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FEEDBACK CONTROL FOR A TURBIDOSTAT MODEL WITH RATIO-DEPENDENT GROWTH RATE†

XIAOYU HU, ZUXIONG LI* AND XINGGUO XIANG

ABSTRACT. In this paper, a turbidostat model with ratio-dependent growth rate and impulsive state feedback control is considered. We obtain sufficient conditions of the globally asymptotically stable of the system without impulsive state feedback control. We also obtain that the system with impulsive state feedback control has periodic solution of order one. Sufficient conditions for existence and stability of periodic solution of order one are given. In some cases, it is possible that the system exists periodic solution of order two. Our results show that the control measure is effective and reliable.

AMS Mathematics Subject Classification : 34C05.
Key words and phrases : turbidostat, impulsive effect, state feedback control, globally asymptotical stability, periodic solution.

1. Introduction

The chemostat is a basic piece of laboratory apparatus used in cultivating micro-organisms. As a tool in biotechnology, the chemostat plays an important role in microbiology and population biology. People use the mathematical models to describe the change process of microorganisms' concentration in the culture room in order to study the growth process of cultivating indoor microorganisms. The chemostat has been used widely to study bacterial metabolism, population genetics, and plasmid stability, mostly because these are the simplest and most easily constructed types of continuous cultures. The ecological environment created by a chemostat is one of the few completely controlled experimental systems for testing microbial growth and competition. As for the mathematical models on the culture of the microorganisms, many papers have investigated them, for

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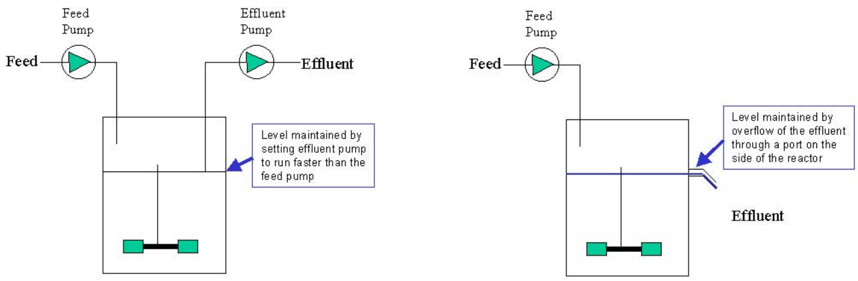

FIGURE 1. *Two type of Chemostat*

example, mathematical [2, 3, 8, 9, 12, 20] and experimental [6, 7, 19] models exhibit the competitive exclusion principle only one species survives. Several modifications of the chemostat have been made to ensure the coexistence of species on a single nutrient [4, 5, 6, 19]. The turbidostat is another basic piece of laboratory apparatus used to cultivate micro-organisms. In the turbidostat, the input medium is intermittent as it is mainly required to control the rise in turbidity due to cell growth. The turbidity is preselected on the basis of biomass density in cultures and can be maintained by intermittent flow of medium and washout of cells. So the desired rate of cell growth can be maintained by adjusting the level of concentrations with respect to the growth-limiting factor and other constituents. Flegr [5] shows coexistence of two species in the turbidostat by numerical results, De Leenheer and Smith [4] later by analysis. However, there are a lot of factors such as temperature, dissolved oxygen content affecting the growth and reproduction of the microorganisms in the process of bio-reacts. These factors do not control by continuous culture systems such as chemostat and turbidostat. In some cases, it is necessary to control the concentration of microorganisms when it reaches to a critical value. Impulsive differential equation with state feedback control fits like a glove to solve the above mentioned question.

The volume of the chemostat can be controlled either by using a pump (seen on the left of Figure 1) or an overflow system (seen on the right of Figure 1). In some cases, a series of adverse effects such as production inhibition are caused because of the high concentration of the microorganisms. To regulate the concentration of the microorganism, we modify overflow system of the chemostat into the right system of Figure 2. When the concentration which can be measured by optoelectronic devices and other ways, reaches a certain threshold, it can be regulated through replenishing water to reduce the concentration of the microorganisms.

Based on the principle of the right system of Figure 2, we propose a mathematical model concerning the turbidostat with ratio-dependent growth rate and impulsive state feedback control. Meng [15], Jiao [11], Shi [16] have investigated and studied the models with impulsive effect. Zeng [21], Jiang [10] and

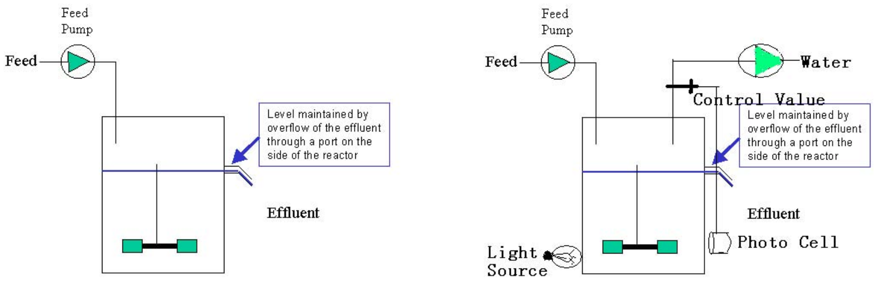

FIGURE 2. *The sketch map of chemostat and modified chemostat*

Tang [18] discussed prey-predator models with impulsive state feedback control and obtained the complete expression of the periodic solution of these systems. However, few papers have discussed the turbidostat system using the impulsive differential equation with state feedback control.

In this paper, we will prove that the model without impulsive effect has a globally asymptotically stable positive equilibrium under certain condition. we will also discuss the existence and stability of periodic solution of the turbidostat models with ratio-dependent growth rate and impulsive state feedback control according to the existence criteria [21] and the stability theorem [17] of periodic solution of the general impulsive autonomous system. This paper is organized as follows. The model and some preliminary results are presented in the next section. The qualitative analysis of the system without impulsive effect is given in Section 3. In Section 4, the existence and stability of periodic solution of order one for differential equation with ratio-dependent growth rate and impulsive state feedback control are investigated. Numerical simulations are given in Section 5. Finally, some conclusion and biological discussions are provided in Section 6.

2. The model and Preliminaries

The general model of continuously culturing microorganism in a chemostat is described by the following differential equation [4]:

$$\begin{cases} \dot{S} = D(S_{in} - S) - \dfrac{1}{\delta} f(S)x, \\ \dot{x} = f(S)x - Dx. \end{cases} \tag{2.1}$$

In model (2.1), S denotes the concentration of substrate at time t and x denotes the concentration of the microorganism in the chemostat at time t; D and $S_{in} > 0$ denote, respectively, the dilution rate of the chemostat and the concentration of the input substrate; The constant $\delta > 0$ is the yield constant; f is uptake function. Based on the Michaelis-Menten or Holling type II function, Arditi and

Ginzburg [1] proposed ratio-dependent uptake function of the form:

$$F\left(\frac{x}{y}\right) = \frac{c(x/y)}{m + (x/y)} = \frac{cx}{my + x}.$$

By the above uptake function, we replace $f(S) = \mu_m S/(k_m + S)$ with $F(\frac{S}{x}) = \frac{\mu_m (S/x)}{k_m + (S/x)} = \frac{\mu_m S}{k_m x + S}$ and (2.1) has the following form:

$$\begin{cases} \dot{S} = D(S_{in} - S) - \dfrac{\mu_m S x}{\delta(k_m x + S)}, \\ \dot{x} = \dfrac{\mu_m S x}{k_m x + S} - Dx. \end{cases} \tag{2.2}$$

where $\mu_m > 0$ is called the maximal specific growth rate of the microorganisms, $k_m > 0$ is the saturation constant. The dilution D of the turbidostat is controlled by setting $D = d + kx$ in [13], where $d > 0, k > 0$.

In some cases, it is not necessary to adopt control measures if the concentration of the microorganisms is lower than a critical value. But the high concentration of the microorganisms will often have caused a series of adverse effects such as the substrate inhibition. To achieve increasing production volume, product concentration and product yield, we always maintain the growth of microorganisms in the scope. In this system, when the flow velocity of the culture medium is lower than the growth rate of microorganism, the concentration of microorganism increases, once the concentration reaching to a critical value which can be detected by optoelectronic devices, we should take measures to decrease the concentration of microorganism. Once the concentration is decreased, it takes a period of time to reach the critical value once more. Therefore, the feedback control can be realized by the impulsive state control.

In this paper, we consider that the substrate with the microorganism is discharged impulsively due to the added water and other regulator solution on the basis of continuous cultivation in the turbidostat when x reaches the critical value (denoted by $x_m > 0$). Therefore, (2.2) can be modified as follows by introducing the impulsive state feedback control:

$$\begin{cases} \left.\begin{aligned} &\dot{S} = (d + kx)(S_{in} - S) - \frac{\mu_m S x}{\delta(k_m x + S)}, \\ &\dot{x} = x\Big(\frac{\mu_m S}{k_m x + S} - (d + kx)\Big), \end{aligned}\right\} x < x_m, \\ \left.\begin{aligned} &\triangle S = -D_1 S, \\ &\triangle x = -D_1 x, \end{aligned}\right\} x = x_m, \\ S(0) = S_{10}, x(0) = x_{10} \end{cases} \tag{2.3}$$

where $0 < D_1 < 1$ is constant and is also the fraction of the concentration of microorganism that decreases due to the feedback control when the concentrate x reach x_m. For simplicity, we nondimensionalize system (2.3) with the following scaling:

$$S \to S/S_{in}, x \to x/(\delta S_{in}), k \to k/\delta.$$

With this scaling, system (2.3) takes the form

$$\begin{cases} \left.\begin{aligned} &\dot{S}=(d+kx)(1-S)-\frac{mSx}{ax+S}, \\ &\dot{x}=x\Big(\frac{mS}{ax+S}-(d+kx)\Big), \end{aligned}\right\} x<h, \\ \left.\begin{aligned} &\triangle S=-bS, \\ &\triangle x=-bx, \end{aligned}\right\} x=h, \\ S(0)=S_0, x(0)=x_0 \end{cases} \tag{2.4}$$

where

$$m=\mu_m>0, a=\frac{k_m}{S_{in}}>0, h=\frac{x_m}{S_{in}}>0, 0<b=D_1<1.$$

In this paper, we mainly discuss the existence of periodic solution of system (2.4) by the existence criteria of the general impulsive autonomous system. Due to the constraints of the layout, we omit the relevant definitions and theorems (see Definition 2.1-2.4 and theorem 2.1-2.2 of [14]).

3. Qualitative analysis of system (2.4) without impulsive effect

In this section, we will study the qualitative characteristic of system (2.4) without the impulsive effect. If no impulsive effect is introduced, then system (2.4) is

$$\begin{cases} \dot{S}=(d+kx)(1-S)-\dfrac{mSx}{ax+S}, \\ \dot{x}=x\Big(\dfrac{mS}{ax+S}-(d+kx)\Big). \end{cases} \tag{3.1}$$

By $(d+kx)(1-S)-\frac{mSx}{ax+S}=0$ and $x\Big(\frac{mS}{ax+S}-(d+kx)\Big)=0$, we can obtain system (3.1) has a boundary equilibrium $(1,0)$ and a positive equilibrium (S^*,x^*) if $m>d$ and $a\neq 1$, where

$$S^*=\frac{(a-1)(2k+d)+k+m-\sqrt{[(a-1)d+k+m]^2+4k(a-1)(m-d)}}{2(a-1)k},$$

$$x^*=\frac{-[(a-1)d+k+m]+\sqrt{[(a-1)d+k+m]^2+4k(a-1)(m-d)}}{2(a-1)k}.$$

System (3.1) has a boundary equilibrium $(1,0)$ and a positive equilibrium $(\frac{k+d}{k+m},\frac{m-d}{k+m})$ if $m>d$ and $a=1$. System (3.1) only has a boundary equilibrium $(1,0)$ in region $R^+=\{(S,x)|S\geq 0, x\geq 0\}$ if $m<d$.

The Jacobian matrix at equilibrium $E=(S,x)$ is given by

$$J_{(S,x)}=\begin{pmatrix} -d-kx-\dfrac{amx^2}{(ax+S)^2} & (1-S)k-\dfrac{mS^2}{(ax+S)^2} \\ \dfrac{amx^2}{(ax+S)^2} & \dfrac{mS}{ax+S}-d-kx-x\big(\dfrac{mSa}{(ax+S)^2}+k\big) \end{pmatrix}.$$

We only give some results about system (3.1) and omit their proof since some things are already known or obvious.

Theorem 3.1. *If* $m > d, a \neq 1$, *then* $B(1,0)$ *is a saddle point, the positive equilibrium* $A(S^*, x^*)$ *is globally asymptotically stable node and* $\lim_{t\to\infty} S(t) = S^*, \lim_{t\to\infty} x(t) = x^*$, *where* $S^* = \frac{(a-1)(2k+d)+k+m-\sqrt{[(a-1)d+k+m]^2+4k(a-1)(m-d)}}{2(a-1)k}$, $x^* = \frac{-[(a-1)d+k+m]+\sqrt{[(a-1)d+k+m]^2+4k(a-1)(m-d)}}{2(a-1)k}$. *If* $m > d, a = 1$, *then* $B(1,0)$ *is a saddle point, the positive equilibrium* $A(\frac{k+d}{k+m}, \frac{m-d}{k+m})$ *is globally asymptotically stable node.*

By the Jacobian matrix at equilibrium $B(1,0)$, we easily obtain the following theorem:

Theorem 3.2. *If* $m < d$, *then the boundary equilibrium* $B(1,0)$ *is globally asymptotically stable.*

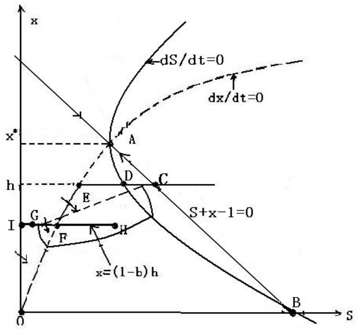

FIGURE 3. *Illustration of (2.4) for the case* $h < x^*, S_G \leq S_F$, *and* $S_F \leq S_H \leq S_{HM}$.

4. Existence and stability of periodic solutions

4.1. Existence of periodic solution of order one. From discussions of Section 3, we can see that (S^*, x^*) is a stable node when $m > d$. For the initial points which satisfy $x(0) < x^*$ and $\frac{dS}{dt}|_{(S_0,x_0)} \geq 0$ if $h \geq x^*$, then all the solutions of (2.4) tend to the equilibrium (S^*, x^*) and no impulsive will occur. It is not necessary to control by impulsive control. So we mainly pay attention to the case that the following assumption holds: (H) $\;h < x^*, x_0 < x^*$ and $S(0) \leq 1$.

We apply the existence criteria (Theorem 1 of [21]) to prove periodic solution of order one of system (2.4). We first need to construct a closed region such that all the solutions of (2.4) enter the closed region and retain there. We will illustrate the ideas by using Figure 3-5.

From Figure 3-5, we can see that the line $x = h$ interacts the isoclinal line $\frac{mS}{ax+S}-(d+kx) = 0$, that is $dx/dt = 0$, and $dS/dt = (d+kx)(1-S)-\frac{mSx}{ax+S} = 0$ at the points $E(S_E, h), D(S_D, h)$, respectively. Here $S_E = \frac{ah(d+kh)}{m-d-kh}$,

$$S_D = \frac{(1-ah)(d+kh) - mh + \sqrt{[(1-ah)(d+kh)-mh]^2 + 4ah(d+kh)^2}}{2(d+kh)}.$$

The impulsive set $M \subseteq \overline{EC}, \overline{EC} = \{(S,x)|S_E \leq S \leq 1\}$. The impulsive functions I_1 and I_2 map the impulsive set M as $N = I(M) \subseteq \overline{GH}, \overline{GH} = \{(S,x)|(1-b)S_E \leq S \leq 1-b, x = (1-b)h\}$, here $G = G(S_G,(1-b)h), H = H(S_H,(1-b)h), S_G = (1-b)S_E, S_H = 1-b$.

From the third equation of (2.4), we know that $S^+ = (1-b)S$ if $x = h$ and furthermore $S_G = (1-b)S_E \leq S_E$. According to the value of S_G and S_F, we have the following cases:

Case 1: $S_G \leq S_F$ and $S_F \leq S_H \leq S_{HM}$ (see figure 3), where S_F is the solution of the equation

$$\frac{mS_F}{a(1-b)h+S_F} - d - k(1-b)h = 0,$$

that is,

$$S_F = \frac{ah(1-b)[d+(1-b)kh]}{m-d-(1-b)kh},$$

and S_{HM} is the solution of the following equation

$$[d+(1-b)kh](1-S_{HM}) - \frac{m(1-b)hS_{HM}}{a(1-b)h+S_{HM}} = 0,$$

that is,

$$S_{HM} = \frac{[d+(1-b)kh][1-a(1-b)h] - m(1-b)h}{2[d+(1-b)kh]} + \frac{\sqrt{\{[d+(1-b)kh][1-a(1-b)h]-m(1-b)h\}^2+4a[d+(1-b)kh]^2}}{2[d+(1-b)kh]}.$$

Case 2: $S_G < S_H < S_F$ (Figure 4).

Case 3: $S_F < S_G < S_H < S_{HM}$ (Figure 5).

Theorem 4.1. *Suppose that* $m > d, 0 < h < x^*, x_0 < h$ *and* $0 < S_0 < 1$ *then system (2.4) has a periodic solution of order one.*

Proof. We prove the theorem by the following case:

Case a): $m > d, 0 < h < x^*, x_0 < h$ and $(d+kx_0)(1-S_0) - \frac{mx_0S_0}{ax_0+S_0} \geq 0$. Under the above conditions, phase set $N = I(M)$ of system (2.4) must be one of Case 1, Case 2 or Case 3. the trajectories of system (2.4) staring from the region shown in Case 1 must interact with the segment $\overline{EC}$. Next, we construct the closed region Ω_1. From the qualitative characteristic of system (3.1), we know that $\frac{dS}{dt} < 0$ for $S = 1$, $\frac{dx}{dt} = 0$ for $x = 0$, and $\frac{dS}{dt} > 0$ for $S = 0$. The straight line

$x = (1-b)h$ interacts $S = 0$ and $\frac{mS}{ax+S} - d - kx = 0$ at the points $H(0, (1-b)h)$ and $F(S_F, (1-b)h)$, respectively. Therefore, we have $\overline{BC}, \overline{CE}, \widehat{EF}, \overline{FG}, \overline{GI}, \overline{IO}$, and $\overline{OB}$. We can see that the trajectories of system (2.4) enter and retain the region Ω_1 formed by $\overline{BC}, \overline{CE}, \widehat{EF}, \overline{FG}, \overline{GI}, \overline{IO}$, and $\overline{OB}$ (Figure 3). By Theorem 2.2, we can obtain that system (2.4) has a periodic solution of order one if the conditions of theorem and Case 1 hold.

Next, we will prove that system (2.4) has a periodic solution of order one if the conditions of Case a) and Case 2 hold. We now construct the closed region Ω_2. Similar to the discussions of Case 1, we can easily obtain that the closed region $\overline{BC}, \overline{CE}, \widehat{EF}, \overline{FH}, \overline{HG}, \overline{GI}, \overline{IO}$ and $\overline{OB}$ (see Figure 4). By Theorem 2.2, we know that system (2.4) has a periodic solution of order one if the conditions of Case a) and Case 2 are satisfied.

Finally, we prove that system (2.4) has a periodic solution of order one if the conditions of Case a) and Case 3 are satisfied. We construct the closed region consisted of $\widehat{EF}, \overline{FG}, \overline{GH}, \widehat{HI}, \overline{IC}$ and $\overline{CE}$ (see Figure 5). Here, $\widehat{EF}$ is the same as that of above construction. The arc $\widehat{HI}$ can be view as a trajectory of system (2.4) which passes through the point H and interacts the isocline equation $dS/dt = 0$ at the point I. The vertical line passing through the point I interacts the line $x = h$ at the point C. By the qualitative characteristic of the system, we can know that all the solution enter the closed region and retain there. By Theorem 2.2, we can obtain that system (2.4) has a periodic solution of order one if the conditions of Case a) and Case 3 are satisfied.

Case b): $m > d, 0 < h < x^*, x_0 < h$ and $(d + kx_0)(1 - S_0) - \frac{mx_0S_0}{ax_0+S_0} \leq 0$. By qualitative analysis of system (2.3), we can see that the trajectories of system (2.4) starting for the region $\{(S_0, x_0)|x_0 < h < x^*, (d+kx_0)(1-S_0) - \frac{mx_0S_0}{ax_0+S_0} \leq 0\}$ will enter the region $\{(S_0, x_0)|x_0 < h < x^*, (d+kx_0)(1-S_0) - \frac{mx_0S_0}{ax_0+S_0} \geq 0\}$ after some time.

To sum up, we have proved that system (2.4) have a periodic solution of order one under the condition of theorem 4.1. □

4.2. Stability of periodic solution of order one. In the following, we analysis the stability of periodic solution of order one in system (2.4). Firstly, we give one lemma to discuss the stability of this positive periodic solution of system (2.4).

Lemma 4.1. *The $T-$periodic solution $x = \xi(t), y = \eta(t)$ of the system*

$$\begin{cases} \left.\begin{array}{l} \frac{dS}{dt} = P(S,x), \\ \frac{dx}{dt} = Q(S,x), \end{array}\right\} if \quad \phi(S,x) \neq 0, \\ \left.\begin{array}{l} \triangle S = \alpha(S,x), \\ \triangle x = \beta(S,x), \end{array}\right\} if \quad \phi(S,x) = 0, \end{cases} \tag{4.1}$$

is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

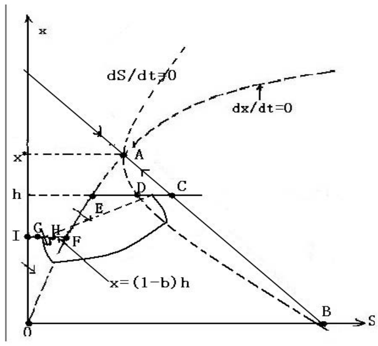

FIGURE 4. *Illustration of (2.4) for the case $h < x^*$, and $S_G < S_H < S_F$.*

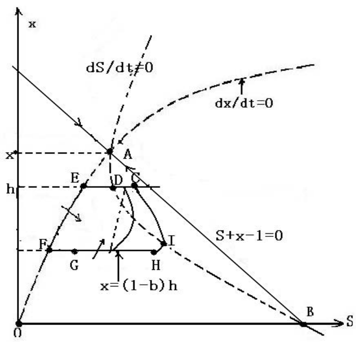

FIGURE 5. *Illustration of (2.4) for the case $h < x^*$, and $S_F < S_G < S_H < S_{HM}$.*

$$\mu_2 = \Pi_{k=1}^{q} \Delta_k \exp\left[\int_0^T \Big(\frac{\partial P}{\partial S}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial x}(\xi(t),\eta(t))\Big)dt\right], \tag{4.2}$$

with

$$\Delta_k = \frac{P_+(\frac{\partial\beta}{\partial x}\frac{\partial\phi}{\partial S} - \frac{\partial\beta}{\partial S}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial S}) + Q_+(\frac{\partial\alpha}{\partial S}\frac{\partial\phi}{\partial x} - \frac{\partial\alpha}{\partial x}\frac{\partial\phi}{\partial S} + \frac{\partial\phi}{\partial x})}{P\frac{\partial\phi}{\partial S} + Q\frac{\partial\phi}{\partial x}}$$

and $P, Q, \frac{\partial\alpha}{\partial S}, \frac{\partial\alpha}{\partial x}, \frac{\partial\beta}{\partial S}, \frac{\partial\beta}{\partial x}, \frac{\partial\phi}{\partial S}$ and $\frac{\partial\phi}{\partial x}$ are calculated at the point $(\xi(\tau_k), \eta(\tau_k))$, $P_+ = P(\xi(\tau_k^+), \eta(\tau_k^+))$, $Q_+ = Q(\xi(\tau_k^+), \eta(\tau_k^+))$, $\phi(S, x)$ is a sufficiently smooth function with $grad\phi(S, x) \neq 0$, and $\tau_k (k \in N)$ is the time of the kth jump.

The proof of this lemma is referred to [17].

In the following, we suppose this periodic solution of system (2.4) with period T passes through the points $E_1^+((1-b)\xi_0, (1-b)h) \in \overline{GH}$ and $E_1(\xi_0, h) \in \overline{EC}$ ($\overline{GH}, \overline{EC}$ see figure 3-5). As the expression and the period of this solution are unknown, we discuss the stability of this positive periodic solution by Lemma 4.1. In our case,

$$P(S,x) = (d+kx)(1-S) - \frac{mSx}{ax+S},\ Q(S,x) = x\Big(\frac{mS}{ax+S} - d - kx\Big),$$
$$\alpha(S,x) = -bS,\ \beta(S,x) = -bx,\ \phi(S,x) = x-h,\ (\xi(T),\eta(T)) = (\xi_0,h),$$
$$(\xi(T^+),\eta(T^+)) = ((1-b)\xi_0,(1-b)h).$$

Then

$$\begin{aligned}
&\frac{\partial P}{\partial S} = -d - kx - \frac{amx^2}{(ax+S)^2},\\
&\frac{\partial Q}{\partial x} = \frac{mS}{ax+S} - d - kx - x\left(\frac{mSa}{(ax+S)^2} + k\right),\\
&\frac{\partial \alpha}{\partial S} = -b,\ \ \frac{\partial \alpha}{\partial x} = 0,\ \ \frac{\partial \beta}{\partial S} = 0,\ \ \frac{\partial \beta}{\partial x} = -b,\ \ \frac{\partial \phi}{\partial S} = 0,\ \ \frac{\partial \phi}{\partial x} = 1,\\
&\Delta_k = \frac{P_+(\frac{\partial\beta}{\partial x}\frac{\partial\phi}{\partial S} - \frac{\partial\beta}{\partial S}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial S}) + Q_+(\frac{\partial\alpha}{\partial S}\frac{\partial\phi}{\partial x} - \frac{\partial\alpha}{\partial x}\frac{\partial\phi}{\partial S} + \frac{\partial\phi}{\partial x})}{P\frac{\partial\phi}{\partial S} + Q\frac{\partial\phi}{\partial x}}\\
&\quad\ = \frac{(1-b)^2\left(\frac{m(1-b)\xi_0}{ah(1-b)+(1-b)\xi_0} - d - k(1-b)h\right)}{\frac{m\xi_0}{ah+\xi_0} - d - kh}.
\end{aligned}$$

Set $G(t) = \frac{\partial P}{\partial S}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial x}(\xi(t),\eta(t))$, then

$$\begin{aligned}
\mu_2 &= \Delta_1 \exp\left[\int_0^T \big(\frac{\partial P}{\partial S}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial x}(\xi(t),\eta(t))\big)dt\right]\\
&= \frac{(1-b)^2\Big(\frac{m(1-b)\xi_0}{ah(1-b)+(1-b)\xi_0} - d - k(1-b)h\Big)}{\frac{m\xi_0}{ah+\xi_0} - d - kh}\exp\Big(\int_0^T G(t)dt\Big).
\end{aligned}$$

Because $(S(t),x(t))$ is periodic solution of system (2.4), we can obtain that $\int_0^T G(t)dt < 0$, that is, $\exp(\int_0^T G(t)dt) < 1$. Obviously, $|\mu_2| < 1$ if

$$\left|\frac{(1-b)^2\Big(\frac{m(1-b)\xi_0}{ah(1-b)+(1-b)\xi_0} - d - k(1-b)h\Big)}{\frac{m\xi_0}{ah+\xi_0} - d - kh}\right| < 1$$

holds. Therefore, we have the following theorem:

Theorem 4.2. *System (2.4) with the conditions of Theorem 4.1 has a periodic solution of order one. Furthermore, this periodic solution of order one is stable if* $\left|\frac{(1-b)^2\Big(\frac{m(1-b)\xi_0}{ah(1-b)+(1-b)\xi_0} - d - k(1-b)h\Big)}{\frac{m\xi_0}{ah+\xi_0} - d - kh}\right| < 1$ *holds.*

4.3. Periodic solution of order two. In this subsection, we discuss existence of periodic solution of order two of system (2.4).

If $(\overline{S},\overline{x})$ is a periodic solution of (2.4), then $(\overline{S}_0,\overline{x}_0)\in N\subseteq\overline{GH}$ and $(\overline{S}_1,\overline{x}_1)\in M\subseteq\overline{EC}$. We can obtain that $\overline{S}_0\leq\overline{S}_1$ because of $S^+=(1-b)S$ by the third equation of system (2.4). Let (S,x) is the arbitrary solution of system (2.4) , the first interaction point of trajectory and the set $M(x=h)$ is (S_1,h) and the corresponding interaction points are $(S_2,h),(S_3,h),\cdots$, respectively. Therefore, under the effect of impulsive function I, the corresponding point after pulse are $(S_1^+,(1-b)h),(S_2^+,(1-b)h),(S_3^+,(1-b)h),\cdots$.

We know that $\frac{dx}{dt}>0$ for $S>S_F$ and $\frac{dx}{dt}<0$ for $S<S_F$ by the qualitative analysis of system (2.4), here $S_F=\dfrac{ah(1-b)[d+(1-b)kh]}{m-d-(1-b)kh}$. So we consider the following cases:

Case 1: It is easily obtained that the trajectory jumps to the point $(S_1^+,(1-b)h)$ from the point (S_1,h) if the periodic solution is in the region $\{(\overline{S},\overline{x})|\overline{S}>S_F\}$. Without loss of generality, we let $S_1<\overline{S}_1$, then $S_1^+<\overline{S}_0$. The trajectory from the point $(S_1^+,(1-b)h)$ will interact the set M at the point (S_2,h) and then jump to the point $(S_2^+,(1-b)h),\cdots$. By the above discussion, there is one of the following sequences:

Case a). $S_1\leq S_2\leq S_3\leq\cdots\leq\overline{S}_1$, and Case b). $\overline{S}_1\geq S_1\geq S_2\geq S_3\geq\cdots$.

From the sequences we can know that the trajectories tend to be periodic. Since the sequence is monotone, it is clearly that order two periodic solution does not exist in the case by the definition 2.4.

Case 2. If the set N is in the region $\{(\overline{S},\overline{x})|\overline{S}<S_F\}$, without loss of generality, let $S_1>S_0$ (see Figure 6), then we can obtain that $S_0\leq S_2<S_1$ or $S_2\leq S_0<S_1$. Therefore, there exists a periodic solution of order two if $S_0=S_2$ holds.

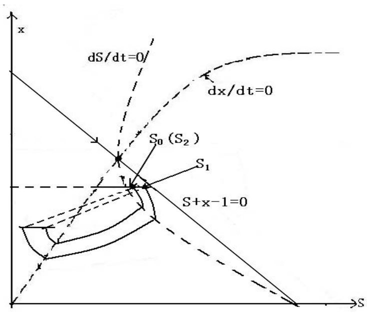

FIGURE 6. *Existence of order two periodic solution*

If $S_0<S_2<S_1$ holds, we can obtain $0<S_0<S_2<\cdots<S_{2k}<S_{2k+1}<\cdots<S_3<S_1<1$, while if $S_2<S_0<S_1$ holds, then we have $0<\cdots<S_{2k}<S_{2(k-1)}<\cdots<S_2<S_0<S_1<S_3<\cdots<S_{2k+1}<\cdots<1$. Similar to the above discussion and the proof of Proposition 3.2 of [10], we can obtain that there is no periodic solution of order k ($k \geq 3$) in system (2.4) and the system is not chaotic.

5. Numerical simulation

Now we consider the following example:

$$\begin{cases} \left.\begin{array}{l} \dot{S} = (3+2x)(1-S) - \dfrac{10Sx}{ax+S}, \\ \dot{x} = x\Big(\dfrac{10S}{ax+S} - (3+2x)\Big), \end{array}\right\} x < h, \\ \left.\begin{array}{l} \triangle S = -bS, \\ \triangle x = -bx, \end{array}\right\} x = h, \end{cases} \tag{5.1}$$

In numerical simulation, let firstly $a = 2, b = 0.5, d = 3, k = 2, m = 10, S(0) = 0.6, x(0) = 0.2$. If $h = 0.5 > x^* = \frac{-[(a-1)d+k+m]+\sqrt{[(a-1)d+k+m]^2+4k(a-1)(m-d)}}{2(a-1)k} \doteq 0.44075$, then the time series and phase portrait can be seen in Figure 8. By analysis of Section 4, we know that no impulse occurs for system (2.4) if $h > x^*$ holds. As shown in Figure 7, numerical simulation also suggests that system (2.4) with the coefficients above admits no impulse to occur. By Theorem 4.1, we know that system (2.4) has a periodic solution of order one under conditions of Theorem 4.1. As shown in Figure 8, if $m = 10, a = 2, b = 0.5, d = 3, k = 2, h = 0.4 < x^*, S(0) = 0.6, x(0) = 0.2$, then system (2.4) has a periodic solution of order one, which verifies theoretical results in this paper. For $a < 1$, our theorems can also be verified by numerical simulations.

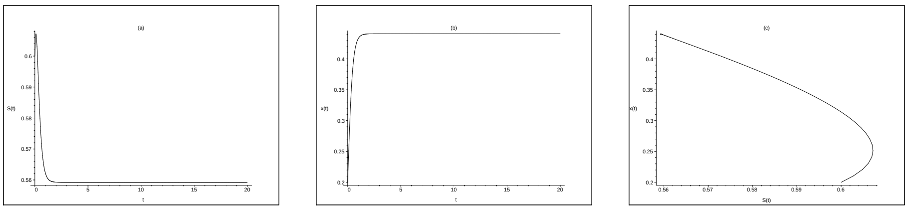

FIGURE 7. *Time series and portrait phase of system (2.4) when* $m = 10, a = 2, b = 0.5, d = 3, k = 2, h = 0.5, S(0) = 0.6, x(0) = 0.2$.

6. Conclusion

In this paper, we build a turbidostat model with ratio-dependent growth rate and impulsive state feedback control. Firstly, we investigated qualitative characteristic of the system without impulsive effect, and obtain sufficient condition of globally asymptotically stable of the system. We obtain that the system with ratio-dependent growth rate and impulsive state feedback control has a periodic solution of order one. Sufficient conditions for existence and stability of periodic

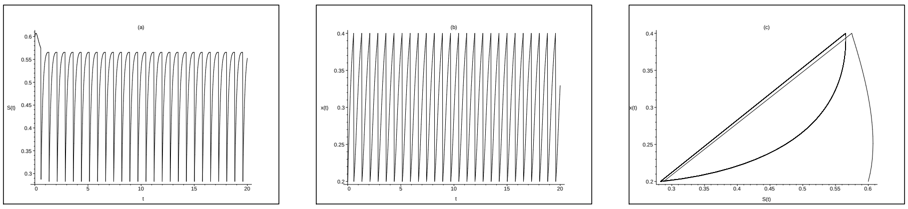

FIGURE 8. *Time series and portrait phase of system (2.4) when* $m = 10, a = 2, b = 0.5, d = 3, k = 2, h = 0.4, S(0) = 0.6, x(0) = 0.2$.

solution of order one are given. The case, in which it is possible that there is a periodic solution of order two, is also given in Subsection 4.3.

The results show that the turbidostat with ratio-dependent growth rate and impulsive state feedback control tends to a stable state or periodic, and the behavior of impulsive state feedback control on the concentrate of microorganism plays an important role on the periodic or stable state of system (2.4). A state feedback measure for controlling concentrate of microorganism is taken when the concentrate of microorganism reaches an appropriate critical value. This measure is effective based on the fact that the system has stable periodic solution under some conditions by the analysis of Section 4. According to the theoretical results, the production of the microorganism will tend to a stable production level or the production will be periodic. The key to the production by applying the system with feedback control is to give the suitable feedback state (the value of h) and the control parameters (b) according to specific situation. It is seen from Figure 8 that there are positive periodic trajectories under the impulsive state feedback control. Therefore, the periodic production of the microorganism can be achieved if the value of h and the suitable initial value of the substrate and the microorganism are taken.

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Xiaoyu Hu is an undergraduate of Hubei University for Nationalities. Her research interests are Mathematical biology and Mathematical modeling.

Department of Mathematics, Hubei University for Nationalities, Enshi 445000, China.
e-mail: 752937362@qq.com

Zuxiong Li received M.Sc. from Guangxi Normal University, and Ph.D. at Dalian University of Technology. Since 2011 he has been at Hubei University for Nationalities. His research interests are theory and application of ordinary differential equations, theory and application of functional differential equations and mathematical biology.

Department of Mathematics, Hubei University for Nationalities, Enshi 445000, China.
e-mail: lizx0427@126.com

Xingguo Xiang is an undergraduate of Hubei University for Nationalities. His research interests are Mathematical biology and Mathematical modeling.

Department of Mathematics, Hubei University for Nationalities, Enshi 445000, China.
e-mail: 1182922100@qq.com