J. Appl. Math. & Informatics Vol. **31***(2013), No. 3 - 4, pp. 365 - 372*
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A NOTE ON THE WEIGHTED $q-$HARDY-LITTLEWOOD-TYPE MAXIMAL OPERATOR WITH RESPECT TO $q-$VOLKENBORN INTEGRAL IN THE $p-$ADIC INTEGER RING

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Abstract. The essential aim of this paper is to define weighted q-Hardy-littlewood-type maximal operator by means of p-adic q-invariant distribution on $\mathbb{Z}_p$. Moreover, we give some interesting properties concerning this type maximal operator.

AMS Mathematics Subject Classification : 11B65, 11B68, 11B73.
Key words and phrases : q-Volkenborn integral, Hardy-littlewood theorem, p-adic analysis, q-analysis.

1. Preliminaries

Recently, q-analysis has served as a structure between mathematics and physics. Therefore, there is a significant increase of activity in the area of the q-analysis due to applications of the q-analysis in mathematics, statistics and physics.

p-adic numbers also play a vital and important role in mathematics. p-adic numbers were invented by the German mathematician Kurt Hensel [10], around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community.

The p-adic q-integral (or q-Volkenborn integral) are originally constructed by Kim [4]. The q-Volkenborn integral is used in Mathematical Physics for example the functional equation of the q-Zeta function, the q-Stirling numbers and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L function.

Received March 16, 2012. Revised August 24, 2012. Accepted December 7, 2012.
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T. Kim, by using q-Volkenborn integral, introduced a novel Lebesgue-Radon-Nikodym type theorem. He is given some interesting properties concerning Lebesgue-Radon-Nikodym theorem. After, Jang [9] defined q-extension of Hardy-Littlewood-type maximal operator by means of q-Volkenborn integral on $\mathbb{Z}_p$. Next, Kim, Choi and Kim [7] defined weighted Lebesgue-Radon-Nikodym theorem. They also gave some interesting properties of this type theorem.

By the same motivation of the above studies, in this paper, we construct weighted q-Hardy-littlewood-type maximal operator by the means of p-adic q-integral on $\mathbb{Z}_p$. We also give some interesting properties of this type operator.

Imagine that p be a fixed prime number. Let $\mathbb{Q}_p$ be the field of p-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$. Thus,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leq a_n \leq p-1 \right\}.$$

Then $\mathbb{Z}_p$ is an integral domain, which is defined by

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n : 0 \leq a_n \leq p-1 \right\},$$

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ as an indeterminate. The p-adic absolute value $|.|_p$, is normally defined by

$$|x|_p = \frac{1}{p^r}, \ (x \in \mathbb{C}_p),$$

where $x = p^r \frac{s}{t}$ with $(p,s) = (p,t) = (s,t) = 1$ and $r \in \mathbb{Q}$.

A p-adic Banach space B is a $\mathbb{Q}_p$-vector space with a lattice B^0 ($\mathbb{Z}_p$-module) separated and complete for p-adic topology, ie.,

$$B^0 \simeq \varprojlim_{n \in \mathbb{N}} B^0 / p^n B^0.$$

For all $x \in B$, there exists $n \in \mathbb{Z}$, such that $x \in p^n B^0$. Define

$$v_B(x) = \sup_{n \in \mathbb{N} \cup \{+\infty\}} \left\{ n : x \in p^n B^0 \right\}.$$

It satisfies the following properties:

$$\begin{aligned} v_B(x+y) &\geq \min(v_B(x), v_B(y)), \\ v_B(\beta x) &= v_p(\beta) + v_B(x), \text{ if } \beta \in \mathbb{Q}_p. \end{aligned}$$

Then, $\|x\|_B = p^{-v_B(x)}$ defines a norm on B, such that B is complete for $\|.\|_B$ and B^0 is the unit ball.

A measure μ on $\mathbb{Z}_p$ with values in a p-adic Banach space B is a continuous linear map

$$f \mapsto \int f(x)\,\mu = \int_{\mathbb{Z}_p} f(x)\,\mu(x)$$

from $C^0(\mathbb{Z}_p, \mathbb{C}_p)$, (continuous function on $\mathbb{Z}_p$) to B. We know that the set of locally constant functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ is dense in $C^0(\mathbb{Z}_p, \mathbb{C}_p)$ so. Explicitly, for all $f \in C^0(\mathbb{Z}_p, \mathbb{C}_p)$, the locally constant functions

$$f_n = \sum_{i=0}^{p^n-1} f(i)\, 1_{i+p^n\mathbb{Z}_p} \to\ f \text{ in } C^0.$$

Now if $\mu \in \boldsymbol{D}_0(\mathbb{Z}_p, \mathbb{Q}_p)$, set $\mu(i + p^n\mathbb{Z}_p) = \frac{1}{p^n}$. Then $\int_{\mathbb{Z}_p} f\mu$ is given by the following "Riemann sums"

$$\int_{\mathbb{Z}_p} f\mu = \lim_{n\to\infty} \sum_{i=0}^{p^n-1} f(i)\,\mu(i + p^n\mathbb{Z}_p).$$

T. Kim defined μ_q as follows:

$$\mu_q(\xi + dp^n\mathbb{Z}_p) = \frac{q^\xi}{[dp^n]_q}$$

and this can be extended to a distribution on $\mathbb{Z}_p$. This distribution yields an integral in the case $d = 1$. So, q-Volkenborn integral was defined by T. Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi)\, d\mu_q(\xi) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{\xi=0}^{p^n-1} f(\xi)\, q^\xi, \tag{1}$$

(for details, see [4], [5]) where $[x]_q$ is a q-extension of x which is defined by

$$[x]_q = \frac{1-q^x}{1-q},$$

note that $\lim_{q\to1} [x]_q = x$ cf. [2], [3], [4], [5], [9].

Let d be a fixed positive integer with $(p, d) = 1$. Note that

$$\begin{aligned} X &= X_d = \varprojlim_n \mathbb{Z}/dp^n\mathbb{Z}, \\ X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^n\mathbb{Z}_p &= \{x \in X \mid x \equiv a \,(\text{mod } dp^n)\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^n$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$,

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \int_X f(x)\, d\mu_q(x),$$

(for details, see [8]).

By the meaning of q-Volkenborn integral, we consider below strongly p-adic q-invariant distribution μ_q on $\mathbb{Z}_p$ in the form

$$\left|[p^n]_q \mu_q (a + p^n \mathbb{Z}_p) - \left[p^{n+1}\right]_q \mu_q \left(a + p^{n+1} \mathbb{Z}_p\right)\right|_p < \delta_n,$$

where $\delta_n \to 0$ as $n \to \infty$ and δ_n is independent of a. Let $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, for any $x \in \mathbb{Z}_p$, we assume that the weight function $\omega(x)$ is defined by $\omega(x) = \omega^x$, where $\omega \in \mathbb{C}_p$ with $|1 - \omega|_p < 1$.

We define the weighted measure on $\mathbb{Z}_p$ as follows:

$$\mu_{f,q}^{(\omega)} (a + p^n \mathbb{Z}_p) = \int_{a + p^n \mathbb{Z}_p} \omega^\xi f(\xi)\, d\mu_q(\xi), \tag{2}$$

where the integral is the q-Volkenborn integral. By (2), we easily note that $\mu_{f,q}^{(\omega)}$ is a strongly weighted measure on $\mathbb{Z}_p$. Thus, we give the following proposition.

Proposition 1.1. *For $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have*

$$\mu_{\alpha f + \beta g, q}^{(\omega)} (a + p^n \mathbb{Z}_p) = \alpha \mu_{f,q}^{(\omega)} (a + p^n \mathbb{Z}_p) + \beta \mu_{g,q}^{(\omega)} (a + p^n \mathbb{Z}_p),$$

where α, β are positive constants. Moreover,

$$\left|[p^n]_q \mu_{f,q}^{(\omega)} (a + p^n \mathbb{Z}_p) - \left[p^{n+1}\right]_q \mu_{f,q}^{(\omega)} \left(a + p^{n+1} \mathbb{Z}_p\right)\right|_p \leq C p^{-n},$$

where C is a positive constant.

Proof. By using Equation (2), it is not difficult to see the following:

$$\begin{aligned}
\mu_{\alpha f + \beta g, q}^{(\omega)} (a + p^n \mathbb{Z}_p) &= \int_{a + p^n \mathbb{Z}_p} \omega^\xi (\alpha f + \beta g)(\xi)\, d\mu_q(\xi) \\
&= \alpha \int_{a + p^n \mathbb{Z}_p} \omega^\xi f(\xi)\, d\mu_q(\xi) + \beta \int_{a + p^n \mathbb{Z}_p} \omega^\xi g(\xi)\, d\mu_q(\xi) \\
&= \alpha \mu_{f,q}^{(\omega)} (a + p^n \mathbb{Z}_p) + \beta \mu_{g,q}^{(\omega)} (a + p^n \mathbb{Z}_p),
\end{aligned}$$

where $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ and α, β are positive constants. Also, we show the following:

$$\begin{aligned}
&\left|[p^n]_q \mu_{f,q}^{(\omega)} (a + p^n \mathbb{Z}_p) - \left[p^{n+1}\right]_q \mu_{f,q}^{(\omega)} \left(a + p^{n+1} \mathbb{Z}_p\right)\right|_p \\
= &\left|\sum_{x=0}^{p^n - 1} \omega^x f(x) q^x - \sum_{x=0}^{p^n} \omega^x f(x) q^x\right|_p \\
\leq &\left|\frac{f(p^n) \omega^{p^n} q^{p^n}}{p^n}\right|_p |p^n|_p \\
\leq &\ C p^{-n},
\end{aligned}$$

where C is a positive constant. Thus we arrive at the desired result. □

Let $UD(\mathbb{Z}_p, \mathbb{C}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$ with supnorm

$$\|f\|_\infty = \sup_{x\in\mathbb{Z}_p} |f(x)|_p .$$

The difference quotient $\Delta_1 f$ of f is the function of two variables given by

$$\Delta_1 f(m,x) = \frac{f(x+m) - f(x)}{m}, \text{ for all } x,\, m \in \mathbb{Z}_p,\, m \neq 0.$$

A function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is said to be a Lipschitz function if there exists a constant $M > 0$ (the Lipschitz constant of f) such that

$$|\Delta_1 f(m,x)| \leq M \text{ for all } m \in \mathbb{Z}_p \setminus \{0\} \text{ and } x \in \mathbb{Z}_p.$$

The $\mathbb{C}_p$ linear space consisting of all Lipschitz functions is denoted by $Lip(\mathbb{Z}_p, \mathbb{C}_p)$. This space is a Banach space with the respect to the norm

$$\|f\|_1 = \|f\|_\infty \bigvee \|\Delta_1 f\|_\infty$$

(for more information, see [1], [2], [3], [4], [5], [6], [9]). The main aim of this paper is to define weighted q-extension of Hardy Littlewood type maximal operator. Moreover, we show the boundedness of the weighted q-Hardy-littlewood-type maximal operator in the p-adic integer ring.

2. The weighted q-Hardy-littlewood-type maximal operator.

In view of (2) and the definition of p-adic q-integral on $\mathbb{Z}_p$, we now start with the following theorem.

Theorem 2.1. *Let μ_q be a strongly p-adic q-invariant in the p-adic integer ring and $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. Then for any $n \in \mathbb{Z}$ and any $\xi \in \mathbb{Z}_p$, we have*

(1) $\int_{a+p^n\mathbb{Z}_p} \omega^{\frac{\xi}{p^n}} f(\xi)\, q^{-\frac{\xi}{p^n}} d\mu_{q^{p^{-n}}}(\xi) = \frac{\omega^{\frac{a}{p^n}}}{[p^n]_{q^{p^{-n}}}} \int_{\mathbb{Z}_p} \omega^\xi f(a+p^n\xi)\, q^{-\xi} d\mu_q(\xi).$

(2) $\int_{a+p^n\mathbb{Z}_p} \omega^{\frac{\xi}{p^n}} d\mu_{q^{p^{-n}}}(\xi) = \frac{(1-q)\omega^{\frac{a}{p^n}} q^{\frac{a}{p^n}}}{(1-\omega q)[p^n]_{q^{p^{-n}}}} \left(1 + \frac{\log \omega}{\log q}\right).$

Proof. (1) We see use of equations (1) and (2)

$$\begin{aligned}
&\int_{a+p^n\mathbb{Z}_p} \omega^{\frac{\xi}{p^n}} f(\xi)\, q^{-\frac{\xi}{p^n}} d\mu_{q^{p^{-n}}}(\xi) \\
=\;& \lim_{m\to\infty} \frac{1}{[p^{m+n}]_{q^{p^{-n}}}} \sum_{\xi=0}^{p^m-1} \omega^{\frac{a+p^n\xi}{p^n}} f(a+p^n\xi)\, q^{-\frac{a+p^n\xi}{p^n}} q^{\frac{a+p^n\xi}{p^n}} \\
=\;& \omega^{\frac{a}{p^n}} \lim_{m\to\infty} \frac{1}{[p^n]_{q^{p^{-n}}} [p^m]_q} \sum_{\xi=0}^{p^m-1} \omega^\xi q^{-\xi} f(a+p^n\xi)\, q^\xi \\
=\;& \frac{\omega^{\frac{a}{p^n}}}{[p^n]_{q^{p^{-n}}}} \int_{\mathbb{Z}_p} \omega^\xi f(a+p^n\xi)\, q^{-\xi} d\mu_q(\xi).
\end{aligned}$$

(2) By the same method of (1), we easily see the following

$$\begin{aligned}
\int_{a+p^n\mathbb{Z}_p} \omega^{\frac{\xi}{p^n}} d\mu_{q^{p^{-n}}}(\xi) &= \lim_{m\to\infty} \frac{1}{[p^{m+n}]_{q^{p^{-n}}}} \sum_{\xi=0}^{p^m-1} \omega^{\frac{a+\xi p^n}{p^n}} q^{\frac{a+\xi p^n}{p^n}} \\
&= \frac{\omega^{\frac{a}{p^n}} q^{\frac{a}{p^n}}}{[p^n]_{q^{p^{-n}}}} \lim_{m\to\infty} \frac{1}{[p^m]_q} \sum_{\xi=0}^{p^m-1} \omega^{\xi} q^{\xi} \\
&= \frac{\omega^{\frac{a}{p^n}} q^{\frac{a}{p^n}} (1-q)}{[p^n]_{q^{p^{-n}}} (1-\omega q)} \lim_{m\to\infty} \frac{1-(\omega q)^{p^m}}{1-q^{p^m}} \\
&= \frac{\omega^{\frac{a}{p^n}} q^{\frac{a}{p^n}} (1-q)}{[p^n]_{q^{p^{-n}}} (1-\omega q)} \left(1 + \frac{\log \omega}{\log q}\right).
\end{aligned}$$

Since $\lim_{m\to\infty} q^{p^m} = 1$ for $|1-q|_p < 1$, our assertion follows. $\square$

Now, we are ready to give definition of weighted q-extension of Hardy-littlewood-type maximal operator related to p-adic q-integral on $\mathbb{Z}_p$ with a strong p-adic q-invariant distribution μ_q in the p-adic integer ring.

Definition 2.2. Let μ_q be a strongly p-adic q-invariant distribution in the p-adic integer ring and $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. Then weighted q-extension of the Hardy-littlewood-type maximal operator with respect to p-adic q-integral on $x + p^n\mathbb{Z}_p$ is defined by

$$M_{p,q}^{(\omega)} f(x) = \sup_{n\in\mathbb{Z}} \frac{1}{\mu_{1,q^{p^{-n}}}^{(w^{p^{-n}})}(x + p^n\mathbb{Z}_p)} \int_{a+p^n\mathbb{Z}_p} \omega^{\frac{x}{p^n}} q^{-\frac{x}{p^n}} f(x)\, d\mu_{q^{\frac{1}{p^n}}}(x)$$

for all $x \in \mathbb{Z}_p$.

We recall that famous Hardy-littlewood maximal operator M_μ is defined by

$$M_\mu f(a) = \sup_{a\in Q} \frac{1}{\mu(Q)} \int_Q |f(x)|\, d\mu(x), \tag{3}$$

where $f : \mathbb{R}^k \to \mathbb{R}^k$ is a locally bounded Lebesgue measurable function, μ is a Lebesgue measure on $(-\infty, \infty)$ and the supremum is taken over all cubes Q which are parallel to the coordinate axes. Note that the boundedness of the Hardy-Littlewood maximal operator serves as one of the most important tools used in the investigation of the properties of variable exponent spaces (see [9]). The essential aim of Theorem 2.1 is to deal with the weighted q-extension of the classical Hardy-Littlewood maximal operator in the space of p-adic Lipschitz functions on $\mathbb{Z}_p$ and to find the boundedness of them. By the meaning of Definition 2.2, we get the following theorem.

Theorem 2.3. *Let $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ and $x \in \mathbb{Z}_p$, we get*

$$(1)\quad M_{p,q}^{(\omega)} f(x) = \frac{1-\omega q}{1-q} \frac{\log q}{\log(\omega q)} \sup_{n\in\mathbb{Z}} q^{-\frac{x}{p^n}} \int_{\mathbb{Z}_p} \omega^{\xi} f(x+p^n\xi) q^{-\xi} d\mu_q(\xi).$$

$$(2)\quad \left| M_{p,q}^{(\omega)} f(x) \right|_p \le \sup_{n\in\mathbb{Z}} \left| \frac{1-wq}{(1-q) q^{\frac{x}{p^n}}} \frac{\log q}{\log(\omega q)} \right|_p \|f\|_1 \left\| \left(\frac{q}{\omega}\right)^{-(\cdot)} \right\|_{L^1},$$

where $\left\| \left(\frac{q}{\omega}\right)^{-(\cdot)} \right\|_{L^1} = \int_{\mathbb{Z}_p} \left| \left(\frac{q}{\omega}\right)^{-\xi} \right|_p d\mu_q(\xi)$.

Proof. (1) Because of Theorem 2.1 and Definition 2.2, we see

$$\begin{aligned}
M_{p,q}^{(\omega)} f(a) &= \sup_{n\in\mathbb{Z}} \frac{1}{\mu_{1,q^{p^{-n}}}^{(w^{p^{-n}})}(x+p^n\mathbb{Z}_p)} \int_{a+p^n\mathbb{Z}_p} \omega^{\frac{x}{p^n}} q^{-\frac{x}{p^n}} f(x)\, d\mu_{q^{\frac{1}{p^n}}}(x) \\
&= \sup_{n\in\mathbb{Z}} \frac{(1-\omega q)[p^n]_{q^{p^{-n}}} \log q}{(1-q)\omega^{\frac{x}{p^n}} q^{\frac{x}{p^n}} \log(\omega q)} \frac{\omega^{\frac{x}{p^n}}}{[p^n]_{q^{p^{-n}}}} \int_{\mathbb{Z}_p} \omega^{\xi} f(a+p^n\xi) q^{-\xi} d\mu_q(\xi) \\
&= \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \sup_{r\in\mathbb{Z}} \frac{1}{q^{\frac{x}{p^n}}} \int_{\mathbb{Z}_p} \omega^{\xi} f(a+p^n\xi) q^{-\xi} d\mu_q(\xi).
\end{aligned}$$

(2) On account of (1), we can derive the following

$$\begin{aligned}
\left| M_{p,q}^{(\omega)} f(a) \right|_p &= \left| \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \sup_{n\in\mathbb{Z}} \frac{1}{q^{\frac{x}{p^n}}} \int_{\mathbb{Z}_p} \omega^{\xi} f(a+p^n\xi) q^{-\xi} d\mu_q(\xi) \right|_p \\
&\le \left| \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \right|_p \sup_{n\in\mathbb{Z}} \left| q^{-\frac{x}{p^n}} \int_{\mathbb{Z}_p} f(a+p^n\xi) \left(\frac{q}{\omega}\right)^{-\xi} d\mu_q(\xi) \right|_p \\
&\le \left| \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \right|_p \sup_{n\in\mathbb{Z}} \left| q^{-\frac{x}{p^n}} \right|_p \int_{\mathbb{Z}_p} |f(a+p^n\xi)|_p \left| \left(\frac{q}{\omega}\right)^{-\xi} \right|_p d\mu_q(\xi) \\
&\le \left| \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \right|_p \sup_{n\in\mathbb{Z}} \left| q^{-\frac{x}{p^n}} \right|_p \|f\|_1 \int_{\mathbb{Z}_p} \left| \left(\frac{q}{\omega}\right)^{-\xi} \right|_p d\mu_q(\xi) \\
&= \left| \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \right|_p \sup_{n\in\mathbb{Z}} \left| q^{-\frac{x}{p^n}} \right|_p \|f\|_1 \left\| \left(\frac{q}{\omega}\right)^{-(\cdot)} \right\|_{L^1}.
\end{aligned}$$

Thus, we complete the proof of theorem. □

We note that Theorem 2.3 (2) shows the supnorm-inequality for the weighted q-Hardy-Littlewood-type maximal operator in the p-adic integer ring, in a word, Theorem 2.3 (2) shows the following inequality

$$\left\| M_{p,q}^{(\omega)} f \right\|_{\infty} = \sup_{x\in\mathbb{Z}_p} \left| M_{p,q}^{(\omega)} f(x) \right|_p \le K \|f\|_1 \left\| \left(\frac{q}{\omega}\right)^{-(\cdot)} \right\|_{L^1}, \tag{4}$$

where $K = \left| \frac{(1-\omega q)\log q}{(1-q)\log(\omega q)} \right|_p \sup_{n\in\mathbb{Z}} \left| q^{-\frac{x}{p^n}} \right|_p$. By the equation (4), we get the following Corollary, which is the boundedness for weighted q-Hardy-Littlewood-type maximal operator in the p-adic integer ring.

Corollary 2.4. *$M_{p,q}^{(\omega)}$ is a bounded operator from $UD\left(\mathbb{Z}_p, \mathbb{C}_p\right)$ into $L^{\infty}\left(\mathbb{Z}_p, \mathbb{C}_p\right)$, where $L^{\infty}\left(\mathbb{Z}_p, \mathbb{C}_p\right)$ is the space of all p-adic supnorm-bounded functions with the*

$$\|f\|_{\infty} = \sup_{x\in\mathbb{Z}_p} \left|f\left(x\right)\right|_p ,$$

for all $f \in L^{\infty}\left(\mathbb{Z}_p, \mathbb{C}_p\right)$.

Acknowledgement

The authors would like to thank the referee for his/her valuable suggestions and comments on this work.

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