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THE VERTEX AND EDGE PI INDICES OF GENERALIZED HIERARCHICAL PRODUCT OF GRAPHS

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Abstract. Pattabiraman and Paulraja [K. Pattabiraman, P. Paulraja, Vertex and edge PI indices of the generalized hierarchical product of graphs, Discrete Appl. Math. 160 (2012) 1376- 1384] obtained exact formulas for the *vertex and edge PI* indices of *generalized hierarchical product* of graphs. The aim of this note is to improve the main results of this paper.

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1. Introduction

Throughout this paper all graphs considered are finite, simple and connected. The distance $d_G(u, v)$ between the vertices u and v of a graph G is equal to the length of a shortest path that connects u and v. Suppose G is a graph with vertex and edge sets $V = V(G)$ and $E = E(G)$, respectively. Suppose $e = ab \in E(G)$. The number of edges of G whose distance to the vertex u is smaller than the distance to the vertex v is denoted by $m_u^G(e)$. The edge PI index of G, $\mathrm{PI}_e(G)$, of a graph G is defined as $\mathrm{PI}_e(G) = \sum_{e=uv\in E(G)}(m_u^G(e) + m_v^G(e))$ [4, 5]. In a similar way, the quantities $n_a^G(e)$ is defined as the number of vertices closer to a than to b. In other words, $n_a^G(e) = |\{u \in V(G)|d(u, a) < d(u, b)\}|$. The vertex PI index of G, $\mathrm{PI}_v(G)$, is defined as the summation of $[n_u^G(uv) + n_v^G(uv)]$ over all edges of G [6, 7].

The edges $e = uv$ and $f = xy$ of G are said to be equidistant edges if $min\{d_G(u, x), d_G(u, y)\} = min\{d_G(v, x), d_G(v, y)\}$. For $e = uv$ in G, the number of equidistant vertices of e is denoted by $N_G(e)$ and the number of equidistant

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edges of e is denoted by $M_G(e)$. Then the above definitions are equivalent to

$$\mathrm{PI}_v(G) = |V(G)||E(G)| - \sum_{e\in E(G)} N_G(e), \qquad \mathrm{PI}_e(G) = |E(G)|^2 - \sum_{e\in E(G)} M_G(e).$$

Suppose G and H are graphs and $U \subseteq V(G)$. The generalized hierarchical product, denoted by $G(U)\sqcap H$, is the graph with vertex set $V(G)\times V(H)$ and two vertices (g,h) and (g',h') are adjacent if and only if $g = g' \in U$ and $hh' \in E(H)$ or, $gg' \in E(G)$ and $h = h'$. This graph operation introduced recently by Barriére et al. [2, 3] and found some applications in computer science.

Most of our notation is standard and taken mainly from [1, 9]. The path graph with n vertices is denoted by P_n.

2. Main results

Let $G = (V,E)$ be a graph and $U \subseteq V$. We need some notation than taken from [8]. We encourage the interested readers to consult this paper and references therein for more information on this topic. Following Pattabiraman and Paulraja [8], an $u - v$ path through U in $G(U)$ is an $u - v$ path in G containing some vertex $w \in U$ (vertex w could be the vertex u or v). Let $d_{G(U)}(u,v)$ denote the length of a shortest $u - v$ path through U in G. Notice that, if one of the vertices u and v belong to U, then $d_{G(U)}(u,v) = d_G(u,v)$. A vertex $x \in V(G(U))$ is said to be equidistant from $e = uv \in E(G(U))$ through U in $G(U)$, if $d_{G(U)}(u,x) = d_{G(U)}(v,x)$. For an edge e in $G(U)$, let $N_{G(U)}(e)$ denote the number of equidistant vertices of e through U in $G(U)$. Then $\mathrm{PI}_v(G(U))$ can be defined as follows:

$$\mathrm{PI}_v(G(U)) = \sum_{e\in E(G(U))} (|V(G(U))| - N_{G(U)}(e)).$$

For $e \in E(G)$ and $S \subseteq V(G)$, let $N_{\langle S\rangle}(e)$ denote the number of equidistant vertices of e (in G) contained in S. The edges $e = uv$ and $f = xy$ of $G(U)$ are said to be equidistant edges through U in $G(U)$ if $min\{d_{G(U)}(u,x), d_{G(U)}(u,y)\} = min\{d_{G(U)}(v,x), d_{G(U)}(v,y)\}$. Let $M_{G(U)}(e)$ denote the number of equidistant edges of e through U in $G(U)$. Then $\mathrm{PI}_e(G(U))$ is defined as follows:

$$\mathrm{PI}_e(G(U)) = \sum_{e\in E(G(U))} (|E(G(U))| - M_{G(U)}(e)).$$

Let $G_i = (V_i,E_i)$, $1 \le i \le N$, be a graph with vertex set V_i having a distinguished or root vertex 0. Following Barriére et al. [2, 3], the hierarchical product $H = G_N \sqcap ... \sqcap G_2 \sqcap G_1$ is the graph with vertices the $N-$tuples $x_N...x_3x_2x_1$, $x_i \in V_i$, and edges defined by the adjacencies:

$$x_N...x_3x_2x_1 \sim \begin{cases} x_N...x_3x_2y_1 & \text{if } y_1 \sim x_1 \text{ in } G_1,\\ x_N...x_3y_2x_1 & \text{if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0,\\ x_N...y_3x_2x_1 & \text{if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0,\\ \quad\vdots & \quad\vdots\\ y_N...x_3x_2x_1 & \text{if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = x_2 = ... = x_{N-1} = 0.\end{cases}$$

A path graph with n vertices, is denoted by P_n and a caterpillar is a tree in which all the vertices are within distance 1 of a central path. By definition of hierarchical product, it is clear that if P_m is a path graph and S_n is a rooted star graph with root vertex r such that $deg(r) > 1$ then $P_m \sqcap S_n$ is a caterpillar with order nm and generally, the hierarchical product of an arbitrary sequence of acyclic graphs is again an acyclic graph. Therefore, we can write:

Lemma 2.1. *If $G_1, G_2, \ldots, G_n$ are trees with orders $m_1, \ldots, m_n$, respectively, then*

$$\mathrm{PI}_v(G_n \sqcap ... \sqcap G_2 \sqcap G_1) = (\prod_{i=1}^{n} m_i - 1)\prod_{i=1}^{n} m_i,$$
$$\mathrm{PI}_e(G_n \sqcap ... \sqcap G_2 \sqcap G_1) = (\prod_{i=1}^{n} m_i - 1)(\prod_{i=1}^{n} m_i - 2).$$

Let $G_1, G_2, \ldots, G_n$ be connected rooted graphs with root vertices $r_1, \cdots, r_n$, respectively and $e = (a_n, ..., a_{i+1}, u, r_{i-1}, ..., r_1)(a_n, ..., a_{i+1}, v, r_{i-1}, ..., r_1)$ is an edge of H such that $uv \in E(G_i)$. In order to simplify our notation, we will denote $n_{(a_n,...,a_{i+1},u,r_{i-1},...,r_1)}(e)$ by $n_1(e)$, $n_{(a_n,...,a_{i+1},v,r_{i-1},...,r_1)}(e)$ by $n_2(e)$, $m_{(a_n,...,a_{i+1},u,r_{i-1},...,r_1)}(e)$ by $m_1(e)$ and $m_{(a_n,...,a_{i+1},v,r_{i-1},...,r_1)}(e)$ by $m_2(e)$.

In what follows, let $\prod_i^j f_i = 1$ and $\sum_i^j f_i = 0$ for each $i, j \in \{0, 1, 2, ...\}$, that $i - j = 1$. Furthermore, let $\prod_i^j f_i = \sum_i^j f_i = 0$ for every $i, j \in \{0, 1, 2, ...\}$, such that $i - j > 1$. Also, for a sequence of graphs, $G_1, G_2, \ldots, G_n$, we set $|V_{i,j}| = \prod_{k=i}^j |V(G_k)|$ and $|V_{i,j}^l| = \prod_{k=i,k\neq l}^j |V(G_k)|$.

The main results of [8] are Theorems 2.2 and 3.1. We claim that these results are incorrect. We first explain the reason that makes Theorem 2.2 to be incorrect. In [8, Eq. 2.3], the authors claim that for each edge $e' = (u_r, v_i)(u_s, v_i) \in G(U) \sqcap H$ such that $v_i \in V(H)$ and $e = u_r u_s \in E(G)$, we have $N_{G(U)\sqcap H}(e') = |V(H)|N_{G(U)}(e)$. In Figure 2, a counterexample for this argument is presented. Notice that if $U = \{r\}$, $e' = (y, 1)(z, 1)$ then $N_{G(U)\sqcap H}(e') = 6$, but $|V(H)|N_{G(U)}(e) = 2$, which is impossible. In Figure 3, a family of enough large counterexamples are presented. In this figure, $H = P_m$, $U = \{x\}$ and $|V(G)| = 2n+1$. Then $\mathrm{PI}_v(G(U)\sqcap H) = 2nm(2nm+2m+n-2)+m(m-1)$. But, [8, Theorem 2.2] implies that $\mathrm{PI}_v(G(U)\sqcap H) = 2nm(3nm+2m-1)+m(m-1)$. Then $|2nm(2nm+2m+n-2)+m(m-1)-(2nm(3nm+2m-1)+m(m-1))| = 2nm(nm-n+1) > 0$, leads to another contradiction.

In the following theorem a correct form of [8, Theorem 2.2] is presented.

Theorem 2.2. *Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \cdots, r_n$, respectively. Then*

$$\mathrm{PI}_v(G_n \sqcap ... \sqcap G_2 \sqcap G_1) = \sum_{i=1}^{n} |V_{1,n}^i|\mathrm{PI}_v(G_i) + \sum_{i=1}^{n-1} |V_{i+1,n}|(|E(G_i)| - N_{r_i})$$

$$\times \sum_{j=i+1}^{n} (|V(G_j)|-1)|V_{1,j-1}|,$$

where $N_{r_i} = |\{uv \in E(G_i) \mid d_{G_i}(u, r_i) = d_{G_i}(v, r_i)\}|$.

Proof. Let $H = G_n \sqcap ... \sqcap G_2 \sqcap G_1$ and $e = (a_n, ..., a_{i+1}, u, r_{i-1}, ..., r_1)(a_n, ..., a_{i+1}, v, r_{i-1}, ..., r_1)$ be an edge of H such that $uv \in E(G_i)$, and $a_j \in V(G_j)$. It follows from the edge structure of H that, if $d_{G_i}(u, r_i) \neq d_{G_i}(v, r_i)$ then

$$n_1^H(e)+n_2^H(e) = (n_v^{G_i}(uv)+n_u^{G_i}(uv)) \prod_{j=1}^{i-1} |V(G_j)| + \sum_{j=i+1}^{n} (|V(G_j)|-1) \prod_{k=1}^{j-1} |V(G_k)|$$

and if $d_{G_i}(u, r_i) = d_{G_i}(v, r_i)$ then

$$n_1^H(e) + n_2^H(e) = (n_v^{G_i}(uv) + n_u^{G_i}(uv)) \prod_{j=1}^{i-1} |V(G_j)|.$$

Thus, the summation of $[n_u^H(uv) + n_v^H(uv)]$ over all edges of copies of G_i, is equal to:

$$(\prod_{j=1, j\neq i}^{n} |V(G_j)|)\mathrm{PI}_v(G_i)+(|E(G_i)|-N_{r_i})(\prod_{j=i+1}^{n} |V(G_j)|) \sum_{j=i+1}^{n} (|V(G_j)|-1) \prod_{k=1}^{j-1} |V(G_k)|.$$

Therefore,

$$\begin{aligned} \mathrm{PI}_v(H) &= \sum_{i=1}^{n} \Big[(\prod_{j=1, j\neq i}^{n} |V(G_j)|)\mathrm{PI}_v(G_i) \\ &\quad + (|E(G_i)| - N_{r_i})(\prod_{j=i+1}^{n} |V(G_j)|) \sum_{j=i+1}^{n} (|V(G_j)| - 1) \prod_{k=1}^{j-1} |V(G_k)| \Big] \\ &= \sum_{i=1}^{n} (\prod_{j=1, j\neq i}^{n} |V(G_j)|)\mathrm{PI}_v(G_i) \\ &\quad + \sum_{i=1}^{n-1} (\prod_{j=i+1}^{n} |V(G_j)|)(|E(G_i)| - N_{r_i}) \sum_{j=i+1}^{n} (|V(G_j)| - 1) \prod_{k=1}^{j-1} |V(G_k)|, \end{aligned}$$

which proves the theorem. □

Corollary 2.3. *Suppose* $G_1, G_2, \ldots, G_n$ *are connected rooted graphs with root vertices* $r_1, \cdots, r_n$*, respectively. We also assume that* r_i*,* $1 \leq i \leq n$*, lies on no odd cycle of* G_i*. Then*

$$\begin{aligned} \mathrm{PI}_v(G_n \sqcap ... \sqcap G_2 \sqcap G_1) = \sum_{i=1}^{n} |V_{1,n}^i| \mathrm{PI}_v(G_i) + \sum_{i=1}^{n-1} |V_{i+1,n}||E(G_i)| \\ \times \sum_{j=i+1}^{n} (|V(G_j)| - 1)|V_{1,j-1}|. \end{aligned}$$

We now prove that the [8, Theorem 3.1] is incorrect. We first explain the reason that makes this Theorem to be incorrect. In [8, Eq. 3.8 and 3.9], the authors claim that for each edge $e' = (u_r, v_i)(u_s, v_i) \in G(U) \sqcap H$ such that $v_i \in V(H)$ and $e = u_r u_s \in E(G)$, we have $M_{G(U)\sqcap H}(e') = |V(H)|M_{G(U)}(e) + |E(H)|N_{\langle U\rangle}(e)$. In Figure 4, a counterexample for this argument is presented. Notice that if $U = \{x, y, z\}$ and e' is corresponding edge of e in $G(U) \sqcap H$ then $M_{G(U)\sqcap H}(e') = 7$, but $|V(H)|M_{G(U)}(e) + |E(H)|N_{\langle U\rangle}(e) = 9$, which is impossible. On the other hand, by [8, Theorem 3.1] $\mathrm{PI}_e(G(U) \sqcap H) = 168$, that is incorrect. The correct value of PI_e is 164.

In the following theorem a correct form of [8, Theorem 3.1] is presented.

Theorem 2.4. *Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then*

$$\begin{aligned}\mathrm{PI}_e(G_n \sqcap \ldots \sqcap G_2 \sqcap G_1) &= \sum_{i=1}^{n} |V_{i+1,n}|\mathrm{PI}_e(G_i)\\ &+ \sum_{i=1}^{n} |V_{i+1,n}| \left(\sum_{j=1}^{i-1} |E(G_j)||V_{j+1,i-1}|\right) \mathrm{PI}_v(G_i)\\ &+ \sum_{i=1}^{n} \Big((|E(G_i)| - N_{r_i})|V_{i+1,n}| \sum_{j=i+1}^{n} \big((|V(G_j)| - 1)\\ &\times \sum_{k=1}^{j-1} |E(G_k)||V_{k+1,j-1}| + |E(G_j)|\big)\Big),\end{aligned}$$

where $N_{r_i} = |\{uv \in E(G_i) \mid d_{G_i}(u, r_i) = d_{G_i}(v, r_i)\}|$.

Proof. Let $H = G_n \sqcap \ldots \sqcap G_2 \sqcap G_1$. By the edge structure of H, it is not difficult to see that, for every edge $e = (a_n, ..., a_{i+1}, u, r_{i-1}, ..., r_1)(a_n, ..., a_{i+1}, v, r_{i-1}, ..., r_1)$ of H such that $uv \in E(G_i)$ and $a_j \in V(G_j)$ (for $j = i + 1, i + 2, ..., n$), if $d_{G_i}(u, r_i) \neq d_{G_i}(v, r_i)$ then

$$\begin{aligned}m_1^H(e) + m_2^H(e) &= m_u^{G_i}(uv) + m_v^{G_i}(uv) + (n_u^{G_i}(uv) + n_v^{G_i}(uv)) \sum_{j=1}^{i-1} |E(G_j)|\\ &\times \prod_{k=j+1}^{i-1} |V(G_k)| + \sum_{j=i+1}^{n} \Big((|V(G_j)| - 1) \sum_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|\Big)\end{aligned}$$

and if $d_{G_i}(u, r_i) = d_{G_i}(v, r_i)$ then

$$m_1^H(e) + m_2^H(e) = m_u^{G_i}(uv) + m_v^{G_i}(uv) + (n_u^{G_i}(uv) + n_v^{G_i}(uv)) \sum_{j=1}^{i-1} |E(G_j)| \prod_{k=j+1}^{i-1} |V(G_k)|.$$

Thus, the summation of $[m_u^H(uv) + m_v^H(uv)]$ over all edges of copies of G_i, is equal to:

$$(\prod_{j=i+1}^{n} |V(G_j)|)\mathrm{PI}_e(G_i) + (\prod_{j=i+1}^{n} |V(G_j)|)(\sum_{j=1}^{i-1} |E(G_j)| \prod_{k=j+1}^{i-1} |V(G_k)|)\mathrm{PI}_v(G_i)$$

$$+ (|E(G_i)| - N_{r_i})(\prod_{j=i+1}^{n} |V(G_j)|)$$
$$\times \sum_{j=i+1}^{n} \big((|V(G_j)| - 1)\sum_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|\big)$$

and therefore

$$\begin{aligned}\mathrm{PI}_e(H) &= \sum_{i=1}^{n} \Big[(\prod_{j=i+1}^{n} |V(G_j)|)\mathrm{PI}_e(G_i) \\ &+ (\prod_{j=i+1}^{n} |V(G_j)|)(\sum_{j=1}^{i-1} |E(G_j)| \prod_{k=j+1}^{i-1} |V(G_k)|)\mathrm{PI}_v(G_i) \\ &+ (|E(G_i)| - N_{r_i})(\prod_{j=i+1}^{n} |V(G_j)|) \sum_{j=i+1}^{n} \big((|V(G_j)| - 1) \\ &\times \sum_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|\big)\Big] \\ &= \sum_{i=1}^{n}(\prod_{j=i+1}^{n} |V(G_j)|)\mathrm{PI}_e(G_i) \\ &+ \sum_{i=1}^{n}(\prod_{j=i+1}^{n} |V(G_j)|)(\sum_{j=1}^{i-1} |E(G_j)| \prod_{k=j+1}^{i-1} |V(G_k)|)\mathrm{PI}_v(G_i) \\ &+ \sum_{i=1}^{n}(|E(G_i)| - N_{r_i})(\prod_{j=i+1}^{n} |V(G_j)|) \\ &\times \sum_{j=i+1}^{n} \Big((|V(G_j)| - 1)\sum_{k=1}^{j-1} |E(G_k)| \prod_{l=k+1}^{j-1} |V(G_l)| + |E(G_j)|\Big),\end{aligned}$$

as desired. □

Corollary 2.5. *Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. We also assume that r_i lies on no odd cycle of G_i, $i = 1, 2, ..., n$. Then*

$$\begin{aligned}\mathrm{PI}_e(G_n \sqcap ... \sqcap G_2 \sqcap G_1) &= \sum_{i=1}^{n} |V_{i+1,n}|\mathrm{PI}_e(G_i) + \sum_{i=1}^{n} |V_{i+1,n}| \left(\sum_{j=1}^{i-1} |E(G_j)||V_{j+1,i-1}|\right) \\ &\times \mathrm{PI}_v(G_i) + \sum_{i=1}^{n} \Big(|E(G_i)||V_{i+1,n}| \sum_{j=i+1}^{n} ((|V(G_j)| - 1) \\ &\times \sum_{k=1}^{j-1} |E(G_k)||V_{k+1,j-1}| + |E(G_j)|)\Big).\end{aligned}$$

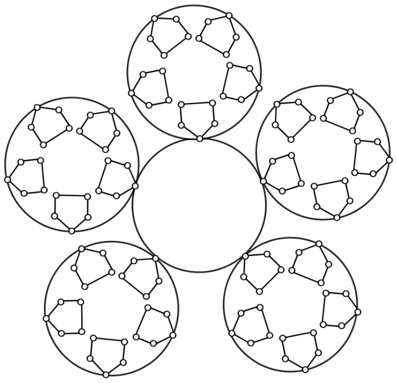

FIGURE 1. The Hierarchical Product of Three Copies of C_5

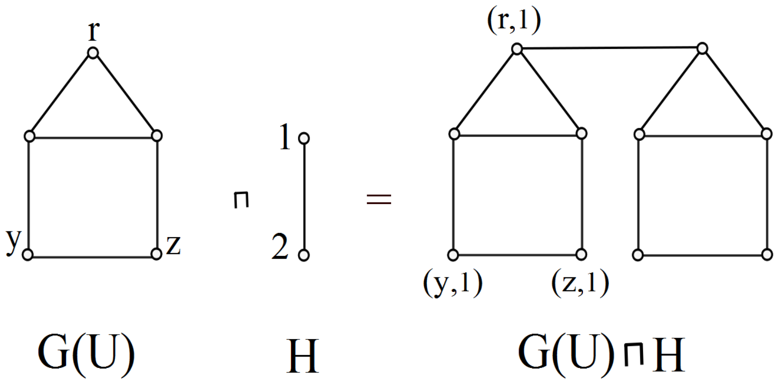

FIGURE 2. The Hierarchical Product of $G(U)$ and H

Example 2.6. Consider a rooted cycle graph C_m with root vertex r. By definition of this graph, Figure 1, it is clear that

$$N_r = \begin{cases} 1 & 2 \nmid m \\ 0 & 2 \mid m \end{cases}, \quad \mathrm{PI}_v(C_m) = \begin{cases} m(m-1) & 2 \nmid m \\ m^2 & 2 \mid m \end{cases}, \quad \mathrm{PI}_e(C_m) = \begin{cases} m(m-1) & 2 \nmid m \\ m(m-2) & 2 \mid m \end{cases}.$$

So, by Theorems 2.2 and 2.4, we calculate that

1. $$\mathrm{PI}_v(\underbrace{C_m \sqcap \cdots \sqcap C_m}_{n}) = \begin{cases} m^{2n} - m^n & 2 \nmid m \\ nm^{n+1} + \frac{m}{m-1}\big(m^{2n} - nm^{n+1} + (n-1)m^n\big) & 2 \mid m \end{cases},$$

2. $$\mathrm{PI}_e(\underbrace{C_m \sqcap \cdots \sqcap C_m}_{n}) = \begin{cases} \frac{m^{2n+1}}{m-1} - \frac{m^{n+3}}{(m-1)^2} + m^{n+1}(1 + \frac{1}{(m-1)^2}) + \frac{m}{m-1} & 2 \nmid m \\ \frac{1}{(m-1)^2}\big(m^{2n+2} - 2m^{n+1}(2m-1) + m(3m-2)\big) & 2 \mid m \end{cases}.$$

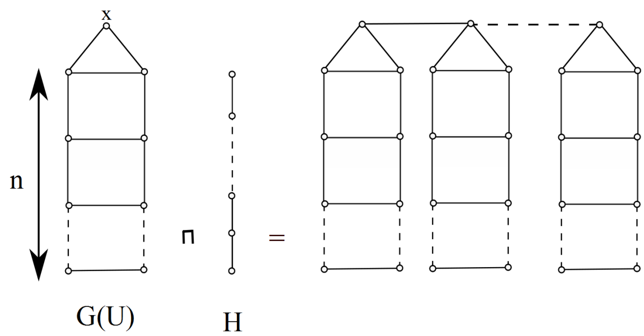

Figure 3. The Hierarchical Product of $G(U)$ and H

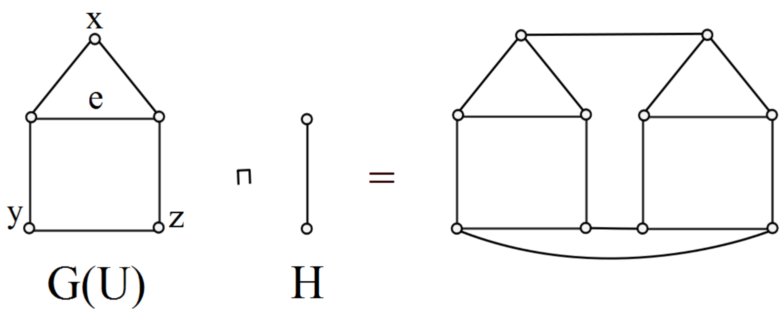

Figure 4. The Generalized Hierarchical Product of $G(U)$ and H

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