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COUPLED COMMON FIXED POINT THEOREMS FOR A CONTRACTIVE CONDITION OF RATIONAL TYPE IN ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to establish some coupled coincidence point theorems for a pair of mappings having a strict mixed g-monotone property satisfying a contractive condition of rational type in the framework of partially ordered metric spaces. Also, we present a result on the existence and uniqueness of coupled common fixed points. The results presented in the paper generalize and extend several well-known results in the literature.

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1. Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a large number of applications. The Banach Contraction Principle is popular tool in solving existence problems in many problems of mathematical analysis. Since its simplicity and usefulness, there are a lot of generalizations of this principle in the literature. Ran and Reurings [14] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Nieto and Rodŕiguez-López [13] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point, coupled fixed point and coupled common fixed point results in partially ordered metric spaces, metric spaces and others (see [1]-[15]).

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The purpose of this paper is to establish some coupled coincidence point results in partially ordered metric spaces for a pair of mappings having strict mixed g-monotone property satisfying a contractive condition of rational type. Also, we present a result on the existence and uniqueness of coupled common fixed points.

Definition 1.1. Let (X, d) be a metric space, $F : X \times X \to X$ and $g : X \to X$. F and g are said to *commute* if $F(gx, gy) = g(F(x, y))$, for all $x, y \in X$.

Definition 1.2. Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the *strict mixed g-monotone property* if $F(x, y)$ is strictly increasing in x and strictly decreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 < gx_2 \Rightarrow F(x_1, y) < F(x_2, y),$$

and

$$y_1, y_2 \in X, gy_1 < gy_2 \Rightarrow F(x, y_1) > F(x, y_2).$$

If g =identity mapping in Definition 1.2, then the mapping F is said to have the *strict mixed monotone property.*

Definition 1.3. Let $(X, \leq)$ be a partially ordered set and $F : X \to X$. The mapping F is said to be *increasing* if for $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$ and *decreasing* if for $x, y \in X$, $x \leq y$ implies $F(x) \geq F(y)$.

Definition 1.4. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$, and $F(y, x) = gy$.

If g =identity mapping in Definition 1.4, then $(x, y) \in X \times X$ is called a *coupled fixed point.*

2. Main Results

In this section, we prove some coupled coincidence and common fixed point theorems in the context of ordered metric spaces.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are self mappings on X, F has the strict mixed g-monotone property on X. Suppose that there exists two elements $x_0, y_0 \in X$ with $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$ and there exists $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that*

$$d(F(x,y), F(u,v)) \leq \alpha \left(\frac{d(gx, F(x,y))d(gu, F(u,v))}{d(gx, gu)} \right) + \beta(d(gx, gu)), \qquad (1)$$

is satisfied for all $x, y, u, v \in X$ with $gx \geq gu$ and $gy \leq gv$. Further suppose that F is continuous, $F(X \times X) \subseteq g(X)$, g is continuous non-decreasing and commutes with F. Then there exists $x, y \in X$ such that $F(x, y) = g(x)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Since $F(X \times X) \subseteq g(X)$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n), \forall n \geq 0. \tag{2}$$

We claim that for all $n \geq 0$,

$$gx_n < gx_{n+1}, \tag{3}$$

and

$$gy_n > gy_{n+1}. \tag{4}$$

We shall use the mathematical induction. Let $n = 0$. Since $gx_0 < F(x_0, y_0)$ and $gy_0 > F(y_0, x_0)$, in view of $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 < gx_1$ and $gy_0 > gy_1$, that is, (3) and (4) hold for $n = 0$. Suppose that (3) and (4) hold for some $n > 0$. As F has the strict mixed g-monotone property and $gx_n < gx_{n+1}$ and $gy_n > gy_{n+1}$, from (2), we get

$$gx_{n+1} = F(x_n, y_n) < F(x_{n+1}, y_n) < F(x_{n+1}, y_{n+1}) = gx_{n+2}, \tag{5}$$

and

$$gy_{n+1} = F(y_n, x_n) > F(y_{n+1}, x_n) > F(y_{n+1}, x_{n+1}) = gy_{n+2}. \tag{6}$$

Now from (5) and (6), we obtain that $gx_{n+1} < gx_{n+2}$ and $gy_{n+1} > gy_{n+2}$. Thus by the mathematical induction, we conclude that (3) and (4) hold for all $n \geq 0$. Therefore

$$gx_0 < gx_1 < gx_2 < \ldots < gx_n < gx_{n+1} < \ldots, \tag{7}$$

and

$$gy_0 > gy_1 > gy_2 > \ldots > gy_n > gy_{n+1} > \ldots. \tag{8}$$

Since $gx_n > gx_{n-1}$ and $gy_n < gy_{n-1}$, from (1) and (2), we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \alpha\left(\frac{d(gx_n, F(x_n, y_n))d(gx_{n-1}, F(x_{n-1}, y_{n-1}))}{d(gx_n, gx_{n-1})}\right) + \\ &\quad \beta(d(gx_n, gx_{n-1})) \\ &= \alpha\left(\frac{d(gx_n, gx_{n+1})d(gx_{n-1}, gx_n)}{d(gx_n, gx_{n-1})}\right) + \beta(d(gx_n, gx_{n-1})) \\ &= \alpha\left(d(gx_n, gx_{n+1})\right) + \beta(d(gx_n, gx_{n-1})), \end{aligned}$$

which implies that $d(gx_{n+1}, gx_n) \leq \frac{\beta}{1-\alpha} d(gx_n, gx_{n-1})$.

Similarly, we have

$$d(gy_{n+1}, gy_n) \leq \frac{\beta}{1-\alpha} d(gy_n, gy_{n-1}).$$

Hence $d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq \frac{\beta}{1-\alpha}(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}))$. Set $\{\varrho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)\}$ and $\delta = \frac{\beta}{1-\alpha} < 1$, we have

$$0 \leq \varrho_n \leq \delta \varrho_{n-1} \leq \delta^2 \varrho_{n-2} \leq \ldots \leq \delta^n \varrho_0$$

which implies that

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty}[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] = 0. \tag{9}$$

Thus $\lim_{n\to\infty} d(gx_{n+1}, gx_n) = 0$ and $\lim_{n\to\infty} d(gy_{n+1}, gy_n) = 0$.

Now, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. For each $m \geq n$, we have

$$d(gx_m, gx_n) \leq d(gx_m, gx_{m-1}) + d(gx_{m-1}, gx_{m-2}) + \ldots + d(gx_{n+1}, gx_n)$$

and

$$d(gy_m, gy_n) \leq d(gy_m, gy_{m-1}) + d(gy_{m-1}, gy_{m-2}) + \ldots + d(gy_{n+1}, gy_n).$$

Therefore

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) &\leq \varrho_{m-1} + \varrho_{m-2} + \ldots + \varrho_n \\ &\leq (\delta^{m-1} + \delta^{m-2} + \ldots + \delta^n)\varrho_0 \\ &\leq \frac{\delta^n}{1-\delta}\varrho_0 \end{aligned}$$

which implies that $\lim_{m,n\to\infty}[d(gx_m, gx_n) + d(gy_m, gy_n)] = 0$. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$.

Since X is a complete metric space, there is $(x, y) \in X \times X$ such that $gx_n \to x$ and $gy_n \to y$. Since g is continuous, $g(gx_n) \to gx$ and $g(gy_n) \to gy$. As F is continuous, $F(gx_n, gy_n) \to F(x, y)$ and $F(gy_n, gx_n) \to F(y, x)$. As, F commutes with g, we have $F(gx_n, gy_n) = gF(x_n, y_n) = g(gx_{n+1}) \to gx$ and $F(gy_n, gx_n) = gF(y_n, x_n) = g(gy_{n+1}) \to gy$. By the uniqueness of the limit, we get $gx = F(x, y)$ and $gy = F(y, x)$. Thus F and g have a coupled coincidence point. □

Now, we shall prove the existence and uniqueness of a coupled common fixed point. Note that, if $(X, \leq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order relation:

$$\text{for } (x, y), (u, v) \in X \times X, (u, v) \leq (x, y) \Leftrightarrow x \leq u, y \geq v.$$

Theorem 2.2. *In addition to hypotheses of Theorem 2.1, suppose that for every $(x, y), (z, t) \in X \times X$, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.*

Proof. From Theorem 2.1, the set of coupled coincidence points of F and g is non-empty. Suppose that (x, y) and (z, t) are coupled coincidence points of F and g, that is, $gx = F(x, y)$, $gy = F(y, x)$, $gz = F(z, t)$ and $gt = F(t, z)$. We shall show that $gx = gz$ and $gy = gt$. By the assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ as $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$ for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $t_0 = t$ and on the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$, $\{gt_n\}$. Then as in Theorem 2.1, we can show that $gx_n \to gx = F(x, y)$, $gy_n \to gy = F(y, x)$, $gz_n \to gz = F(z, t)$, $gt_n \to gt = F(t, z)$, for all $n \geq 1$. Since $(F(x, y), F(y, x)) = (gx, gy)$ and $(F(u, v), F(v, u)) = (gu_1, gv_1)$ are comparable, then $gx \geq gu_1$ and $gy \leq gv_1$. Now, we shall show that (gx, gy) and (gu_n, gv_n) are comparable, that is, $gx \geq u_n$ and $gy \leq gv_n$ for all n. Suppose that it holds for some $n \geq 0$, then by the strict mixed g-monotone property of F, we have $gu_{n+1} = F(u_n, v_n) \leq F(x, y) = gx$ and $gv_{n+1} = F(v_n, u_n) \geq F(y, x) = gy$. Hence $gx \geq gu_n$ and $gy \leq gv_n$ hold for all n. Thus from (1), we have

$$\begin{aligned} d(gx, gu_{n+1}) &= d(F(x, y), F(u_n, v_n)) \\ &\leq \alpha\left(\frac{d(gx, F(x, y))d(gu_n, F(u_n, v_n))}{d(gx, gu_n)}\right) + \beta(d(gx, gu_n)) \\ &= \beta(d(gx, gu_n)). \end{aligned} \tag{10}$$

Similarly, we can prove that $d(gy, gv_{n+1}) \leq \beta d(gy, gv_n)$. Hence

$$\begin{aligned} d(gx, gu_{n+1}) + d(gy, gv_{n+1}) &\leq \beta[d(gx, gu_n) + d(gy, gv_n)] \\ &\leq (\beta)^2[d(gx, gu_{n-1}) + d(gy, gv_{n-1})] \\ &\quad \dots \\ &\leq (\beta)^{n+1}[d(gx, gu_0) + d(gy, gv_0)]. \end{aligned}$$

On taking limit, $n \to \infty$, we get $\lim_{n\to\infty}[d(gx, gu_{n+1}) + d(gy, gv_{n+1})] = 0$. Thus $\lim_{n\to\infty} d(gx, gu_{n+1}) = 0$ and $\lim_{n\to\infty} d(gy, gv_{n+1}) = 0$.

Similarly, we can prove that $\lim d(gz, gu_n) = 0 = \lim d(gt, gv_n)$. Finally, we have $d(gx, gz) \leq d(gx, gu_n) + d(gu_n, gz)$ and $d(gy, gt) \leq d(gy, gv_n) + d(gv_n, gt)$. Taking $n \to \infty$ in these inequalities, we get $d(gx, gz) = 0 = d(gy, gt)$, that is $gx = gz$ and $gy = gt$. Since $gx = F(x, y)$ and $gy = F(y, x)$, by the commutativity of F and g, we have

$$\begin{aligned} &g(g(x)) = g(F(x, y)) = F(gx, gy), \text{ and} \\ &g(gy) = g(F(y, x)) = F(gy, gx). \end{aligned} \tag{11}$$

Denote $gx = p$ and $gy = q$. Then $gp = F(p, q)$ and $gq = F(q, p)$. Thus (p, q) is a coupled coincidence point. Then from (11), with $z = p$ and $t = q$, it follows $gp = gx$ and $gq = gy$, that is, $gp = p$ and $gq = q$. Hence $p = gp = F(p, q)$ and $q = gq = F(q, p)$. Therefore, (p, q) is a coupled common fixed point of F and g. To prove the uniqueness, assume that (r, s) is another coupled common fixed point. Then by (11), we have $r = gr = gp = p$ and $s = gs = gq = q$. Hence we get the result. □

Theorem 2.3. *In addition to hypotheses of Theorem 2.1, if gx_0 and gy_0 are comparable. Then F and g have a unique coupled coincidence point, that is, there exists a $(x, y) \in X \times X$ such that $gx = F(x, y) = F(y, x) = gy$.*

Proof. By Theorem 2.1, we can construct two sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $gx_n \to gx$ and $gy_n \to gy$, where (x, y) is a coincidence point of F and g. Suppose $gx_0 \leq gy_0$. We shall show that $gx_n \leq gy_n$, where $gx_n = F(x_{n-1}, y_{n-1})$, $gy_n = F(y_{n-1}, x_{n-1})$, for all n. Suppose it holds for some $n > 0$. Then by strict mixed g-monotone property of F, we have $gx_{n+1} = F(x_n, y_n) \leq F(y_n, x_n) = gy_{n+1}$. From (1), we have

$$\begin{aligned} d(gx_{n+1}, gy_{n+1}) &= d(F(x_n, y_n), F(y_n, x_n)) \\ &\leq \alpha\left(\frac{d(gx_n, F(x_n, y_n))d(gy_n, F(y_n, x_n))}{d(gx_n, gy_n)}\right) + \beta(d(gx_n, gy_n)) \\ &= \alpha\left(\frac{d(gx_n, gx_{n+1})d(gy_n, gy_{n+1})}{d(gx_n, gy_n)}\right) + \beta(d(gx_n, gy_n)). \end{aligned}$$

On taking $n \to \infty$, we obtain $d(gy, gx) \leq (\beta)d(gy, gx)$. Since $\beta < 1$, $d(gy, gx) = 0$. Hence $F(x, y) = gx = gy = F(y, x)$.

A similar arguments can be used if $gy_0 \leq gx_0$. □

Remark 2.1. If $g = I$(Identity mapping) in above Theorems, then we have Theorems 2.1 and 2.2 of Ciric, Olatinwo, Gopal and Akinbo [9].

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