ell \cdot 3 \cdot 5 \cdot 7$, $2^\ell \cdot 3 \cdot 5p$, $2^\ell \cdot 3 \cdot 7p$, $2^\ell \cdot 5 \cdot 7p$ or $2^\ell \cdot 3 \cdot 5 \cdot 7p$ where $1 \leq \ell \leq 5$ and $p = 37$, 43, 47, 53, 59, 61, 67, 73, 79, 83, 139, 167, 223, 239, 373, 419, 479, 839, 3359. By checking the orders of finite simple groups (see [5]), we have $M \cong A_5$ or $\mathrm{PSL}_2(7)$ of order $2^2 \cdot 3 \cdot 5$ or $2^3 \cdot 3 \cdot 7$. By Proposition 2.2, M is semiregular. But, in both cases $|M|$ has divisor 2^2 and M is not semiregular, a contradiction.

(8) Suppose that there exists a cubic symmetric graph X of order $76p$ with $p \geq 29$. Since there is no cubic symmetric graph of order 76, P is not normal in A. Let $N_A(P)$ be the normalizer of P in A. Since $np+1 \geq 30$ and $np+1 \mid 2^6 \cdot 3 \cdot 19$, we have $(n, p) = (1, 31)$, $(1, 37)$, $(1, 47)$, $(3, 101)$, $(1, 113)$, $(1, 151)$, $(1, 191)$, $(1, 227)$, $(7, 521)$, $(1, 607)$, $(1, 911)$, $(1, 1823)$ and $np+1 = 2^5$, $2 \cdot 19$, $2^4 \cdot 3$, $2^4 \cdot 19$, $2 \cdot 3 \cdot 19$, $2^3 \cdot 19$, $2^6 \cdot 3$, $2^2 \cdot 3 \cdot 19$, $2^6 \cdot 3 \cdot 19$, $2^5 \cdot 19$, $2^4 \cdot 3 \cdot 19$, $2^5 \cdot 3 \cdot 19$. In particular, this implies that if $p \neq 37, 113, 227$, then X is at least 2-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

Suppose that M is elementary abelian. Then $M \cong \mathbb{Z}_2$ or $\mathbb{Z}_{19}$, and so $\underline{X}$ is symmetric with order $4p$ or $38p$ where $p =$ 31, 37, 47, 101, 113, 151, 191, 227, 521, 607, 911, 1823 (in particular, if $p \neq 37, 113, 227$, $\underline{X}$ is at least 2-arc-transitive). The cases except for $p \neq 37$ are impossible by [9, Theorem 6.2] and [13]. Now, assume $p = 37$. If A has a normal subgroup, say K, of order 19, then the quotient graph of X corresponding to the orbits of K is symmetric with order $2^2 \cdot 37 = 148$, which is impossible by [4]. Hence, $M \cong \mathbb{Z}_2$ and $\underline{X}$ is isomorphic to one of the two cubic 1-regular graphs F_{1406A} and F_{1406B} by [3]. Hence $A/M = \mathrm{Aut}(F_{1406A})$ or $\mathrm{Aut}(F_{1406B})$. Let L/M be a minimal normal subgroup of A/M. Since $|A/M| = 2 \cdot 19 \cdot 37 \cdot 3 = 4218$, A/M is solvable and so $L/M \cong \mathbb{Z}_{19}$ or $\mathbb{Z}_{37}$. Since $|M| = 2$, the Sylow 19- or 37-subgroup of L is characteristic in L and hence normal in A, a contradiction. Thus, like in the proof of (2), M is a non-abelian simple group and has order $2^\ell \cdot 3 \cdot 19$, $2^\ell \cdot 3p$, $2^\ell \cdot 19p$ or $2^\ell \cdot 3 \cdot 19p$ where $1 \leq \ell \leq 6$ and $p =$ 31, 37, 47, 101, 113, 151, 191, 227, 521, 607, 911, 1823. However, there is no simple group with such orders (see [5]). □

Remark 2.2. After writing this paper, the author was acknowledged that Alaeiyan and Hosseinipoor [1] already classified cubic s-regular graphs of orders $12p$ and $12p^2$.

References

1. M. Alaeiyan and M.K. Hosseinipoor, *A classification of the cubic s-regular graphs of orders* $12p$ *and* $12p^2$, Acta. Univ. Apul. **25** (2011), 153-158.
2. Y. Cheng and J. Oxley, *On weakly symmetric graphs of order twice a prime*, J. of Combin. Theory Ser. B **42** (1987), 196-211.
3. M.D.E. Conder, *Trivalent (cubic) symmetric graphs on up to* 2048 *vertices*, 2006, (http://www.math.auckland.ac.nz/ ∼conder/symmcubic2048list.txt).
4. M.D.E. Conder and P. Dobcsányi, *Trivalent symmetric graphs on up to* 768 *vertices*, J. Combin. Math. Combin. Comput. **40** (2002), 41-63.
5. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *An ATLAS of finite groups,* Oxford University press, Oxford, 1985.
6. W. Feit and J.G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1936), 775-1029.
7. Y.Q. Feng and J.H. Kwak, *One-regular cubic graphs of order a small number times a prime or a prime square* J. Aust. Math. Soc. **76** (2004), 345-356.

8. Y.Q. Feng and J.H. Kwak, *Classifying cubic symmetric graphs of order* $10p$ *or* $10p^2$, Science in China Series A: Math. **49** (2006) 300-319.
9. Y.Q. Feng and J.H. Kwak, *Cubic symmetric graphs of order a small number times a prime or a prime square*, J. of Combin. Theory Ser. B **97** (2007), 627-646.
10. Y.Q. Feng and J.H. Kwak, *Cubic symmetric graphs of order twice an odd prime-power*, J. Aust. Math. Soc. **81** (2006), 153-164.
11. Y.Q. Feng, J.H. Kwak and K. Wang, *Classifying cubic symmetric graphs of order* $8p$ *or* $8p^2$, European J. Combin. **26** (2005), 1033-1052.
12. Y.Q. Feng, J.H. Kwak and M.Y. Xu, *Cubic s-regular graphs of order* $2p^3$, J. Graph Theory **52** (2006), 341-352.
13. Y.Q. Feng and J.X. Zhou, *Cubic vertex-transitive graphs of order* $2pq$, J. Graph Theory **65** (2010), 285-302.
14. J.M. Oh, *A classification of cubic s-regular graphs of order* $16p$, Discrete Math. **309** (2009), 3150-3155.
15. J.M. Oh, *Arc-transitive elementary abelian covers of the Pappus graph*, Discrete Math. **309** (2009), 6590-6611.
16. P. Lorimer, *Vertex-transitive graphs: Symmetric graphs of prime valency*, J. Graph Theory **8** (1984), 55-68.
17. D.J. Robinson, *A course in the theory of groups,* Springer-Verlag, Berlin, 1979.
18. W.T. Tutte, *A family of cubical graphs*, Proc. Cambridge Philos. Soc. **43** (1947), 459-474.

Ju-Mok Oh received M.Sc. and Ph.D at Pohang University of Science and Technology. Since 2012 he has been at Kangnung-Wonju National University. His research interests include graph theory by using finite group theory and enumerative combinatorics.

Department of Mathematics, Kangnung-Wonju National University, Kangnung 210-702, Korea.
e-mail: jumokoh@gwnu.ac.kr