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EXPONENTIAL INEQUALITIES AND COMPLETE CONVERGENCE OF EXTENDED ACCEPTABLE RANDOM VARIABLES†

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Abstract. Giuliano Antonini et al.(2008) have introduced the concept of extended acceptability and the results show that the extended acceptability structure has no effect on the exponential inequality except replacing a constant $M = 1$ with a constant $M > 0$. We discuss the complete convergence for extended acceptable random variables by using the exponential inequality.

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1. Introduction

Giuliano Antonini et al.(2008) recently have introduced the concept of acceptability as follows; A finite sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be acceptable if for any real λ,

$$E\exp(\lambda\sum_{i=1}^{n} X_i) \leq \Pi_{i=1}^{n} E\exp(\lambda X_i). \tag{1.1}$$

An infinite sequence $\{X_n, n \geq 1\}$ of random variable is acceptable if every finite subcollection is acceptable. They also mentioned that a sequence of negatively dependent random variables with a finite Laplace transform or finite moment generating function near zero provides us an example of acceptable random variables. In addition, Liu(2009) introduced the concept of extended negative orthant dependence by extending the negative orthant dependence as follows;

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A sequence of random variables $\{X_i, i \geq 1\}$ is said to be extended negatively orthant dependent($ENOD$) if there exists a constant $M > 0$ such that both

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq M\Pi_{i=1}^n P(X_i \leq x_i) \tag{1.2}$$

and

$$P(X_1 > x_1, \cdots, X_n > x_n) \leq M\Pi_{i=1}^n P(X_i > x_i) \tag{1.3}$$

hold for each $n = 1, 2, \cdots$ and all $x_1, \cdots, x_n$. Recall that the sequence $\{X_i, i \geq 1\}$ is said to be negatively orthant dependent(NOD) if both (1.2) and (1.3) hold when $M = 1$; it is called positively orthant dependent(POD) if (1.2) and (1.3) hold both in the reverse direction when $M = 1$. Obviously, NOD(See Joag-Dev and Proschan(1983) and Baek et al.(2011)) sequence must be an $ENOD$ sequence.

We defined an extended acceptability from the definitions of acceptability and extended orthant dependence as follows.

Definition 1.1. A finite sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be extended acceptable if there exists a constant $M > 0$ such that for any real λ

$$E\exp(\lambda \sum_{i=1}^n X_i) \leq M\Pi_{i=1}^n E\exp(\lambda X_i). \tag{1.4}$$

An infinite sequence $\{X_n, n \geq 1\}$ of random variables is extended acceptable if every finite subcollection is extended acceptable. A sequence $\{X_i, i \geq 1\}$ of random variables is obviously acceptable if (1.4) holds when $M = 1$ and hence an acceptable sequence must be an extended acceptable sequence. In addition, (1.2) and (1.3) obviously satisfy (1.4). Therefore, the $ENOD$ random variables are extended acceptable random variables.

From the similar method in Giuliano Antonini etc., a sequence of $ENOD$ random variables with a finite Laplace transform or finite moment generating function near zero provides us an example of extended acceptable random variables. In particular, there have been many investigations on the exponential inequality for dependent random variables. For examples, Kim et al.(2007) and Xing et al.(2010),and Wang et al.(2010) had established an exponential inequality for dependent random variables. Sung et al.(2011) obtained an exponential inequality for identically distributed acceptable random variables as follows.

Theorem 1.2. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed acceptable random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$. Then for any $0 < \epsilon \leq K\delta$,*

$$P(|S_n - ES_n| \geq n\epsilon) \leq 2\exp(-\frac{n\epsilon^2}{4K}), \tag{1.5}$$

where $S_n = X_1 + \cdots + X_n$.

The main goal of our paper is to extend Theorem 1.1 to extended acceptable random variables; discuss the above result for extended acceptable random variables and in addition complete convergence of extended acceptable. This paper is organized as follows. In section 2, we provide the establish the exponential inequalities for sum of extended acceptable random variables and in section 3, we obtain a result dealing with the complete convergence for these random variables by using the exponential inequality.

2. Exponential inequalities for extended acceptable random variables

First we extend Sung et al. s'(2011) results on acceptable structure to the extended acceptability cases.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed and extended acceptable random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$. Then there exists a constant $M > 0$ such that for any $0 < \epsilon \leq K\delta$*

$$P(S_n - ES_n \geq n\epsilon) \leq M\exp(-\frac{n\epsilon^2}{4K}) \tag{2.1}$$

and

$$P(|S_n - ES_n| \geq n\epsilon) \leq 2M\exp(-\frac{n\epsilon^2}{4K}), \tag{2.2}$$

where $S_n = X_1 + \cdots + X_n$.

Proof. The proof is similar to that of Theorem 2.1 in Sung et al.(2011). Suppose that $0 < \epsilon \leq K\delta$, then by Markov's inequality, the definition of extended acceptable random variables and Sung et al's Lemma 2.1, for any $0 < \lambda \leq \delta/2$,

$$\begin{aligned} P(\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon) &= P(\exp(\lambda\sum_{i=1}^{n}(X_i - EX_i)) > \exp(\lambda n\epsilon)) \\ &\leq \exp(-\lambda n\epsilon)E\exp(\lambda\sum_{i=1}^{n}(X_i - EX_i)) \\ &\leq M\exp(-\lambda n\epsilon)\prod_{i=1}^{n}\exp(\lambda(X_i - EX_i)) \\ &\leq M\exp(-\lambda n\epsilon)\prod_{i=1}^{n}\exp(K\lambda^2) = M\exp(-\lambda n\epsilon + K\lambda^2 n). \end{aligned} \tag{2.3}$$

We take $\lambda = \epsilon/(2K)$ in the last term and note that $\epsilon/(2K) \leq \delta/2$ by condition $0 < \epsilon \leq K\delta$. Thus, we get that

$$P(\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon) \leq M\exp(-\frac{n\epsilon^2}{4K}). \tag{2.4}$$

Since a sequence $\{-X_n, n \geq 1\}$ is also extended acceptable, by replacing X_i with $-X_i$ in the above statement, we obtain that

$$P(-\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon) \leq M\exp(-\frac{n\epsilon^2}{4K}). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\begin{aligned} & P(|\sum_{i=1}^{n}(X_i - EX_i)| > n\epsilon) \\ = \; & P(\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon) + P(-\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon) \leq 2M\exp(-\frac{n\epsilon^2}{4K}). \end{aligned}$$

□

Remark 2.1. Theorem 2.1 of Sung et al.(2011) is a special case of Theorem 2.2 when $M = 1$.

From Theorem 2.1, we can get the result of Corollary 2.2 as follows.

Corollary 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed and extended acceptable random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$. Set $\epsilon_n = 2(K\alpha(\log n)/n)^{1/2}$, where $\alpha > 0$ and $K = 2(E|X_1|^4)^{1/2}E(e^{\delta|X_1|})$. Then there exists a constant $M > 0$ such that for $\alpha > 1$*

$$\sum_{n=1}^{\infty} P|S_n - ES_n| > n\epsilon_n) \leq 2M\sum_{n=1}^{\infty}\exp(-\alpha\log n) < \infty.$$

Proof. Let $\epsilon_n = 2(K\alpha(\log n)/n)^{1/2}$, where $\alpha > 1$ and $K = 2(E|X_1|^4)^{1/2}E(e^{\delta|X_1|})$. Then $\epsilon_n/(K\delta) \leq 1$ for all large n. Hence, the result follows directly from Theorem 2.1. □

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of extended acceptable random variables with $Ee^{\delta|X_i|} < \infty$ for some $\delta > 0$ and for each $i \geq 1$ and $\{g_n, n \geq 1\}$ be a sequence of positive numbers with $G_n = \sum_{i=1}^{n} g_i$ for each $n \geq 1$. For fixed $n \geq 1$, if there exists a positive number T such that*

$$E\exp(tX_i) \leq \exp(\frac{1}{2}g_i t^2),\ 0 \leq t \leq T,\ i = 1, 2, \cdots, n, \tag{2.6}$$

then, for some constant $M > 0$

$$P(S_n \geq x) \leq \begin{cases} M\exp(-\frac{x^2}{2G_n}),\ 0 \leq x < G_n T, \\ M\exp(-\frac{Tx}{2}),\ x \geq G_n T. \end{cases} \tag{2.7}$$

Proof. For each x, by Markov's inequality we see that

$$P(S_n \geq x) \leq \exp(-tx)E\exp(tS_n),\ t > 0. \tag{2.8}$$

By (1.4) there exists a constant $M > 0$ such that for $0 < t \le T$

$$E\exp(tS_n) = E(\prod_{i=1}^{n}\exp(tX_i)) \le M\prod_{i=1}^{n}E\exp(tX_i) \le M\exp(\frac{G_nt^2}{2}). \qquad (2.9)$$

It follows from (2.8) and (2.9) that

$$P(S_n \ge x) \le M\inf_{0<t\le T}\exp(\frac{G_nt^2}{2} - tx) = M\exp(\inf_{0<t\le T}(\frac{G_nt^2}{2} - tx)). \qquad (2.10)$$

For fixed $x \ge 0$, if $T > \frac{x}{G_n} \ge 0$, then

$$\exp(\inf_{0<t\le T}(\frac{G_nt^2}{2} - tx)) = \exp(-\frac{x^2}{2G_n}) \qquad (2.11)$$

and if $T < \frac{x}{G_n}$, then

$$\exp(\inf_{0<t\le T}(\frac{G_nt^2}{2} - tx)) = \exp(\frac{G_nt^2}{2} - Tx) \le \exp(-\frac{Tx}{2}). \qquad (2.12)$$

From (2.10)-(2.12), we obtain the result of (2.7). □

Theorem 2.3. *Let $\{X_n, n \ge 1\}$ be a sequence of extended acceptable random variables with $EX_i = 0$, $Ee^{\delta|X_i|} < \infty$ for some $\delta > 0$ and $|X_i| \le b$ for each $i \ge 1$, where b is a positive constant. Denote $B_n^2 = \sum_{i=1}^n EX_i^2$ for each $n \ge 1$. Then, for any $\epsilon > 0$ and for some constant $M > 0$*

$$P(S_n \ge \epsilon) \le M\exp\{-\frac{\epsilon^2}{2(2B_n^2 + b\epsilon)}\} \qquad (2.13)$$

and

$$P(|S_n| \ge \epsilon) \le 2M\exp\{-\frac{\epsilon^2}{2(2B_n^2 + b\epsilon)}\}. \qquad (2.14)$$

Proof. Clearly, for any $0 < t \le \frac{1}{b}$ $|tX_i| \le 1$ by assumption $|X_i| \le b$. Thus, by the fact that $e^y > 1 + y, y > 0$ we obtain

$$E\exp|tX_i| = 1 + \sum_{n=2}^{\infty}\frac{E(tX_i)^n}{n!} \le 1 + t^2EX_i^2 \le \exp(t^2EX_i^2). \qquad (2.15)$$

By Markov's inequality, Definition 1.1 and (2.15)

$$P(S_n \ge \epsilon) \le \exp(-t\epsilon)E\exp(tS_n)$$
$$\le Me^{-t\epsilon}\prod_{i=1}^{n}E\exp(tX_i) \le M\exp(-t\epsilon + t^2B_n^2),$$

which yields (2.13) by taking $t = \epsilon/(2B_n^2 + b\epsilon)$. Since $\{-X_n, n \ge 1\}$ is also a sequence of extended acceptable random variables, it follows from (2.13) that

$$P(S_n \le -\epsilon) = P(-S_n \ge \epsilon) \le M\exp\{-\frac{\epsilon^2}{2(2B_n^2 + b\epsilon)}\}. \qquad (2.16)$$

Combining (2.13) and (2.16) yields (2.14). □

Theorem 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of extended acceptable random variables. If there exist sequences of real numbers $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ such that $a_i \leq X_i \leq b_i$ for each $i \geq 1$. Then, for any $\epsilon > 0$ and some constant $M > 0$*

$$P(S_n - ES_n \geq n\epsilon) \leq M \exp\{-\frac{2n^2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\} \tag{2.17}$$

and

$$P(|S_n - ES_n| \geq n\epsilon) \leq 2M \exp\{-\frac{2n^2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\}, \ n \geq 1 \tag{2.18}$$

Proof. For any $h > 0$ by Markov's inequality, we have

$$P(S_n - ES_n \geq n\epsilon) \leq E \exp[h(S_n - ES_n - n\epsilon)] \tag{2.19}$$

It follows from Definition 1.1 that for some constant $M > 0$

$$\begin{aligned} E \exp[h(S_n - ES_n - n\epsilon)] &= \exp(-hn\epsilon) E \prod_{i=1}^n \exp[h(X_i - EX_i)] \\ &\leq M \exp(-hn\epsilon) \prod_{i=1}^n \exp[h(X_i - EX_i)]. \end{aligned} \tag{2.20}$$

Hoeffding(1963) proved that if $a \leq X \leq b$, then for any $h > 0$

$$E \exp[h(X - EX)] \leq \exp[h^2(b-a)^2/8]. \tag{2.21}$$

By (2.20) and (2.21)

$$P(S_n - ES_n \geq n\epsilon) \leq M \exp(-hn\epsilon + \frac{1}{8}h^2 \sum_{i=1}^n (b_i - a_i)^2) \tag{2.22}$$

It is easily seen that the right hand side of (2.22) has its minimum at $h = \frac{4n\epsilon}{\sum_{i=1}^n (b_i - a_i)^2}$. Inserting this value in (2.22) we obtain (2.17). Since $\{-X_n\}$ is also a sequence of extended acceptable random variables by (2.17) we obtain

$$P(S_n - ES_n \leq -n\epsilon) \leq M \exp\{-\frac{2n^2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\}, \ n \geq 1. \tag{2.23}$$

From (2.17) and (2.21), we obtain the result of (2.18). □

3. Complete convergence for extended acceptable random variables

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed and extended acceptable random variables with $EX_1 = 0$ and $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$. Then $n^{-1}(S_n - ES_n) \to 0$ completely as $n \to \infty$.*

Proof. By using Theorem 2.1, we can be obtained the result of Theorem 3.1 and the proof is omitted. □

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of extended acceptable random variables with $EX_i = 0$, $Ee^{\delta|X_i|} < \infty$ for some $\delta > 0$ and $|X_i| \leq b$ for each $i \geq 1$, where b is a positive constant. If $\sum_{i=1}^{\infty} EX_i^2 < \infty$ then for any $r > 0$*

$$n^{-r} S_n \to 0 \text{ completely } as\ n \to \infty. \tag{3.1}$$

Proof. It follows from (2.14) that for any $\epsilon > 0$ and some constant $M > 0$

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n| \geq n^r \epsilon) &\leq 2M \sum_{n=1}^{\infty} \exp\{-\frac{n^{2r}\epsilon^2}{2(2\sum_{i=1}^{n} EX_i^2 + bn^r \epsilon)}\} \\ &\leq 2\sum_{n=1}^{\infty} [\exp(-C)]^{n^r} < \infty,\end{aligned}$$

which yields (3.1), where C is a positive number not depending on n. □

Theorem 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of extended acceptable random variables with $Ee^{\delta|X_i|} < \infty$ for some $\delta > 0$ and $|X_i| \leq C < \infty$ for each $i \geq 1$, where C is a positive constant. Then, for any $r > \frac{1}{2}$*

$$n^{-r}(S_n - ES_n) \to 0 \text{ completely } as\ n \to \infty.$$

Proof. For any $\epsilon > 0$ and some constant $M > 0$ we obtain

$$\sum_{n=1}^{\infty} P(|S_n - ES_n| \geq n^r \epsilon) \leq 2M \sum_{n=1}^{\infty} [\exp(-\frac{\epsilon^2}{2C^2})]^{n^{2r-1}} < \infty$$

by Theorem 2.4. Hence, $n^{-r}(S_n - ES_n) \to 0$ completely as $n \to \infty$. □

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