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GLOBAL EXPONENTIAL STABILITY OF ALMOST PERIODIC SOLUTIONS OF HIGH-ORDER HOPFIELD NEURAL NETWORKS WITH DISTRIBUTED DELAYS OF NEUTRAL TYPE†

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Abstract. In this paper, we study the global stability and the existence of almost periodic solution of high-order Hopfield neural networks with distributed delays of neutral type. Some sufficient conditions are obtained for the existence, uniqueness and global exponential stability of almost periodic solution by employing fixed point theorem and differential inequality techniques. An example is given to show the effectiveness of the proposed method and results.

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1. Introduction

Since high-order Hopfield neural networks (HHNNs) have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order Hopfield neural networks, the study of high-order Hopfield neural networks has recently gained a lot of attention, moreover there have been extensive results on the problem of the existence and stability of equilibrium points, periodic solutions and almost periodic solutions of high-order Hopfield neural networks in the literature. We refer the reder to [1-12] and the references cited therein. Moreover, time delays may occur in neural procession and signal transmission, which can cause instability and oscillations in system and the distributed delays should be incorporated in the model. In other words,

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it is the often the case that the neural networks model possesses both bounded and unbounded delays (distributed delays).

In addition, because of the complicated dynamic properties of the neural cells in the real world, the existing neural network models in many cases cannot characterize the properties of a neural reaction process precisely. Thus, it is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions. Therefore, neural networks with delays of neutral type are considered to be more accordant with reality. Recently, there are many results on the stability and the existence of periodic solutions to neural networks with delays of neutral type (see [13-19]). However, few papers have been published on the stability and the existence of almost periodic solution to neutral high-order Hopfield neural networks. In this paper, we consider the following high-order Hopfield neural networks with distributed delays of neutral type

$$\begin{aligned} x_i'(t) =& -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)\int_0^\infty d_{ij}(\theta)f_j(x_j(t-\theta))d\theta \\ &+\sum_{j=1}^{n}\alpha_{ij}(t)\int_0^\infty \beta_{ij}(\theta)h_j(x_j'(t-\theta))d\theta \\ &+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\int_0^\infty k_{ij}(\theta)g_j(x_j(t-\theta))d\theta\int_0^\infty k_{il}(\theta)g_l(x_l(t-\theta))d\theta \\ &+I_i(t),\ i=1,2,\dots,n,\ t>0, \end{aligned} \tag{1}$$

where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t, $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural network, d_{ij}, k_{ij} are the kernels, $I_i(t)$ denotes the external input at time t, f_j and g_j are the activation functions of signal transmission.

The initial conditions of (1) are of the form

$$x_i(s) = \phi_i(s),\ s \in (-\infty, 0],\ i = 1, 2, \dots, n,$$

where $\phi_i(\cdot)$ denotes a differential real-value bounded function defined on $(-\infty, 0]$ and satisfies that $\phi_i'(\cdot)$ is bounded on $(-\infty, 0)$ too.

Throughout this paper we assume that:

- (H_1) $c_i(t) > 0, a_{ij}(t), \alpha_{ij}(t), b_{ijl}(t)$ and $I_i(t)$ are almost periodic functions, $i, j, l = 1, 2, \dots, n$.
- (H_2) There exist positive constants $\epsilon_i > 0, \vartheta_i > 0, \varepsilon_i > 0, G_i > 0$ such that $|f_i(x) - f_i(y)| \le \epsilon_i|x-y|,\ |g_i(x) - g_i(y)| \le \varepsilon_i|x-y|,\ |h_i(x)-h_i(y)| \le \vartheta_i|x-y|,\ |g_i(x)| \le G_i$, for all $x, y \in \mathbb{R}$, and $f_i(0) = g_i(0) = h_i(0) = 0,\ i = 1, 2, \dots, n$.

(H_3) For $i,j \in \{1,2,\dots,n\}$, the delay kernels $d_{ij}, k_{ij}, \beta_{ij} : [0,\infty) \to \mathbb{R}$ are continuous and integrable with

$$0 \le \int_0^\infty |k_{ij}(s)|ds \le k_{ij}^M,\ 0 \le \int_0^\infty |d_{ij}(s)|ds \le d_{ij}^M,\ 0 \le \int_0^\infty |\beta_{ij}(s)|ds \le \beta_{ij}^M.$$

Our main purpose of this paper is to study the existence, uniqueness and global exponential stability of almost periodic solution to (1) by employing fixed point theorem and differential inequality techniques. To the best of our knowledge, this is the first paper to investigate the global exponential stability and existence of almost periodic solution to system (1).

2. Existence of almost periodic solution

To obtain the existence of almost periodic solution of system (1), we shall recall the following definitions and lemmas:

Definition 2.1 ([20, 21])**.** Let $u : \mathbb{R} \to \mathbb{R}^n$ be continuous in t, $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u,\varepsilon) = \{\delta : |u(t+\delta) - u(t)| < \varepsilon,\ \forall t \in \mathbb{R}\}$ is relatively dense, i.e, for $\forall \varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \varepsilon$, for $\forall t \in \mathbb{R}$.

For $z = (z_1, z_2, \dots, z_n)^T \in \mathbb{R}^n$, we define $\|z\| = \max_{1\le i\le n} |z_i|$.

Definition 2.2. Let $x \in \mathbb{R}^n$ and $Q(t)$ be a $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = Q(t)x(t) \tag{2}$$

is said to admit an exponential dichotomy on $\mathbb{R}^n$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2) satisfy

$$\|X(t)PX^{-1}(s)\| \le ke^{-\alpha(t-s)}\ (\forall t \ge s),\ \|X(t)(I-P)X^{-1}(s)\| \le ke^{-\alpha(s-t)}\ (\forall s \ge t).$$

Lemma 2.3 ([20, 21])**.** *If the linear system* (2) *admits an exponential dichotomy, then the almost periodic system:*

$$x'(t) = Q(t)x(t) + g(t)$$

has a unique almost periodic solution $x(t)$*, and*

$$x(t) = \int_{-\infty}^t X(t)PX^{-1}(s)g(s)ds - \int_t^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds.$$

Lemma 2.4 ([20, 21])**.** *Let* $c_i(t)$ *be an almost periodic function on* $\mathbb{R}^n$*, and*

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} c_i(s)ds > 0, \quad i = 1,2,\dots,n.$$

Then the linear system

$$x'(t) = \mathrm{diag}(-c_1(t), -c_2(t), \dots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}^n$*.*

For the sake of convenience, we introduce the following notations:

$$c_i^m=\inf_{t\in\mathbb{R}}c_i(t),\quad c_i^M=\sup_{t\in\mathbb{R}}c_i(t),\quad \overline{I_i}=\sup_{t\in\mathbb{R}}|I_i(t)|,\quad \overline{a_{ij}}=\sup_{t\in\mathbb{R}}|a_{ij}(t)|,$$

$$\overline{\alpha_{ij}}=\sup_{t\in\mathbb{R}}|\alpha_{ij}(t)|,\quad \overline{b_{ijl}}=\sup_{t\in\mathbb{R}}|b_{ijl}(t)|,\quad i,j,l=1,2,\dots,n.$$

$$A=\max\left\{\max_{1\le i\le n}\left\{\frac{\overline{I_i}}{c_i^m}\right\},\max_{1\le i\le n}\left\{\overline{I_i}+c_i^M\frac{\overline{I_i}}{c_i^m}\right\}\right\},$$

$$\begin{aligned}B&=\max_{1\le i\le n}\left\{\max\left\{\frac{1}{c_i^m},1+\frac{c_i^M}{c_i^m}\right\}\sum_{j=1}^n\left[\overline{a_{ij}}d_{ij}^M|f_j(0)|+\overline{\alpha_{ij}}\beta_{ij}^M|h_j(0)|\right.\right.\\&\quad\left.\left.+\sum_{l=1}^n\overline{b_{ijl}}G_lG_jk_{ij}^Mk_{il}^M\right]\right\}.\\ \phi_0(t)&=\left(\int_{-\infty}^t e^{-\int_s^t c_1(u)du}I_1(s)ds,\int_{-\infty}^t e^{-\int_s^t c_2(u)du}I_2(s)ds,\dots,\right.\\&\quad\left.\int_{-\infty}^t e^{-\int_s^t c_n(u)du}I_n(s)ds\right)^T,\end{aligned}$$

$$\mathbb{X}=\{\phi|\phi=(\phi_1(t),\phi_2(t),\dots,\phi_n(t))^T\},$$

where $\phi_i:\mathbb{R}\to\mathbb{R}$ is a continuously differential almost periodic function. Then, $\mathbb{X}$ is a Banach space with the norm defined by

$$\|\phi\|_{\mathbb{X}}=\sup_{t\in\mathbb{R}}\|\phi(t)\|_1=\max\{\|\phi\|_0,\|\phi'\|_0\},$$

where

$$\|\phi(t)\|_1=\max\{\|\phi(t)\|_0,\|\phi'(t)\|_0\},\ \ \|\phi(t)\|_0=\max_{1\le i\le n}|\phi_i(t)|,\ \ \|\phi\|_0=\sup_{t\in\mathbb{R}}\|\phi(t)\|_0.$$

Theorem 2.5. *Suppose that* $(H_1)-(H_3)$ *and*

(H_4) *For* $i=1,2,\dots,n,c_i^m>0$ *and*

$$\begin{aligned}\xi&=\max_{1\le i\le n}\left\{\max\left\{\frac{1}{c_i^m},1+\frac{c_i^M}{c_i^m}\right\}\sum_{j=1}^n\left[\overline{a_{ij}}d_{ij}^M\epsilon_j+\overline{\alpha_{ij}}\beta_{ij}^M\vartheta_j\right.\right.\\&\quad\left.\left.+2\varepsilon_j\sum_{l=1}^n\overline{b_{ijl}}G_lk_{ij}^Mk_{il}^M\right]\right\}<1\end{aligned}$$

hold, then there exists a unique continuously differentiable almost periodic solution of system (1) *in the region* $\mathbb{X}_0=\{\phi|\phi\in\mathbb{X},\|\phi-\phi_0\|_{\mathbb{X}}\le\frac{A\xi+B}{1-\xi}\}$.

Proof. For $\forall\phi\in\mathbb{X}$, we consider the almost periodic solution $x^\phi(t)$ of the linear nonhomogeneous almost periodic differential equations

$$x_i'(t)=-c_i(t)x_i(t)+\sum_{j=1}^n a_{ij}(t)\int_0^\infty d_{ij}(\theta)f_j(\phi_j(t-\theta))d\theta$$

$$+\sum_{j=1}^{n}\alpha_{ij}(t)\int_0^{\infty}\beta_{ij}(\theta)h_j(\phi_j^{'}(t-\theta))d\theta$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_0^{\infty}k_{ij}(\theta)g_j(\phi_j(t-\theta))d\theta\int_0^{\infty}k_{il}(\theta)g_l(\phi_l(t-\theta))d\theta$$
$$+I_i(t),\ i=1,2,\dots,n. \tag{3}$$

Noticing that $c_i(t)$, $a_{ij}(t)$, $b_{ijl}(t)$, $d_{ij}(t)$, $k_{ij}(t)$ and $I_i(t)$ are almost periodic functions, and $M[c_i]>0$, it follows from Lemma **??** and Lemma **??** that system (3) has a unique almost periodic solution which can be expressed as follows:

$$\begin{aligned} x^{\phi}(t) &= (x_1^{\phi}(t),x_2^{\phi}(t),\dots,x_n^{\phi}(t))^T \\ &= \Big(\int_{-\infty}^{t}e^{-\int_s^t c_1(u)du}\Big[\sum_{j=1}^{n}a_{1j}(s)\int_0^{\infty}d_{1j}(\theta)f_j(\phi_j(s-\theta))d\theta \\ &\quad+\sum_{j=1}^{n}\alpha_{1j}(s)\int_0^{\infty}\beta_{1j}(\theta)h_j(\phi_j^{'}(s-\theta))d\theta+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{1jl}(s) \\ &\quad\times\int_0^{\infty}k_{1j}(\theta)g_j(\phi_j(s-\theta))d\theta\int_0^{\infty}k_{1l}(\theta)g_l(\phi_l(s-\theta))d\theta+I_1(s)\Big]ds, \\ &\quad\dots,\int_{-\infty}^{t}e^{-\int_s^t c_n(u)du}\Big[\sum_{j=1}^{n}a_{nj}(s)\int_0^{\infty}d_{nj}(\theta)f_j(\phi_j(s-\theta))d\theta \\ &\quad+\sum_{j=1}^{n}\alpha_{nj}(s)\int_0^{\infty}\beta_{nj}(\theta)h_j(\phi_j^{'}(s-\theta))d\theta+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{njl}(s)\int_0^{\infty}k_{nj}(\theta) \\ &\quad\times g_j(\phi_j(s-\theta))d\theta\int_0^{\infty}k_{nl}(\theta)g_l(\phi_l(s-\theta))d\theta+I_1(s)\Big]ds\Big)^T. \end{aligned} \tag{4}$$

Now, we define a mapping $T:\mathbb{X}\to\mathbb{X}$ as follows:

$$(T\phi)(t)=x^{\phi}(t),\ \forall\phi\in\mathbb{X},$$

where x^{ϕ} is defined by (4). By the definition of the norm of Banach space $\mathbb{X}$, we have

$$\begin{aligned} \|\phi_0\|_{\mathbb{X}} &= \max\{\|\phi_0\|_0,\|\phi_0^{'}\|_0\} \\ &= \max\Big\{\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big|\int_{-\infty}^{t}I_i(s)e^{-\int_s^t c_i(u)du}ds\Big|, \\ &\quad \sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big|I_i(t)-\int_{-\infty}^{t}I_i(s)c_i(t)e^{-\int_s^t c_i(u)du}ds\Big|\Big\} \\ &\le \max\Big\{\max_{1\le i\le n}\{\frac{\overline{I_i}}{c_i^m}\},\ \max_{1\le i\le n}\{\overline{I_i}+c_i^M\frac{\overline{I_i}}{c_i^m}\}\Big\}=A. \end{aligned}$$

Hence, for $\forall\phi\in\mathbb{X}_0=\Big\{\phi|\phi\in\mathbb{X},\|\phi-\phi_0\|_{\mathbb{X}}\le\frac{A\xi+B}{1-\xi}\Big\}$, we obtain

$$\|\phi\|_{\mathbb{X}}\le\|\phi-\phi_0\|_{\mathbb{X}}+\|\phi_0\|_{\mathbb{X}}\le\frac{A\xi+B}{1-\xi}+A=\frac{A+B}{1-\xi}. \tag{5}$$

Next, we show that T maps the set $\mathbb{X}_0$ into itself. In fact, for any $\phi\in\mathbb{X}_0$, we obtain by $(H_2)-(H_3)$ that

$$\|T\phi-\phi_0\|_0$$

$$
\begin{aligned}
&= \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \bigg\{ \bigg| \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \bigg[\sum_{j=1}^{n} a_{ij}(s) \int_0^\infty d_{ij}(\theta) f_j(\phi_j(s-\theta))d\theta \\
&\quad + \sum_{j=1}^{n} \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta) h_j(\phi_j^{'}(s-\theta))d\theta + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s) \\
&\quad \times \int_0^\infty k_{ij}(\theta) g_j(\phi_j(s-\theta))d\theta \int_0^\infty k_{il}(\theta) g_l(\phi_l(s-\theta))d\theta \bigg] ds \bigg| \bigg\} \\
&\le \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \bigg\{ \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \bigg[\sum_{j=1}^{n} \bigg(|a_{ij}(s)| \\
&\quad \times \int_0^\infty |d_{ij}(\theta)| \big[|f_j(\phi_j(s-\theta)) - f_j(0)| + |f_j(0)| \big] d\theta \bigg) \\
&\quad + \sum_{j=1}^{n} |\alpha_{ij}(s)| \int_0^\infty |\beta_{ij}(\theta)| \big[h_j(\phi_j^{'}(s-\theta)) - h_j(0)| + |h_j(0)| \big] d\theta \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} |b_{ijl}(s)| \bigg[\int_0^\infty |k_{ij}(\theta)||g_j(\phi_j(s-\theta)) - g_j(0)| d\theta \\
&\quad \times \int_0^\infty |k_{il}(\theta)||g_l(\phi_l(s-\theta))| d\theta \bigg] \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} |b_{ijl}(s)| \int_0^\infty |k_{ij}(\theta)||g_j(0)| d\theta \int_0^\infty |k_{il}(\theta)||g_l(\phi_l(s-\theta)) - g_l(0)| d\theta \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} |b_{ijl}(s)| \int_0^\infty |k_{ij}(\theta)||g_j(0)| d\theta \int_0^\infty |k_{il}(\theta)||g_l(0)| d\theta \bigg] ds \bigg\} \\
&\le \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \bigg\{ \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \bigg[\sum_{j=1}^{n} \overline{a_{ij}} \int_0^\infty |d_{ij}(\theta)| \big(\epsilon_j |\phi_j(s-\theta)| + |f_j(0)| \big) d\theta \\
&\quad + \sum_{j=1}^{n} \overline{\alpha_{ij}} \int_0^\infty |\beta_{ij}(\theta)| \big(\vartheta_j |\phi_j^{'}(s-\theta)| + |h_j(0)| \big) d\theta \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} G_l \varepsilon_j k_{il}^M \int_0^\infty |k_{ij}(\theta)||\phi_j(s-\theta)| d\theta \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} G_j \varepsilon_l k_{ij}^M \int_0^\infty |k_{il}(\theta)||\phi_l(s-\theta)| d\theta + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{ijl} G_j G_l k_{ij}^M k_{il}^M \bigg] ds \bigg\} \\
&\le \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \bigg\{ \int_{-\infty}^{t} e^{-c_i^m(t-s)} \bigg[\sum_{j=1}^{n} \bigg(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \\
&\quad + 2\sum_{l=1}^{n} \overline{b_{ijl}} G_l \varepsilon_j k_{ij}^M k_{il}^M \bigg) \|\phi\|_{\mathbb{X}} \\
&\quad + \sum_{j=1}^{n} \bigg(\overline{a_{ij}} d_{ij}^M |f_j(0)| + \overline{\alpha_{ij}} \beta_{ij}^M |h_j(0)| + \sum_{l=1}^{n} \overline{b_{ijl}} G_j G_l k_{ij}^M k_{il}^M \bigg) \bigg] ds \bigg\}
\end{aligned}
$$

$$\leq \max_{1\leq i\leq n}\left\{\frac{1}{c_i^m}\left[\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M\epsilon_j+\overline{\alpha_{ij}}\beta_{ij}^M\vartheta_j+2\sum_{l=1}^{n}\overline{b_{ijl}}G_l\varepsilon_j k_{ij}^M k_{il}^M\right)\|\phi\|_{\mathbb{X}}\right.\right.$$
$$\left.\left.+\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M|f_j(0)|+\overline{\alpha_{ij}}\beta_{ij}^M|h_j(0)|+\sum_{l=1}^{n}\overline{b_{ijl}}G_jG_l k_{ij}^M k_{il}^M\right)\right]\right\}. \tag{6}$$

Furthermore, we have

$$\begin{aligned}
&\|(T\phi-\phi_0)^{'}\|_0\\
=&\sup_{t\in\mathbb{R}}\max_{1\leq i\leq n}\left\{\left|\sum_{j=1}^{n}a_{ij}(t)\int_0^{\infty}d_{ij}(\theta)f_j(\phi_j(t-\theta))d\theta\right.\right.\\
&+\sum_{j=1}^{n}\alpha_{ij}(t)\int_0^{\infty}\beta_{ij}(\theta)h_j(\phi_j^{'}(t-\theta))d\theta\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_0^{\infty}k_{ij}(\theta)g_j(\phi_j(t-\theta))d\theta\int_0^{\infty}k_{il}(\theta)g_l(\phi_l(t-\theta))d\theta\\
&-\int_{-\infty}^{t}c_i(t)e^{-\int_s^t c_i(u)du}\left[\sum_{j=1}^{n}a_{ij}(s)\int_0^{\infty}d_{ij}(\theta)f_j(\phi_j(s-\theta))d\theta\right.\\
&+\sum_{j=1}^{n}\alpha_{ij}(s)\int_0^{\infty}\beta_{ij}(\theta)h_j(\phi_j^{'}(s-\theta))d\theta\\
&\left.\left.\left.+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(s)\int_0^{\infty}k_{ij}(\theta)g_j(\phi_j(s-\theta))d\theta\int_0^{\infty}k_{il}(\theta)g_l(\phi_l(s-\theta))d\theta\right]ds\right|\right\}\\
\leq&\sup_{t\in\mathbb{R}}\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M\epsilon_j+\overline{\alpha_{ij}}\beta_{ij}^M\vartheta_j+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{ij}^M k_{il}^M\right)\|\phi\|_{\mathbb{X}}\right.\\
&+\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M|f_j(0)|+\overline{\alpha_{ij}}\beta_{ij}^M|h_j(0)|+G_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{ij}^M k_{il}^M\right)\\
&+\int_{-\infty}^{t}c_i^M e^{-c_i^m(t-s)}\left[\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M\epsilon_j+\overline{\alpha_{ij}}\beta_{ij}^M\vartheta_j+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{ij}^M k_{il}^M\right)\|\phi\|_{\mathbb{X}}\right.\\
&\left.\left.+\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M|f_j(0)|+\overline{\alpha_{ij}}\beta_{ij}^M|h_j(0)|+G_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{ij}^M k_{il}^M\right)\right]ds\right\}\\
\leq&\max_{1\leq i\leq n}\left\{\left(1+\frac{c_i^M}{c_i^m}\right)\left[\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M\epsilon_j+\overline{\alpha_{ij}}\beta_{ij}^M\vartheta_j+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{ij}^M k_{il}^M\right)\|\phi\|_{\mathbb{X}}\right.\right.\\
&\left.\left.+\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M|f_j(0)|+\overline{\alpha_{ij}}\beta_{ij}^M|h_j(0)|+G_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{ij}^M k_{il}^M\right)\right]\right\}.
\end{aligned}\tag{7}$$

Thus, it follows from (5)-(7) that

$$\|T\phi-\phi_0\|_{\mathbb{X}}=\max\left\{\frac{1}{c_i^m},1+\frac{c_i^M}{c_i^m}\right\}\left[\sum_{j=1}^{n}\left(\overline{a_{ij}}d_{ij}^M\epsilon_j+\overline{\alpha_{ij}}\beta_{ij}^M\vartheta_j\right.\right.$$

$$+2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \Big) \|\phi\|_{\mathbb{X}} \Big] + B$$
$$\leq \xi \|\phi\|_{\mathbb{X}} + B \leq \frac{A\xi + B}{1-\xi},$$

which implies that $T\phi \in \mathbb{X}_0$. So, the mapping T is a self-mapping from $\mathbb{X}_0$ to $\mathbb{X}_0$. Finally, we prove that T is a contraction mapping of the $\mathbb{X}_0$. In fact, in view of (H_2), for any ϕ, $\psi \in \mathbb{X}_0$, we obtain

$$\begin{aligned}
&\|T\phi - T\psi\|_0 \\
= &\sup_{t\in\mathbb{R}} \max_{1\leq i\leq n} \bigg\{ \bigg| \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \bigg[\sum_{j=1}^{n} a_{ij}(s) \int_0^\infty d_{ij}(\theta) \big[f_j(\phi_j(s-\theta)) \\
&-f_j(\psi_j(s-\theta)) \big] d\theta + \sum_{j=1}^{n} \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta) \big[h_j(\phi_j^{'}(s-\theta)) - h_j(\psi_j^{'}(s-\theta)) \big] d\theta \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s) \int_0^\infty k_{ij}(\theta) \big[g_j(\phi_j(s-\theta)) \\
&-g_j(\psi_j(s-\theta)) \big] d\theta \int_0^\infty k_{il}(\theta) g_l(\phi_l(s-\theta)) d\theta \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s) \int_0^\infty k_{ij}(\theta) g_j(\psi_j(s-\theta)) d\theta \\
&\times \int_0^\infty k_{il}(\theta) [g_l(\phi_l(s-\theta)) - g_l(\psi_l(s-\theta))] d\theta \bigg] ds \bigg| \bigg\} \\
\leq &\sup_{t\in\mathbb{R}} \max_{1\leq i\leq n} \bigg\{ \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \bigg[\sum_{j=1}^{n} \overline{a_{ij}} d_{ij}^M \epsilon_j \|\phi - \psi\|_0 \\
&+\sum_{j=1}^{n} \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \|(\phi-\psi)^{'}\|_0 + 2\sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} G_l \varepsilon_j k_{ij}^M k_{il}^M \|\phi-\psi\|_0 \bigg] ds \bigg\} \\
\leq &\max_{1\leq i\leq n} \bigg\{ \frac{1}{c_i^m} \sum_{j=1}^{n} \bigg[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2\sum_{l=1}^{n} \overline{b_{ijl}} G_l \varepsilon_j k_{ij}^M k_{il}^M \bigg] \bigg\} \|\phi-\psi\|_{\mathbb{X}} \\
\leq &\xi \|\phi - \psi\|_{\mathbb{X}}
\end{aligned}$$

and

$$\begin{aligned}
&\|(T\phi - T\psi)^{'}\|_0 \\
= &\sup_{t\in\mathbb{R}} \max_{1\leq i\leq n} \bigg\{ \bigg| \sum_{j=1}^{n} a_{ij}(t) \int_0^\infty d_{ij}(\theta) \big[f_j(\phi_j(t-\theta)) - f_j(\psi_j(t-\theta)) \big] d\theta \\
&+\sum_{j=1}^{n} \alpha_{ij}(t) \int_0^\infty \beta_{ij}(\theta) \big[h_j(\phi_j^{'}(t-\theta)) - h_j(\psi_j^{'}(t-\theta)) \big] d\theta \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) \int_0^\infty k_{ij}(\theta) \big[g_j(\phi_j(t-\theta)) - g_j(\psi_j(t-\theta)) \big] d\theta
\end{aligned}$$

$$\times \int_0^\infty k_{il}(\theta) g_l(\phi_l(t-\theta))d\theta + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \int_0^\infty k_{ij}(\theta) g_j(\psi_j(t-\theta))d\theta$$

$$\times \int_0^\infty k_{il}(\theta)\big[g_l(\phi_l(t-\theta)) - g_l(\psi_l(t-\theta))\big]d\theta - \int_{-\infty}^t c_i(t) e^{-\int_s^t c_i(u)du}$$

$$\times \Big[\sum_{j=1}^n a_{ij}(s) \int_0^\infty d_{ij}(\theta)\big[f_j(\phi_j(s-\theta)) - f_j(\psi_j(s-\theta))\big]d\theta$$

$$+ \sum_{j=1}^n \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta)\big[h_j(\phi_j^{'}(s-\theta)) - h_j(\psi_j^{'}(s-\theta))\big]d\theta$$

$$+ \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \int_0^\infty k_{ij}(\theta)\big[g_j(\phi_j(s-\theta)) - g_j(\psi_j(s-\theta))\big]d\theta$$

$$\times \int_0^\infty k_{il}(\theta) g_l(\phi_l(s-\theta))d\theta + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \int_0^\infty k_{ij}(\theta) g_j(\psi_j(s-\theta))d\theta$$

$$\times \int_0^\infty k_{il}(\theta)\big[g_l(\phi_l(s-\theta)) - g_l(\psi_l(s-\theta))\big]d\theta\Big]ds\Big|\Big\}$$

$$\leq \max_{1\leq i\leq n} \Big\{ \Big(1 + \frac{c_i^M}{c_i^m}\Big) \Big[\sum_{j=1}^n \Big(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j$$

$$+ 2\sum_{l=1}^n \overline{b_{ijl}} G_l \varepsilon_j k_{ij}^M k_{il}^M \Big)\Big]\Big\} \|\phi - \psi\|_{\mathbb{X}} \leq \xi \|\phi - \psi\|_{\mathbb{X}}.$$

Thus,

$$\|T\phi - T\psi\|_{\mathbb{X}} \leq \xi \|\phi - \psi\|_{\mathbb{X}}.$$

Notice that $\xi < 1$, it means that the mapping T is a contraction mapping. By Banach fixed point theorem, there exists a unique fixed point $z \in \mathbb{X}_0$ such that $Tz = z$, which implies that system (1) has a unique almost periodic solution. This completes the proof. □

3. Global exponential stability of almost periodic solution

Definition 3.1. The almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ of system (1) with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \ldots, \phi_n^*(t))^T$ is said to be globally exponentially stable. If there exist constants $\lambda > 0$ and $M \geq 1$ such that for every solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of system (1) with any initial value $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T$ satisfies

$$\|x(t) - x^*(t)\| = \max\{\|x(t) - x^*(t)\|, \|x'(t) - x^{*'}(t)\|\} \leq M\|\psi\|_{\mathbb{X}} e^{-\lambda t}, \ \forall t > 0,$$

where

$$\|\psi\|_{\mathbb{X}} = \max\Big\{ \sup_{t\in(-\infty,0]} \max_{1\leq i\leq n} |\phi_i(t) - \phi_i^*(t)|, \sup_{t\in(-\infty,0]} \max_{1\leq i\leq n} |\phi_i'(t) - (\phi_i^*)'(t)| \Big\}.$$

Theorem 3.2. *If conditions $(H_1)-(H_4)$ hold, then system (1) has a unique continuously differentiable almost periodic solution $z(t)$ which is globally exponential stable.*

Proof. It follows from Theorem 2.5 that system (1) has a unique almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T \in \mathbb{X}_0$ with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \ldots, \phi_n^*(t))^T$. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ be an arbitrary solution of system (1) with initial value $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T$. Let $y_i(t) = x_i(t) - x_i^*(t)$, $\psi_i(t) = \phi(t) - \phi_i^*(t)$, $i = 1, 2, \ldots, n$, then

$$\begin{aligned}
& y_i^{'}(s) + c_i(s)y_i(s) \\
= & \sum_{j=1}^{n} a_{ij}(s) \int_0^\infty d_{ij}(\theta)\big[f_j\big(y_j(s-\theta) + x_j^*(s-\theta)\big) - f_j(x_j^*(s-\theta))\big]d\theta \\
& + \sum_{j=1}^{n} \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta)\big[h_j\big(y_j^{'}(s-\theta) + (x_j^*)^{'}(s-\theta)\big) - h_j((x_j^*)^{'}(s-\theta))\big]d\theta \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\bigg[\bigg(\int_0^\infty k_{ij}(\theta)\big[g_j\big(y_j(s-\theta) + x_j^*(s-\theta)\big) - g_j(x_j^*(s-\theta))\big]d\theta \\
& \times \int_0^\infty k_{il}(\theta)g_l\big(y_l(s-\theta) + x_l^*(s-\theta)\big)d\theta\bigg) + \bigg(\int_0^\infty k_{ij}(\theta)g_j(x_j^*(s-\theta))d\theta \\
& \times \int_0^\infty k_{il}(\theta)\big[g_l\big(y_l(s-\theta) + x_l^*(s-\theta)\big) - g_l(x_l^*(s-\theta))\big]d\theta\bigg)\bigg], \qquad (8)
\end{aligned}$$

where $i = 1, 2, \ldots, n$. Let H_i and H_i^* be defined by

$$\begin{aligned}
H_i(\eta) = \ & c_i^m - \eta - \sum_{j=1}^{n}\bigg[\overline{a_{ij}}\epsilon_j \int_0^\infty |d_{ij}(\theta)|e^{\eta\theta}d\theta + \overline{\alpha_{ij}}\vartheta_j \int_0^\infty |\beta_{ij}(\theta)|e^{\eta\theta}d\theta \\
& + 2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}}G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)|e^{\eta\theta}d\theta\bigg],
\end{aligned}$$

$$\begin{aligned}
H_i^*(\eta) = \ & c_i^m - \eta - (c_i^M + c_i^m)\sum_{j=1}^{n}\bigg[\overline{a_{ij}}\epsilon_j \int_0^\infty |d_{ij}(\theta)|e^{\eta\theta}d\theta \\
& + \overline{\alpha_{ij}}\vartheta_j \int_0^\infty |\beta_{ij}(\theta)|e^{\eta\theta}d\theta + 2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}}G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)|e^{\eta\theta}d\theta\bigg],
\end{aligned}$$

where $i = 1, 2, \ldots, n$, $\eta \in [0, \infty)$. By (H_4), we obtain that for $i = 1, 2, \ldots, n$,

$$\begin{aligned}
H_i(0) = \ & c_i^m - \sum_{j=1}^{n}\bigg[\int_0^\infty \bigg(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)| + \overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)| \\
& + 2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}}G_l k_{il}^M |k_{ij}(\theta)|\bigg)d\theta\bigg] > 0
\end{aligned}$$

and

$$H_i^*(0) = c_i^m - (c_i^M + c_i^m)\sum_{j=1}^{n}\bigg[\overline{a_{ij}}\epsilon_j \int_0^\infty |d_{ij}(\theta)|d\theta + \overline{\alpha_{ij}}\vartheta_j \int_0^\infty |\beta_{ij}(\theta)|d\theta$$

$$+2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| d\theta \Bigg] > 0.$$

Since H_i, H_i^* are continuous on $[0,\infty)$ and $H_i(\eta), H_i^*(\eta) \to -\infty$ as $\eta \to \infty$, there exist $\eta_i^*, \gamma_i^* > 0$ such that $H_i(\eta_i^*) = H_i^*(\gamma_i^*) = 0$, and $H_i(\eta_i) > 0$ for $\eta_i \in (0, \eta_i^*)$, $(H_i^*(\gamma_i) > 0$ for $\gamma_i \in (0, \gamma_i^*)$. By choosing $\gamma = \min\{\eta_1^*, \eta_2^*, \ldots, \eta_n^*, \gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*\}$, we have

$$H_i(\gamma) \geq 0, \quad H_i^*(\gamma) \geq 0, \quad i = 1, 2, \ldots, n.$$

So, we can choose a positive constant $0 < \lambda < \min\{\gamma, \lambda_0\})$ such that $H_i(\lambda) > 0$ and $H_i^*(\lambda) > 0$, which implies that

$$\frac{1}{c_i^m - \lambda} \sum_{j=1}^{n} \Bigg[\int_0^\infty \Big(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| + 2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \Big) e^{\lambda\theta} d\theta \Bigg] < 1, \tag{9}$$

$$\Big(1 + \frac{c_i^m}{c_i^m - \lambda}\Big) \sum_{j=1}^{n} \Bigg[\int_0^\infty \Big(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| + 2\varepsilon_j \sum_{l=1}^{n} \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \Big) e^{\lambda\theta} d\theta \Bigg] < 1, \tag{10}$$

where $i = 1, 2, \ldots, n$.

Multiplying (8) by $e^{\int_0^s c_i(u)du}$ and integrating on $[0,t]$, we have

$$\begin{aligned} y_i(t) = {} & y_i(0) e^{-\int_0^t c_i(u)du} \\ & + \int_0^t e^{-\int_s^t c_i(u)du} \Bigg[\sum_{j=1}^{n} a_{ij}(s) \int_0^\infty d_{ij}(\theta) \big[f_j\big(y_j(s-\theta) + x_j^*(s-\theta)\big) \\ & - f_j(x_j^*(s-\theta))\big] d\theta + \sum_{j=1}^{n} \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta) \big[f_j\big(y_j^{'}(s-\theta) + (x_j^*)^{'}(s-\theta)\big) \\ & - f_j(x_j^*(s-\theta))\big] d\theta + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s) \Big(\int_0^\infty k_{ij}(\theta) \big[g_j\big(y_j(s-\theta) \\ & + x_j^*(s-\theta)\big) - g_j(x_j^*(s-\theta))\big] d\theta \int_0^\infty k_{il}(\theta) g_l\big(y_l(s-\theta) + x_l^*(s-\theta)\big) d\theta \Big) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s) \Big(\int_0^\infty k_{ij}(\theta) g_j(x_j^*(s-\theta)) d\theta \int_0^\infty k_{il}(\theta) \big[g_j\big(y_j(s-\theta) \\ & + x_j^*(s-\theta)\big) - g_j(x_j^*(s-\theta))\big] d\theta \Big) \Bigg] ds, \ \ i = 1, 2, \ldots, n. \end{aligned} \tag{11}$$

Let

$$M = \max_{1\le i\le n}\left\{c_i^m\bigg(\sum_{j=1}^n\bigg[\int_0^\infty\bigg(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|\right.$$
$$\left.+2\varepsilon_j\sum_{l=1}^n\overline{b_{ijl}}G_lk_{il}^M|k_{ij}(\theta)|\bigg)d\theta\bigg]\bigg)^{-1}\right\}.$$

By (H_4) we have $M>1$. Thus,

$$\frac{1}{M}-\frac{1}{c_i^m-\lambda}\bigg[\sum_{j=1}^n\int_0^\infty\bigg(\bar{a}_{ij}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|$$
$$+2\varepsilon_j\sum_{l=1}^n\overline{b_{ijl}}G_lk_{il}^M|k_{ij}(\theta)|\bigg)e^{\lambda\theta}d\theta\bigg]\le 0 \tag{12}$$

and

$$\|y(t)\|_1=\|\psi(t)\|_1\le\|\psi\|_{\mathbb{X}}\le M\|\psi\|_{\mathbb{X}}e^{-\lambda t},\ \forall t\in(-\infty,0],$$

where $\lambda>0$ as in (9) and (10). We claim that

$$\|y(t)\|_1\le M\|\psi\|_{\mathbb{X}}e^{-\lambda t},\ t>0. \tag{13}$$

To prove (13), we first show for any $p>1$, the following inequality holds

$$\|y(t)\|_1\le pM\|\psi\|_{\mathbb{X}}e^{-\lambda t},\ t>0. \tag{14}$$

If (14) is not true, then there must be some $t_1>0$, and some $i,\iota\in\{1,2,\ldots,n\}$ such that

$$\|y(t_1)\|_1=\max\{\|y(t_1)\|_0,\|y^{'}(t_1)\|_0\}=\max\{|y_i(t_1)|,|y_\iota^{'}(t_1)|\}=pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1} \tag{15}$$

and

$$\|y(t)\|_1\le pM\|\psi\|_{\mathbb{X}}e^{-\lambda t},\ t\in(-\infty,t_1]. \tag{16}$$

By (9)-(11),(12),(16) and (H_2), we get

$$\begin{aligned}
&|y_i(t_1)|\\
\le\ & e^{-\int_0^{t_1}c_i(u)du}\|\psi\|_{\mathbb{X}}+\int_0^{t_1}e^{-\int_s^{t_1}c_i(u)du}\bigg[\sum_{j=1}^n\overline{a_{ij}}\epsilon_j\int_0^\infty|d_{ij}(\theta)|||y_j(s-\theta)||d\theta\\
&+\sum_{j=1}^n\overline{\alpha_{ij}}\vartheta_j\int_0^\infty|\beta_{ij}(\theta)||y_j^{'}(s-\theta)|d\theta\\
&+2\sum_{j=1}^n\sum_{l=1}^n\overline{b_{ijl}}G_lk_{il}^M\varepsilon_j\int_0^\infty|k_{ij}(\theta)||y_j(s-\theta)|d\theta\bigg]ds\\
\le\ & e^{-c_i^mt_1}\|\psi\|_{\mathbb{X}}+\int_0^{t_1}e^{-\int_s^{t_1}c_i(u)du}\bigg[\sum_{j=1}^n\overline{a_{ij}}\epsilon_jpM\|\psi\|_{\mathbb{X}}e^{-\lambda s}\int_0^\infty|d_{ij}(\theta)|e^{\lambda\theta}d\theta
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\overline{\alpha_{ij}}\vartheta_j pM\|\psi\|_{\mathbb{X}}e^{-\lambda s}\int_0^\infty|\beta_{ij}(\theta)|e^{\lambda\theta}d\theta\\
&+2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}G_l\varepsilon_j k_{il}^M pM\|\psi\|_{\mathbb{X}}e^{-\lambda s}\int_0^\infty|k_{ij}(\theta)|e^{\lambda\theta}d\theta\Bigg]ds\\
\le\ & e^{-c_i^m t_1}\|\psi\|_{\mathbb{X}}+pM\|\psi\|_{\mathbb{X}}\Bigg[\sum_{j=1}^{n}\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|\Big)e^{\lambda\theta}d\theta\\
&+2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}G_l\varepsilon_j k_{il}^M\int_0^\infty|k_{ij}(\theta)|e^{\lambda\theta}d\theta\Bigg]\int_0^{t_1}e^{-c_i^m(t_1-s)-\lambda s}ds\\
=\ & e^{-c_i^m t_1}\|\psi\|_{\mathbb{X}}+pM\|\psi\|_{\mathbb{X}}\sum_{j=1}^{n}\Bigg[\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|\Big)e^{\lambda\theta}d\theta\\
&+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{il}^M\int_0^\infty|k_{ij}(\theta)|e^{\lambda\theta}d\theta\Bigg]\frac{e^{-\lambda t_1}}{c_i^m-\lambda}(1-e^{(\lambda-c_i^m)t_1})\\
=\ & pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}\Bigg\{\frac{1}{pM}e^{(\lambda-c_i^m)t_1}+\frac{1}{c_i^m-\lambda}(1-e^{(\lambda-c_i^m)t_1})\\
&\times\Bigg[\sum_{j=1}^{n}\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|\\
&+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{il}^M|k_{ij}(\theta)|\Big)e^{\lambda\theta}d\theta\Bigg]\Bigg\}\\
<\ & pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}\Bigg\{\frac{1}{M}e^{(\lambda-c_i^m)t_1}+\frac{1}{c_i^m-\lambda}(1-e^{(\lambda-c_i^m)t_1})\\
&\times\Bigg[\sum_{j=1}^{n}\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|\\
&+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{il}^M|k_{ij}(\theta)|\Big)e^{\lambda\theta}d\theta\Bigg]\Bigg\}\\
=\ & pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}\Bigg\{\Bigg(\frac{1}{M}-\frac{1}{c_i^m-\lambda}\sum_{j=1}^{n}\Bigg[\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|\\
&+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|\Big)e^{\lambda\theta}d\theta+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_l k_{il}^M\int_0^\infty|k_{ij}(\theta)|e^{\lambda\theta}d\theta\Bigg]\Bigg)e^{(\lambda-c_i^m)t_1}\\
&+\frac{1}{c_i^m-\lambda}\sum_{j=1}^{n}\Bigg[\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|
\end{aligned}
$$

$$+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_lk_{il}^M|k_{ij}(\theta)|\Big)d\theta\Big]$$

$$\leq pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}\frac{1}{c_i^m-\lambda}\sum_{j=1}^{n}\Big[\int_0^\infty\Big(\overline{a_{ij}}\epsilon_j|d_{ij}(\theta)|+\overline{\alpha_{ij}}\vartheta_j|\beta_{ij}(\theta)|$$

$$+2\varepsilon_j\sum_{l=1}^{n}\overline{b_{ijl}}G_lk_{il}^M|k_{ij}(\theta)|\Big)d\theta\Big]<pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}. \tag{17}$$

Direct differentiation of (11) gives

$$\begin{aligned}
y_i^{'}(t)&=-c_i(t)\phi_i(0)e^{-\int_0^t c_i(u)du}\\
&+\sum_{j=1}^{n}a_{ij}(t)\int_0^\infty d_{ij}(\theta)\big[f_j\big(y_j(t-\theta)+x_j^*(t-\theta)\big)-f_j(x_j^*(t-\theta))\big]d\theta\\
&+\sum_{j=1}^{n}\alpha_{ij}(t)\int_0^\infty \beta_{ij}(\theta)\big[h_j\big(y_j^{'}(t-\theta)+(x_j^*)^{'}(t-\theta)\big)-h_j((x_j^*)^{'}(t-\theta))\big]d\theta\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\Big(\int_0^\infty k_{ij}(\theta)\big[g_j\big(y_j(t-\theta)+x_j^*(t-\theta)\big)-g_j(x_j^*(t-\theta))\big]d\theta\\
&\times\int_0^\infty k_{il}(\theta)g_l\big(y_l(t-\theta)+x_l^*(t-\theta)\big)d\theta\Big)\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\Big(\int_0^\infty k_{ij}(\theta)g_j(x_j^*(t-\theta))d\theta\\
&\times\int_0^\infty k_{il}(\theta)\big[g_l\big(y_l(t-\theta)+x_l^*(t-\theta)\big)-g_l(x_l^*(t-\theta))\big]d\theta\Big)\\
&-\int_0^t c_i(t)e^{-\int_s^t c_i(u)du}\Big[\sum_{j=1}^{n}a_{ij}(s)\int_0^\infty d_{ij}(\theta)\big[f_j\big(y_j(s-\theta)+x_j^*(s-\theta)\big)\\
&-f_j(x_j^*(s-\theta))\big]d\theta+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(s)\Big(\int_0^\infty k_{ij}(\theta)[g_j(y_j(s-\theta)\\
&+x_j^*(s-\theta))-g_j(x_j^*(s-\theta))]d\theta\int_0^\infty k_{il}(\theta)g_l(y_l(s-\theta)+x_l^*(s-\theta))d\theta\Big)\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(s)\Big(\int_0^\infty k_{ij}(\theta)g_j(x_j^*(s-\theta))d\theta\\
&\times\int_0^\infty k_{il}(\theta)[g_l(y_l(s-\theta)+x_l^*(s-\theta))-g_l(x_l^*(s-\theta))]d\theta\Big)\Big]ds,
\end{aligned} \tag{18}$$

where $i=1,2,\ldots,n$. Thus, we have by (9),(10),(18) and $(H_2)-(H_3)$ that

$$
\begin{aligned}
|y_\iota^{'}(t_1)| \leq\ & c_\iota^M \|\psi\|_{\mathbb{X}} e^{-\int_0^{t_1} c_\iota(u)du} + \sum_{j=1}^n \overline{a_{\iota j}} \int_0^\infty |d_{\iota j}(\theta)| \epsilon_j |y_j(t_1-\theta)| d\theta \\
& + \sum_{j=1}^n \overline{\alpha_{\iota j}} \int_0^\infty |\beta_{\iota j}(\theta)| \vartheta_j |y_j^{'}(t_1-\theta)| d\theta \\
& + 2\sum_{j=1}^n \sum_{l=1}^n \overline{b_{\iota j l}} G_l k_{\iota l}^M \int_0^\infty |k_{\iota j}(\theta)| \varepsilon_j |y_j(t_1-\theta)| d\theta \\
& + \int_0^{t_1} c_\iota^M e^{-c_\iota^m (t_1-s)} \Bigg[\sum_{j=1}^n \overline{a_{\iota j}} pM \|\psi\|_{\mathbb{X}} e^{-\lambda t_1} \int_0^\infty |d_{\iota j}(\theta)| \epsilon_j |y_j(s-\theta)| d\theta \\
& + \sum_{j=1}^n \overline{\alpha_{\iota j}} pM \|\psi\|_{\mathbb{X}} e^{-\lambda t_1} \int_0^\infty |\beta_{\iota j}(\theta)| \vartheta_j |y_j^{'}(s-\theta)| d\theta \\
& + 2\sum_{j=1}^n \sum_{l=1}^n \overline{b_{\iota j l}} G_l k_{\iota l}^M pM \|\psi\|_{\mathbb{X}} e^{-\lambda t_1} \int_0^\infty |k_{\iota j}(\theta)| \varepsilon_j |y_j(s-\theta)| d\theta \Bigg] ds \\
& = pM\|\psi\|_{\mathbb{X}} e^{-\lambda t_1} \Bigg\{ \frac{c_\iota^M}{pM} e^{(\lambda - c_\iota^m)t_1} + \left(1 + c_\iota^M \int_0^{t_1} e^{(t_1-s)(\lambda - c_\iota^m)} ds\right) \\
& \times \Bigg[\sum_{j=1}^n \int_0^\infty \Bigg(\overline{a_{\iota j}} \epsilon_j |d_{\iota j}(\theta)| + \overline{\alpha_{\iota j}} \vartheta_j |\beta_{\iota j}(\theta)| \\
& + 2\sum_{j=1}^n \sum_{l=1}^n \overline{b_{\iota j l}} G_l \varepsilon_j k_{\iota l}^M |k_{\iota j}(\theta)| \Bigg) e^{\lambda\theta} d\theta \Bigg] \Bigg\} \\
& < pM\|\psi\|_{\mathbb{X}} e^{-\lambda t_1} \Bigg\{ \frac{c_\iota^M}{M} e^{(\lambda - c_\iota^m)t_1} + \left(1 + \frac{c_\iota^M}{c_\iota^m - \lambda}\big(1 - e^{(\lambda - c_\iota^m)t_1}\big)\right) \\
& \times \Bigg[\sum_{j=1}^n \int_0^\infty \Bigg(\overline{a_{\iota j}} \epsilon_j |d_{\iota j}(\theta)| + \overline{\alpha_{\iota j}} \vartheta_j |\beta_{\iota j}(\theta)| \\
& + 2\sum_{j=1}^n \sum_{l=1}^n \overline{b_{\iota j l}} G_l \varepsilon_j k_{\iota l}^M |k_{\iota j}(\theta)| \Bigg) e^{\lambda\theta} d\theta \Bigg] \Bigg\} \\
& = pM\|\psi\|_{\mathbb{X}} e^{-\lambda t_1} \Bigg\{ \Bigg(\frac{1}{M} - \frac{1}{c_\iota^m - \lambda} \Bigg[\sum_{j=1}^n \int_0^\infty \Big(\overline{a_{\iota j}} \epsilon_j |d_{\iota j}(\theta)| + \overline{\alpha_{\iota j}} \vartheta_j |\beta_{\iota j}(\theta)| \Big) \\
& \times e^{\lambda\theta} d\theta + 2\sum_{j=1}^n \sum_{l=1}^n \overline{b_{\iota j l}} G_l \varepsilon_j k_{\iota l}^M \int_0^\infty |k_{\iota j}(\theta)| e^{\lambda\theta} d\theta \Bigg] \Bigg) c_\iota^M e^{(\lambda - c_\iota^m)t_1} \\
& + \left(1 + \frac{c_\iota^M}{c_\iota^m - \lambda}\right) \Bigg[\sum_{j=1}^n \int_0^\infty \Big(\overline{a_{\iota j}} \epsilon_j |d_{\iota j}(\theta)| + \overline{\alpha_{\iota j}} \vartheta_j |\beta_{\iota j}(\theta)|
\end{aligned}
$$

$$+ 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{\iota jl}}G_l\varepsilon_j k_{\iota l}^M|k_{\iota j}(\theta)|\Big)e^{\lambda\theta}d\theta\Big]\Big\}$$
$$< pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}\left(1+\frac{c_\iota^M}{c_\iota^m-\lambda}\right)\Big[\sum_{j=1}^{n}\int_0^\infty\big(\overline{a_{\iota j}}\epsilon_j|d_{\iota j}(\theta)|$$
$$+\overline{\alpha_{\iota j}}\vartheta_j|\beta_{\iota j}(\theta)|\big)e^{\lambda\theta}d\theta + 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{\iota jl}}G_l\varepsilon_j k_{\iota l}^M\int_0^\infty|k_{\iota j}(\theta)|e^{\lambda\theta}d\theta\Big]$$
$$< pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}. \tag{19}$$

In view of (17) and (19), we obtain $\|y(t_1)\|_1 < pM\|\psi\|_{\mathbb{X}}e^{-\lambda t_1}$, which contradicts the equality (15), and so (14) holds. Letting $p\to 1$, then (13) holds. Hence, the almost periodic solution of system (1) is globally exponentially stable. The proof is complete. □

4. An example

In this section, we give one example to illustrate our result. Let

$$f_1(x)=f_2(x)=\frac{1}{10}\sin\big(\frac{1}{2^{\frac12}}x\big),\ h_1(x)=h_2(x)=\frac{1}{20}\sin\big(\frac{1}{2^{\frac32}}x\big),$$
$$g_1(x)=g_2(x)=\frac{1}{20}|\arctan\big(\frac{1}{2^{\frac12}}x\big)|,$$
$$a_{11}(t)=1+\cos(2\pi t),\ a_{12}(t)=2+\cos(2\pi t),\quad a_{21}(t)=2+\cos(2\pi t),$$
$$a_{22}(t)=3+\cos(2\pi t),\quad \alpha_{11}(t)=1+\sin(\pi t),\quad \alpha_{12}(t)=2+\sin^5(2\pi t),$$
$$\alpha_{21}(t)=2+\cos^3(2\pi t),\quad \alpha_{22}(t)=3+\cos(\pi t),\quad c_1(t)=25+5\sin(2\pi t),$$
$$c_2(t)=35+5\sin(2\pi t),\quad I_1(t)=1+\sin(\pi t),\quad I_2(t)=1+\cos(\pi t),$$
$$b_{111}(t)=b_{211}(t)=\frac14+\frac14\sin(\pi t),\quad b_{112}(t)=b_{212}(t)=\frac13+\frac13\cos(\pi t),$$
$$b_{121}(t)=b_{221}(t)=\frac15+\frac15\cos(\pi t),\quad b_{122}(t)=b_{222}(t)=\frac16+\frac16\sin(\pi t),$$
$$(d_{ij}(\theta))_{2\times2}=(k_{ij}(\theta))_{2\times2}=(\beta_{ij}(\theta))_{2\times2}=\begin{pmatrix} e^{-10\theta} & 0\\ 0 & e^{-50\theta}\end{pmatrix}.$$

Then system (1) has exactly one continuously differentiable almost periodic solution, which is globally exponential stable.

Proof. By calculating,

$c_1^m = 20,\ c_1^M = 30,\ c_2^m = 30,\ c_2^M = 40,\ \overline{a_{11}} = \overline{\alpha_{11}} = 2,\ \overline{a_{12}} = \overline{\alpha_{12}} = 3,\ \overline{a_{21}} = \overline{\alpha_{21}} = 3,\ \overline{a_{22}} = \overline{\alpha_{22}} = 4,\ \overline{b_{111}} = \overline{b_{211}} = \frac12,\ \overline{b_{112}} = \overline{b_{212}} = \frac23,\ \overline{b_{121}} = \overline{b_{221}} = \frac25,\ \overline{b_{122}} = \overline{b_{222}} = \frac13,\ \epsilon_1 = \epsilon_2 = \frac{1}{10},\ \vartheta_1 = \vartheta_2 = \frac{1}{20},\ \varepsilon_1 = \varepsilon_2 = \frac{1}{20},\ G_1 = G_2 = \frac{\pi}{40},\ d_{11}^M = k_{11}^M = \beta_{11}^M = \frac{1}{10},\ d_{12}^M = k_{12}^M = \beta_{12}^M = 0,\ d_{21}^M = k_{21}^M = \beta_{21}^M = 0,\ d_{22}^M = k_{22}^M = \beta_{22}^M = \frac{1}{50}$, hence we have

$$\xi = \max_{1\le i\le 2}\left\{\max\left\{\frac{1}{c_i^m}, 1+\frac{c_i^M}{c_i^m}\right\}\right.$$

$$\times \sum_{j=1}^{2} \left[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2\varepsilon_j \sum_{l=1}^{2} \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right] \Big\}$$
$$= \max \left\{ \frac{3}{40} + \frac{\pi}{16000}, \frac{7}{100} + \frac{7\pi}{240000} \right\} < 1.$$

It is obvious that $(H_1) - (H_4)$ are satisfied. By Theorem 2.5 and Theorem 3.2, system (1) has exactly one continuously differentiable almost periodic solution, which is globally exponential stable. □

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