

FOURIER SERIES ACCELERATION AND HARDY-LITTLEWOOD SERIES[†]

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ABSTRACT. We discuss the effects of the δ^2 and Lubkin acceleration methods on the partial sums of Fourier Series. We construct continuous, even Hölder continuous functions, for which these acceleration methods fail to give convergence. The constructed functions include some interesting trigonometric series whose properties were investigated by Hardy and Littlewood.

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1. Introduction

For a measurable function $f \in L^1([-\pi, \pi])$ the Fourier coefficients of f are defined by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} dx$$

for each integer k , and the n^{th} partial sum of the Fourier series is given as

$$S_n f(x) = \sum_{k=-n}^n \hat{f}(k)e^{ikx} \quad (1)$$

For $f \in L^2([-\pi, \pi])$, $S_n f \rightarrow f$ in L^2 and Carleson's theorem gives a.e. convergence. Theorems of Dini-Lipschitz, Lebesgue, and Dirichlet-Jordan (among others) give conditions on f for pointwise convergence (see Zygmund [12] for all of these). In practice, the difficulty is that this convergence can often be quite slow. This is particularly the case for discontinuous functions with jump discontinuities. Thus, it is desirable to attempt to find methods to speed this

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convergence. This paper continues investigations began in [1] and [3] to try to determine properties of functions for which one of the sequence acceleration methods described below may accelerate the convergence of its Fourier series.

For a numerical sequence $\{s_n\}$ which converges to a finite limit s , we say that a transformation of the sequence $\{t_n\}$ accelerates the convergence of $\{s_n\}$ if there exists a positive integer k such that each t_n depends only on s_1, s_2, \dots, s_{n+k} and so that t_n converges to s faster than s_n . In this paper we consider two of the most well-known nonlinear transformations: the δ^2 process in which a sequence s_n is transformed to the sequence

$$\varepsilon_2(s_{n-1}) = s_n - \frac{(s_{n+1} - s_n)(s_n - s_{n-1})}{(s_{n+1} - s_n) - (s_n - s_{n-1})} \quad (2)$$

and the Lubkin [7] transform in which the sequence s_n is transformed to the sequence

$$s_{n+1}^* = s_n + \frac{a_{n+1}(1 - \rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n\rho_{n+1}},$$

where we have set $a_{n+1} = s_{n+1} - s_n$ and $\rho_n = \frac{a_{n+1}}{a_n}$. In either of these we define the fraction on the right to be zero if the denominator is zero. We have used the notation $\varepsilon_2(s_{n-1})$ for the δ^2 transform as it is one of a family of transformations $\varepsilon_k(s_n)$ which we will discuss below. In the case when the s_n are partial sums of a geometric series, both of these transforms produce a constant sequence which is the sum of the series. Shanks [8] shows acceleration of convergence using (2) for partial series which are, in a sense made precise in [8], "nearly geometric." These and related transforms are discussed extensively in Brezinski and Redivo-Zaglia [5] and Sidi [9].

Throughout, we will consider the sequence of partial sums $S_n f(x)$ of a Fourier series as in (1), and apply the δ^2 and Lubkin transforms pointwise to obtain the sequences of functions given by respectively:

$$\begin{aligned} \varepsilon_2(S_{n-1}f)(x) &= S_n f(x) \\ &- \frac{(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x})(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx})}{(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}) - (\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx})} \end{aligned} \quad (3)$$

and

$$S_{n+1}^* f(x) = S_n f(x) + \frac{(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x})(1 - \rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n\rho_{n+1}} \quad (4)$$

where

$$\rho_n = \frac{\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}}{\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}}.$$

Neither of these transformations will accelerate the convergence of the partial sums of the Fourier series of a function with a single jump. In fact, we have

Theorem 1.1. *Suppose that $f \in C^2([- \pi, \pi])$ and that $f(-\pi) \neq f(\pi)$.*

- (a) (Abebe, Graber, Moore [1]). Consider the sequence $\varepsilon_2(S_n f)(x)$ formed by applying the δ^2 process (3) to the sequence $S_n f(x)$. Then $\varepsilon_2(S_n f)(x)$ fails to converge to $f(x)$ at every x of the form $x = 2\pi a$, where $a \in (-\frac{1}{2}, \frac{1}{2})$ is irrational.
- (b) (Bogges, Bunch, Moore [3]). Suppose that $f \in C^2([-\pi, \pi])$ and that $f(-\pi) \neq f(\pi)$. Consider the sequence $S_n^* f(x)$ formed by applying the Lubkin transform (4) to the sequence $S_n f(x)$. Then $S_n^* f(x)$ fails to converge to $f(x)$ at every x of the form $x = 2\pi a$, where $a \in (-\frac{1}{4}, \frac{1}{4})$ is irrational.

For functions as in the hypotheses of the theorems, $|\hat{f}(n)|$ decays like $\frac{1}{|n|}$ as $n \rightarrow \pm\infty$ so that these Fourier series converge slowly. This is disappointing, as these are exactly the types of Fourier series we would like to accelerate. In this paper, we will investigate similar types of results. We will consider a family of Hölder continuous functions and show that for each function f in this family, and each $x \in [-\pi, \pi]$ the transformed partial sums do not converge to $f(x)$.

2. Convergence of the transformed series

Given a sequence $s_n \rightarrow s$, set $\varepsilon_{-1}^{(n)} = 0$ and $\varepsilon_0^{(n)} = s_n$ and for $k, n = 0, 1, \dots$ compute

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + \left(\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)} \right)^{-1}.$$

This is known as the epsilon algorithm and was introduced by Wynn [10] as an efficient computational procedure for computing the Shanks transformation. With this notation, the sequence $\varepsilon_2^{(n-1)}$ is the δ^2 transform of the sequence s_n . Here the sequences $\varepsilon_{2k}^{(n)}$ are of interest; the sequences with odd lower index are just intermediate computations. Consider the partial sums of a Fourier series of a function f :

$$S_n f(x) = \frac{a_0}{2} + \sum_{j=1}^n [a_j \cos(jx) + b_j \sin(jx)].$$

Brezinski [4] (see also Wynn [11]) has proposed the following procedure for accelerating the convergence of the sequence $S_n f(x)$: Consider the conjugate series

$$\tilde{S}_n f(x) = \sum_{j=1}^n [a_j \sin(jx) - b_j \cos(jx)].$$

Add the conjugate series to $S_n f(x)$ to obtain $S_n f(x) + i\tilde{S}_n f(x) = G_n(f)(e^{ix})$.

Then $G_n f(z) = \sum_{j=0}^n c_j z^j$, with $c_0 = \frac{a_0}{2}$ and $c_j = a_j - ib_j$, is the n th partial sum of the formal series $Gf(z) = \sum_{j=0}^{\infty} c_j z^j$. Apply the ε algorithm to the sequence $G_n f(z)$ and take the real part of the resulting $\varepsilon_{2k}^{(n)}$ to obtain a sequence $\text{Re } \varepsilon_{2k}^{(n)}$ which we hope will approximate $\text{Re } Gf(e^{ix}) = f(x)$ more quickly than $S_n f(x)$. There is a relationship between the ε algorithm and Padé approximants:

$\varepsilon_{2k}^{(n)} = [n + k/k]_G$, where $[n + k/k]_G$ denotes the unique rational function with

numerator N of degree $n + k$ and denominator D of degree k such that $DG - N = O(z^{n+k+k+1})$ as $z \rightarrow 0$. Because of this, $\text{Re } \varepsilon_{2k}^{(n)}$ is likely to converge more quickly than $S_n f$. Numerical evidence illustrating the acceleration of convergence is given by Brezinski [4]. Beckermann, Matos, and Wielonsky [2] show that this method accelerates convergence for functions of the form $f = f_1 + f_2$, where f_1 has prescribed discontinuities but is smooth elsewhere, f_2 has quickly decaying Fourier coefficients, and $G(f_1) = \lim_{n \rightarrow \infty} G_n(f_1)$ is a certain type of hypergeometric function. The authors in both [4] and [2] also note that this scheme reduces the Gibbs phenomenon.

We follow the idea of adding a series and conjugate series so that the transforms are applied to analytic functions. We will show that the δ^2 transform and the Lubkin transform do not always behave well when applied to analytic functions. In the remainder of this section, however, we will record a few straightforward results about transforms of analytic functions, which show that in many cases, at the very least, the transformed sequence converges to the original limit.

Theorem 2.1. *For an integrable function f on $[-\pi, \pi]$ let $\sum_{k=1}^n c_k e^{ikx} = S_n f(x)$ denote the partial sums of its Fourier series. Suppose x is a point at which $S_n f(x) \rightarrow f(x)$.*

(a) *If the c_k all lie on a line through the origin in the complex plane, and $x \neq -\pi, \pi$, or 0 then $\varepsilon_2(S_n f)(x)$ converges to $f(x)$.*

(b) *If there exists a $\lambda < 1$ such that $|\frac{c_{n+1}}{c_n}| \leq \lambda$ for every n then $\varepsilon_2(S_n f)(x) \rightarrow f(x)$.*

(c) *Suppose there exists $\beta > 1$ so that $|c_n| = \frac{1}{n^\beta}$ for every n . Then $\varepsilon_2(S_n f)(x) \rightarrow f(x)$.*

(d) *Suppose that all the c_k lie on a line through the origin, $x \neq -\pi, \pi$, or 0 , and $\rho_n = \frac{c_{n+1}}{c_n} e^{ix}$ satisfies $|\rho_{n+1} - \rho_n| \rightarrow 0$ as $n \rightarrow \infty$. Then $S_n^* f(x) \rightarrow f(x)$.*

(e) *Suppose that there exists $\eta < \sqrt{2} - 1$ such that $|\rho_n| \leq \eta$ for every n . Then $S_n^* f(x) \rightarrow f(x)$.*

Note that none of these conclude convergence is accelerated, only that it is not destroyed. These show the necessity of the intricacy of the examples in Section 3.

Proof. Take the absolute value of the fraction on the right hand side of (3). If either c_n or c_{n+1} is 0 there is nothing to show. Otherwise simplify to obtain the expression $\frac{|c_{n+1}|}{|e^{-ix} - \frac{c_{n+1}}{c_n}|}$. For (a), note that $\frac{c_{n+1}}{c_n}$ is a real number and that $c_{n+1} \rightarrow 0$, by the Riemann-Lebesgue lemma. For (b), note that the above expression is bounded by $\frac{|c_{n+1}|}{1-\lambda} \rightarrow 0$. For (c) we have

$$\frac{|c_{n+1}|}{|e^{-ix} - \frac{c_{n+1}}{c_n}|} \leq \frac{|c_{n+1}|}{1 - |\frac{c_{n+1}}{c_n}|} = \frac{(n+1)^{-\beta}}{1 - (\frac{n}{n+1})^\beta} \leq \frac{(n+1)^{-\beta}}{c_\beta \frac{1}{n+1}}$$

for a constant c_β , depending only on β . The latter expression tends to 0. For (d) we take the absolute value of the fraction on the right hand side of (4) and

estimate that for sufficiently large n ,

$$\begin{aligned} & \frac{|c_{n+1}| \left| 1 - \frac{c_{n+2}}{c_{n+1}} e^{ix} \right|}{\left| 1 - 2 \frac{c_{n+2}}{c_{n+1}} e^{ix} + \frac{c_{n+2}}{c_n} e^{2ix} \right|} \\ &= \frac{|c_{n+1}| \left| 1 - \frac{c_{n+2}}{c_{n+1}} e^{ix} \right|}{\left| 1 - \frac{c_{n+2}}{c_{n+1}} e^{ix} - \frac{c_{n+1}}{c_n} e^{ix} + \frac{c_{n+2}}{c_{n+1}} e^{ix} \frac{c_{n+1}}{c_n} e^{ix} + \left(\frac{c_{n+1}}{c_n} - \frac{c_{n+2}}{c_{n+1}} \right) e^{ix} \right|} \\ &\leq \frac{|c_{n+1}| \left| 1 - \frac{c_{n+2}}{c_{n+1}} e^{ix} \right|}{\left| 1 - \frac{c_{n+2}}{c_{n+1}} e^{ix} \right| \left| 1 - \frac{c_{n+1}}{c_n} e^{ix} \right| - |\rho_n - \rho_{n+1}|} = \frac{|c_{n+1}|}{\left| e^{-ix} - \frac{c_{n+1}}{c_n} \right| - \frac{|\rho_n - \rho_{n+1}|}{\left| e^{-ix} - \frac{c_{n+2}}{c_{n+1}} \right|}}. \end{aligned}$$

We obtain (d) by noting that as in (a), $\frac{c_{n+2}}{c_{n+1}}$ and $\frac{c_{n+1}}{c_n}$ are real and $c_{n+1} \rightarrow 0$.

For (e) we are assuming that $|\rho_n| \leq \eta < \sqrt{2} - 1$ for every n . Take the absolute value of the fraction on the right hand side of (4):

$$\frac{|c_{n+1}| |1 - \rho_{n+1}|}{|1 - 2\rho_{n+1} + \rho_n \rho_{n+1}|} \leq \frac{|c_{n+1}| (1 + \eta)}{1 - 2\eta - \eta^2},$$

valid for $\eta < \sqrt{2} - 1$. The denominator $1 - 2\eta - \eta^2 > 0$, and $c_{n+1} \rightarrow 0$, yielding (e). \square

The advantage of these results is that with explicit values of c_k we can determine the exact rate at which $S_n f(x) - \varepsilon_2(S_{n-1} f)(x)$ or $S_n f(x) - S_{n+1}^* f(x)$ tends to 0. In the next two theorems we will lose this advantage but will obtain results with wider scope.

Theorem 2.2. *Suppose $S_n f(x) = \sum_{k=1}^n c_k e^{ikx}$ are the partial sums of the Fourier series of an integrable and analytic function. Suppose $\sum_{k=1}^\infty |c_k| < \infty$. Then at almost every x at which $S_n f(x) \rightarrow f(x)$ we have $\varepsilon_2(S_n f)(x) \rightarrow f(x)$.*

Proof. As in the previous theorem, we estimate the fractions on the right hand side of (3). If either c_n or c_{n+1} is 0 there is nothing to show; otherwise we estimate

$$|S_n f(x) - \varepsilon_2(S_{n-1} f)(x)| = \frac{|c_{n+1}|}{\left| e^{-ix} - \frac{c_{n+1}}{c_n} \right|}.$$

For $\lambda > 1$, $|\{x \in [-\pi, \pi] : \frac{1}{\left| e^{-ix} - \frac{c_{n+1}}{c_n} \right|} > \lambda\}| \leq 2 \arcsin(\frac{1}{\lambda}) < \frac{2\pi}{\lambda}$. The latter bound remains valid for all $\lambda > 0$, and thus, for a fixed $\varepsilon > 0$,

$$|\{x \in [-\pi, \pi] : \frac{|c_{n+1}|}{\left| e^{-ix} - \frac{c_{n+1}}{c_n} \right|} > \varepsilon\}| < \frac{2\pi |c_{n+1}|}{\varepsilon}.$$

By the Borel-Cantelli lemma we have that at almost every x , $\frac{|c_{n+1}|}{\left| e^{-ix} - \frac{c_{n+1}}{c_n} \right|} \leq \varepsilon$ eventually. Taking a countable sequence of $\varepsilon \rightarrow 0$ finishes the proof. \square

The next theorem gives the same conclusion for $S_n^* f$ although the estimation is slightly more difficult. We first need a lemma.

Lemma 2.3. *Suppose that γ, β are nonzero complex numbers. Then for all $\lambda > 0$,*

$$\left| \{x \in [-\pi, \pi] : \left| \frac{1 - \gamma e^{ix}}{1 - 2\gamma e^{ix} + \gamma\beta e^{2ix}} \right| > \lambda\} \right| \leq \frac{24\pi}{|\beta|\lambda} + \frac{9\pi}{\lambda}.$$

Proof. Set $E_\lambda = \{x \in [-\pi, \pi] : \left| \frac{1 - \gamma e^{ix}}{1 - 2\gamma e^{ix} + \gamma\beta e^{2ix}} \right| > \lambda\}$. If $\gamma = \beta$, $\frac{1 - \gamma e^{ix}}{1 - 2\gamma e^{ix} + \gamma\beta e^{2ix}} = \frac{1}{1 - \beta e^{ix}}$ and as in the proof of the previous theorem, $\left| \frac{1}{1 - \beta e^{ix}} \right| > \lambda$ can happen on a set of x of measure at most $\frac{2\pi}{\lambda}$.

If $\gamma \neq \beta$, for z complex write $1 - 2\gamma z + \gamma\beta z^2 = \gamma\beta(z - z_1)(z - z_2)$, where $z_1, z_2 = \frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} - \frac{1}{\gamma\beta}}$. Expanding using partial fractions yields:

$$\left| \frac{1 - \gamma z}{1 - 2\gamma z + \gamma\beta z^2} \right| \leq \left| \frac{1 - \gamma z_1}{\gamma\beta(z_1 - z_2)} \cdot \frac{1}{z - z_1} \right| + \left| \frac{1 - \gamma z_2}{\gamma\beta(z_2 - z_1)} \cdot \frac{1}{z - z_2} \right|.$$

We estimate

$$\begin{aligned} \left| \frac{1 - \gamma z_1}{\gamma\beta(z_1 - z_2)} \right| &= \left| \frac{1 - \frac{\gamma}{\beta} - \gamma\sqrt{\frac{1}{\beta^2} - \frac{1}{\gamma\beta}}}{2\gamma\beta\sqrt{\frac{1}{\beta^2} - \frac{1}{\gamma\beta}}} \right| \\ &\leq \frac{|1 - \frac{\gamma}{\beta}|}{2|\gamma\beta|\sqrt{|\frac{1}{\beta^2} - \frac{1}{\gamma\beta}|}} + \frac{1}{2|\beta|} = \frac{\sqrt{|1 - \frac{\beta}{\gamma}|} + 1}{2|\beta|}. \end{aligned} \tag{5}$$

Similarly, $\left| \frac{1 - \gamma z_2}{\gamma\beta(z_1 - z_2)} \right| \leq \frac{\sqrt{|1 - \frac{\beta}{\gamma}|} + 1}{2|\beta|}$.

Suppose now that $|1 - \frac{\beta}{\gamma}| \leq 10$. Then $\frac{\sqrt{|1 - \frac{\beta}{\gamma}|} + 1}{2|\beta|} \leq \frac{\sqrt{10} + 1}{2|\beta|} < \frac{3}{|\beta|}$ and hence,

$$\begin{aligned} |E_\lambda| &\leq \left| \{x \in [-\pi, \pi] : \frac{3}{|\beta|} \frac{1}{|e^{ix} - z_1|} > \frac{\lambda}{2}\} \right| + \left| \{x \in [-\pi, \pi] : \frac{3}{|\beta|} \frac{1}{|e^{ix} - z_2|} > \frac{\lambda}{2}\} \right| \\ &\leq 2 \cdot 2\pi \frac{6}{|\beta|\lambda} = \frac{24\pi}{|\beta|\lambda}. \end{aligned}$$

Suppose that $|1 - \frac{\beta}{\gamma}| \geq 10$. Then $|\frac{\beta}{\gamma}| \geq 9$, and hence $|\gamma|^2 \leq \frac{1}{9}|\beta\gamma|$. There are two possibilities:

Case 1. $|\gamma\beta| < \frac{1}{3}$. Then $|\gamma|^2 \leq \frac{1}{27} < \frac{1}{25}$. Then for $x \in [-\pi, \pi]$,

$$\frac{|1 - \gamma e^{ix}|}{|1 - 2\gamma e^{ix} + \gamma\beta e^{2ix}|} \leq \frac{1 + |\gamma|}{1 - 2|\gamma| - |\gamma\beta|} < \frac{\frac{6}{5}}{1 - \frac{2}{5} - \frac{1}{3}} = \frac{9}{2},$$

so that if $\lambda > \frac{9}{2}$, $E_\lambda = \emptyset$. For $\lambda \leq \frac{9}{2}$, $|E_\lambda| \leq 2\pi \leq \frac{9\pi}{\lambda}$.

Case 2. $|\gamma\beta| \geq \frac{1}{3}$. Then $\frac{\sqrt{|1 - \frac{\beta}{\gamma}|}}{2|\beta|} = \frac{\sqrt{|\gamma^2 - \gamma\beta|}}{2|\gamma\beta|} \leq \frac{\sqrt{|\gamma|^2 + |\gamma\beta|}}{2|\gamma\beta|} \leq \frac{\sqrt{\frac{10}{9}|\gamma\beta|}}{2|\gamma\beta|} < 1$ and thus, using (5)

$$|E_\lambda| \leq \left| \left\{ x \in [-\pi, \pi] : \frac{1 + \frac{1}{2|\beta|}}{|e^{ix} - z_1|} > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x \in [-\pi, \pi] : \frac{1 + \frac{1}{2|\beta|}}{|e^{ix} - z_2|} > \frac{\lambda}{2} \right\} \right|$$

$$\leq 2 \cdot 2 \left(1 + \frac{1}{2|\beta|} \right) \frac{2\pi}{\lambda} = \frac{8\pi}{\lambda} + \frac{4\pi}{|\beta|\lambda}.$$

□

Theorem 2.4. *Suppose $S_n f(x) = \sum_{k=1}^n c_k e^{ikx}$ are the partial sums of the Fourier series of an integrable analytic function. Suppose that $\sum_{k=1}^\infty |c_k| < \infty$. Then at almost every x at which $S_n f(x) \rightarrow f(x)$ we have $S_n^* f(x) \rightarrow f(x)$.*

Proof. Using the lemma, for $\varepsilon > 0$, and $n = 1, 2, 3, \dots$

$$\left| \left\{ x \in [-\pi, \pi] : \frac{|c_{n+1}| \left| 1 - \frac{c_{n+2}}{c_{n+1}} e^{ix} \right|}{\left| 1 - 2 \frac{c_{n+2}}{c_{n+1}} e^{ix} + \frac{c_{n+2}}{c_{n+1}} \frac{c_{n+1}}{c_n} e^{2ix} \right|} > \varepsilon \right\} \right|$$

$$\leq \frac{24\pi |c_{n+1}|}{\frac{|c_{n+1}|}{c_n} \varepsilon} + \frac{9\pi |c_{n+1}|}{\varepsilon} = \frac{24\pi |c_n|}{\varepsilon} + \frac{9\pi |c_{n+1}|}{\varepsilon}.$$

We sum over n and apply the Borel-Cantelli theorem to conclude that for a.e. x at which $S_n f(x) \rightarrow f(x)$, we have $|S_{n+1}^* f(x) - S_n f(x)| \leq \varepsilon$ eventually. Take a countable sequence of $\varepsilon \rightarrow 0$ to obtain the conclusion of the theorem. □

3. Analyticity is not enough

From the work of Brezinski [4] and the work of Beckermann, Matos, and Wielonsky [2] discussed above, we might conjecture that analyticity implies acceleration of convergence. Here, we show by example that this is false; in fact, even for analytic functions, both transforms may destroy convergence. This is not to say that analyticity does not play a role – just that it might be a factor in a subtle way – but that alone, it is not enough for acceleration, and not even enough for convergence. Our examples show a little more, namely, that Hölder continuity is also not sufficient to insure convergence. We will first construct our examples and then show how the partial sums of their Fourier series behave under transformations (3) and (4).

Theorem 3.1. (Hardy and Littlewood [6]). *For $\frac{1}{2} < \beta \leq \frac{3}{2}$ and $c > 0$ the partial sums of the series $\sum_{n=1}^\infty \frac{1}{n^\beta} e^{i(nx + cn \log n)}$ converge uniformly to a function which is Hölder continuous of order $\beta - \frac{1}{2}$.*

We wish to construct another class of Hölder continuous functions which we will also use. Although this proof essentially follows that of the theorem immediately above (see Zygmund [12], Chapter 5, section 4), we include it for completeness.

Lemma 3.2. *Suppose $1 < \alpha \leq 2$. Set $s_n(x) = \sum_{k=1}^n e^{i(kx+k^\alpha)}$. Then there exists a constant C , depending only on α , such that (i) $s_n(x) \leq Cn^{\frac{\alpha}{2}}$ and (ii) $s'_n(x) \leq Cn^{1+\frac{\alpha}{2}}$.*

Proof. Fix x ; set $g(u) = u^\alpha + ux$. Then $g''(u) = \alpha(\alpha - 1)u^{\alpha-2} \geq \alpha(\alpha - 1)b^{\alpha-2} = \rho$ for $u \in [a, b]$ so by the van der Corput lemma (see [12], pg. 198), for any integers $a < b$,

$$\begin{aligned} \sum_{k=a}^b e^{ig(k)} &\leq (|g'(b) - g'(a)| + 2) \left(\frac{4}{\sqrt{\rho}} + 3 \right) \\ &\leq (C|b^{\alpha-1} - a^{\alpha-1}| + 2) \left(Cb^{\frac{2-\alpha}{2}} + 3 \right) \leq \left(C \frac{(b-a)}{a^{2-\alpha}} + 2 \right) \left(Cb^{\frac{2-\alpha}{2}} + 3 \right), \end{aligned}$$

where in each occurrence C is a constant depending only on α . Pick N so that $2^N \leq n < 2^{N+1}$. Then for each j , $|s_{2^{j+1}}(x) - s_{2^j}(x)| \leq C(2^j 2^{-j(2-\alpha)} + 2)(C2^{(j+1)\frac{2-\alpha}{2}} + 3) \leq C2^{\frac{j\alpha}{2}}$. Similarly, $|s_n(x) - s_{2^N}(x)| \leq C2^{\frac{(N+1)\alpha}{2}}$. Consequently,

$$|s_n(x)| \leq 1 + \sum_{j=0}^{N-1} |s_{2^{j+1}}f(x) - s_{2^j}f(x)| + |s_n f(x) - s_{2^N} f(x)| \leq C \sum_{j=0}^N 2^{\frac{j\alpha}{2}} \leq Cn^{\frac{\alpha}{2}}.$$

Taking the derivative and summing by parts gives:

$$|s'_n(x)| = \left| \sum_{k=1}^n i k e^{i(kx+k^\alpha)} \right| = \left| - \sum_{k=1}^{n-1} s_k(x) + n s_n(x) \right| \leq Cn^{1+\frac{\alpha}{2}}.$$

□

Theorem 3.3. *If $1 < \alpha \leq 2$, $\beta > \frac{1}{2}$, and $0 < \beta - \frac{\alpha}{2} < 1$, then $\sum_{n=1}^\infty \frac{1}{n^\beta} e^{i(nx+n^\alpha)}$ is the Fourier series of a function which is Hölder continuous of order $\beta - \frac{\alpha}{2}$.*

Proof. Set $f(x) = \sum_{n=1}^\infty \frac{1}{n^\beta} e^{i(nx+n^\alpha)}$, which is in L^2 , and for $N \geq 1$, let $S_N f(x)$ denote the N th partial sum of this series. Summing by parts yields

$$S_N f(x) = \sum_{n=1}^N \frac{1}{n^\beta} e^{i(nx+n^\alpha)} = \frac{1}{N^\beta} s_N(x) + \sum_{n=1}^{N-1} \left(\frac{1}{n^\beta} - \frac{1}{(n+1)^\beta} \right) s_n(x),$$

and hence for $M > N$,

$$\begin{aligned} |S_M f(x) - S_N f(x)| &\leq \frac{|s_M(x)|}{M^\beta} + \frac{|s_N(x)|}{N^\beta} + \sum_{n=N}^{M-1} \left(\frac{1}{n^\beta} - \frac{1}{(n+1)^\beta} \right) |s_n(x)| \\ &\leq \frac{M^{\frac{\alpha}{2}}}{M^\beta} + \frac{N^{\frac{\alpha}{2}}}{N^\beta} + C \sum_{n=N}^{M-1} \frac{n^{\frac{\alpha}{2}}}{n^{\beta+1}} \leq \frac{C}{N^{\beta-\frac{\alpha}{2}}}. \end{aligned}$$

Consequently, $S_N f(x)$ converges uniformly so that f is continuous.

Let $h > 0$. Summing by parts gives $f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^\beta} - \frac{1}{(n+1)^\beta} \right) s_n(x)$. Let N be the greatest integer $\leq \frac{1}{h}$. Then

$$|f(x+h) - f(x)| \leq \sum_{n=1}^N \left| \frac{1}{n^\beta} - \frac{1}{(n+1)^\beta} \right| |s_n(x+h) - s_n(x)| + \sum_{n=N+1}^{\infty} \left| \frac{1}{n^\beta} - \frac{1}{(n+1)^\beta} \right| |s_n(x+h) - s_n(x)| = I + II.$$

Using the mean value theorem and (ii) of Lemma 3.2, we can estimate: $I \leq C \sum_{n=1}^N \frac{1}{n^{\beta+1}} n^{\frac{\alpha}{2}+1} h \leq ChN^{-\beta+\frac{\alpha}{2}+1} \leq Ch^{\beta-\frac{\alpha}{2}}$. We use (i) of Lemma 3.2 to estimate $II \leq C \sum_{n=N+1}^{\infty} \frac{1}{n^{\beta+1}} n^{\frac{\alpha}{2}} \leq C \frac{1}{N^{\beta-\frac{\alpha}{2}}} = Ch^{\beta-\frac{\alpha}{2}}$. \square

The next theorem gives conditions which guarantee that $\varepsilon_2(S_n f)(x)$ fails to converge at any x .

Theorem 3.4. *Suppose f is an integrable function on $[-\pi, \pi]$, analytic on the interior of the unit disk, and let $S_N f(x) = \sum_{n=0}^N c_n e^{inx}$ denote the partial sums of its Fourier series. For each n write $c_n = a_n e^{i\theta_n}$ where $a_n \geq 0$, and θ_n is real. For $n \geq 1$ set $\gamma_n = \theta_n - \theta_{n-1}$. Suppose that there exists a constant M such that*

- (a) $|1 - \frac{a_n}{a_{n+1}}| < Ma_n$ for every n , and
- (b) $\gamma_n \rightarrow \infty$ and $|\gamma_{n+1} - \gamma_n| < Ma_n$ for every n . Then $\varepsilon_2(S_n f)(x)$ fails to converge at any x .

Proof. By (3)

$$|S_n f(x) - \varepsilon_2(S_{n-1} f)(x)| = \frac{|c_n|}{|e^{ix} - \frac{c_n}{c_{n+1}}|} = \frac{a_n}{|e^{ix+\gamma_{n+1}} - \frac{a_n}{a_{n+1}}|}.$$

We consider the denominator

$$\left| e^{i(x+\theta_{n+1}-\theta_n)} - \frac{a_n}{a_{n+1}} \right| \leq \left| e^{i(x+\theta_{n+1}-\theta_n)} - 1 \right| + \left| 1 - \frac{a_n}{a_{n+1}} \right| \tag{6}$$

and estimate each term. The second term is bounded by Ma_n for every n by hypothesis. To estimate the first term, consider a fixed $x \in [-\pi, \pi]$. Then for each positive integer l , there exists k such that $x + \gamma_k \leq 2l\pi$ but $x + \gamma_{k+1} > 2l\pi$. Then

$$|x + \gamma_k - 2l\pi| < |x + \gamma_{k+1} - (x + \gamma_k)| = |\gamma_{k+1} - \gamma_k| < Ma_k.$$

so that

$$\left| e^{i(x+\gamma_k)} - 1 \right| = \left| e^{i(x+\gamma_k)} - e^{i2\pi l} \right| \leq |x + \gamma_k - 2\pi l| \leq Ma_k.$$

Since $\gamma_k = \theta_{k+1} - \theta_k$ this immediately gives the estimate $|e^{i(x+\theta_{k+1}-\theta_k)} - 1| < Ma_k$, which by (6) immediately leads to

$$\frac{1}{2M} \leq \frac{|a_k|}{|e^{i(x+\theta_{k+1}-\theta_k)} - \frac{a_k}{a_{k+1}}|}$$

for an infinite number of k . □

Example 1. Consider the Hardy-Littlewood series $\sum_{n=1}^{\infty} \frac{1}{n^\beta} e^{i(nx+cn \log n)}$ with $\frac{1}{2} < \beta \leq 1$ and $c > 0$. Here $a_n = \frac{1}{n^\beta}$ and $\gamma_n = cn \log n - c(n-1) \log(n-1)$. Then

$$\left| 1 - \frac{a_n}{a_{n+1}} \right| = \left| 1 - \left(\frac{n+1}{n} \right)^\beta \right| \leq \frac{C_\beta}{n} \leq \frac{C_\beta}{n^\beta} = C_\beta a_n. \tag{7}$$

Also, $\gamma_n = cn \log(1 + \frac{1}{n-1}) + c \log(n-1) \rightarrow \infty$ as $n \rightarrow \infty$ and by the mean value theorem, there exists ξ between $n-1$ and $n+1$ so that

$$|\gamma_{n+1} - \gamma_n| = c|(n+1) \log(n+1) + (n-1) \log(n-1) - 2n \log n| \leq \frac{2c}{\xi} \leq \frac{C}{n^\beta} = C a_n.$$

Thus, the Hardy-Littlewood series satisfies the hypotheses of the theorem, and hence $\varepsilon_2(S_n f)(x)$ fails to converge at any x .

Similarly consider the series $\sum_{n=1}^{\infty} \frac{1}{n^\beta} e^{i(nx+n^\alpha)}$ of Theorem 3.3 and suppose $\frac{1}{2} < \beta \leq 1$, $1 < \alpha \leq 2$, $0 < \beta - \frac{\alpha}{2} < 1$ and $\alpha + \beta \leq 2$. Here again $a_n = \frac{1}{n^\beta}$ which satisfies (7). Now $\gamma_n = n^\alpha - (n-1)^\alpha$ and estimating just as with the Hardy-Littlewood series gives $|\gamma_{n+1} - \gamma_n| \leq \frac{C_\alpha}{n^{2-\alpha}} \leq \frac{C_\alpha}{n^\beta} = C_\alpha a_n$. The theorem applies and we conclude that $\varepsilon_2(S_n f)(x)$ fails to converge at any x

Example 2. In a similar way we show that there exists functions f which are analytic on the unit disk, Hölder continuous on the boundary such that the transformed sums $S_n^* f(x)$ diverge at every point. We consider the modification of the Hardy-Littlewood series given by $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^\beta} e^{i\theta_n} e^{inx}$, where $\frac{1}{2} < \beta \leq 1$ and for $n = 1, 2, 3, \dots$, $\theta_n = n \log n$, for n even and $\theta_n = \frac{\theta_{n-1} + \theta_{n+1}}{2}$ for n odd. (Put $\theta_0 = 0$.) By breaking the sum into even and odd terms, we can write f as a sum of two functions, each of which is represented by a series that is essentially the Hardy-Littlewood series itself. We conclude that f is Hölder continuous of order $\beta - \frac{1}{2}$. In this case we have $c_n = a_n e^{i\theta_n}$ with $a_n = \frac{1}{n^\beta}$ and $\gamma_n = \theta_n - \theta_{n-1}$ satisfies $\lim_{n \rightarrow \infty} \gamma_n = \infty$ and $\gamma_{n+2} = \gamma_{n+1}$ if n is even. Then

$$|S_n f(x) - S_{n+1}^* f(x)| = \frac{a_{n+1} \left| 1 - \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+2})} \right|}{\left| 1 - 2 \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+2})} + \frac{a_{n+2}}{a_n} e^{i(2x+\gamma_{n+2}+\gamma_{n+1})} \right|}. \tag{8}$$

We estimate the denominator of this expression when n is even (so $\gamma_{n+1} = \gamma_{n+2}$):

$$\begin{aligned} & \left| 1 - 2 \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+2})} + \frac{a_{n+2}}{a_n} e^{i(2x+\gamma_{n+2}+\gamma_{n+1})} \right| \\ & \leq \left| 1 - \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+1})} - \frac{a_{n+1}}{a_n} e^{i(x+\gamma_{n+1})} + \frac{a_{n+2}}{a_n} e^{2i(x+\gamma_{n+1})} \right| + \left| \frac{a_{n+1}}{a_n} - \frac{a_{n+2}}{a_{n+1}} \right| \\ & = \left| 1 - \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+2})} \right| \left| 1 - \frac{a_{n+1}}{a_n} e^{i(x+\gamma_{n+1})} \right| + \left| \frac{a_{n+1}}{a_n} - \frac{a_{n+2}}{a_{n+1}} \right| = I + II. \end{aligned}$$

Noting that $a_{n+1} = (n+1)^{-\beta}$ and $|1 - \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+2})}| \geq |1 - \frac{a_{n+2}}{a_{n+1}}| = |1 - (\frac{n+1}{n+2})^\beta| \geq \frac{c_\beta}{n+2}$, (where c_β is a constant depending only on β) yields

$$II = \left| \frac{a_{n+1}}{a_n} - \frac{a_{n+2}}{a_{n+1}} \right| = \left| \left(\frac{n}{n+1} \right)^\beta - \left(\frac{n+1}{n+2} \right)^\beta \right| \leq \frac{C}{(n+1)(n+2)} \leq Ca_{n+1} \left| 1 - \frac{a_{n+2}}{a_{n+1}} e^{i(x+\gamma_{n+2})} \right|.$$

To estimate I, we argue as in the proof of Theorem 3.4. We first estimate:

$$\left| 1 - \frac{a_{n+1}}{a_n} e^{i(x+\gamma_{n+1})} \right| \leq \left| 1 - \frac{a_n}{a_{n+1}} \right| + \left| 1 - e^{i(x+\gamma_{n+1})} \right| = I_a + I_b.$$

Then $I_a = |1 - (\frac{n+1}{n})^\beta| \leq \frac{C_\beta}{n} \leq Ca_{n+1}$. To estimate I_b , fix $x \in [-\pi, \pi]$. Since $\gamma_n \rightarrow \infty$, then for each positive integer l there exists a k such that $x + \gamma_k \leq 2l\pi$ but $x + \gamma_{k+1} > 2l\pi$, (here necessarily k is even) and consequently,

$$\begin{aligned} I_b &= |e^{2\pi l} - e^{i(x+\gamma_{k+1})}| \leq |x + \gamma_{k+1} - 2l\pi| \leq |\gamma_{k+1} - \gamma_k| \\ &= |\theta_{k+1} + \theta_{k-1} - 2\theta_k| = \left| \frac{\theta_{k+2} + \theta_k}{2} + \frac{\theta_k + \theta_{k-2}}{2} - 2\theta_k \right| = \left| \frac{\theta_{k+2} + \theta_{k-2}}{2} - \theta_k \right| \\ &= \left| \frac{(k+2)\log(k+2) + (k-2)\log(k-2)}{2} - k \log k \right| \leq \frac{4}{\xi} \leq \frac{C}{k} \leq Ca_{k+1} \end{aligned}$$

where ξ is between $k-2$ and $k+2$ and C is an absolute constant.

Combining the estimates for I_a and I_b yields $|1 - \frac{a_{k+1}}{a_k} e^{i(x+\gamma_{k+1})}| \leq Ca_{k+1}$ for an infinite number of k . Then for these k , $I \leq Ca_{k+1} |1 - \frac{a_{k+2}}{a_{k+1}} e^{i(x+\gamma_{k+2})}|$, which combined with the estimate for II gives an estimate for the denominator in the fraction on the right hand side of (8). Consequently, there exists a constant C such that for each $x \in [-\pi, \pi]$, there are an infinite number of k for which $|S_k f(x) - S_{k+1}^* f(x)| \geq \frac{1}{C}$.

Similarly, fix α and β which satisfy $1 < \alpha \leq 2$, $\frac{1}{2} < \beta \leq 1$, $\alpha + \beta \leq 2$, and $0 < \beta - \frac{\alpha}{2} < 1$, and consider a series of the form $f(x) = \sum_{n=1}^\infty \frac{1}{n^\beta} e^{i\theta_n} e^{inx}$, where for $n = 1, 2, 3, \dots$ we put $\theta_n = n^\alpha$ if n is even and $\theta_n = \frac{\theta_{n+1} + \theta_{n-1}}{2}$ if n is odd. Following the arguments of the above example shows that f is a function which is Hölder continuous of order $\beta - \frac{\alpha}{2}$ and that there exists an absolute constant so that for each $x \in [-\pi, \pi]$, $|S_{k+1}^* f(x) - S_k f(x)| \geq \frac{1}{C}$ for an infinite number of k .

In Figure 1, we consider the series $\sum_{n=1}^\infty \frac{1}{n^{.75}} e^{i(nx+n^{1.2})}$, and graph the first 30 terms of the real part of the series, $S_{30}f$, as well as the real part of $\varepsilon_2(S_{29}f)$. In this case, using the notation of Example 1, we have that $\gamma_{30} = 30^{(1.2)} - 29^{(1.2)} \approx 2.36$ and $\gamma_{31} = 31^{(1.2)} - 30^{(1.2)} \approx 2.38$ so that for x between -2.36 and -2.38 , $x + \gamma_{30} \leq 0$, and $x + \gamma_{31} > 0$, and the difference $|S_{30}f(x) - \varepsilon_2(S_{29}f)(x)|$ is bounded below.

In Figure 2, we consider the series from Example 2 with $\beta = .75$: $f(x) = \sum_{n=1}^\infty \frac{1}{n^{.75}} e^{i\theta_n} e^{inx}$. The graphs represent the real parts of $S_{30}f$ and S_{31}^*f . Here,

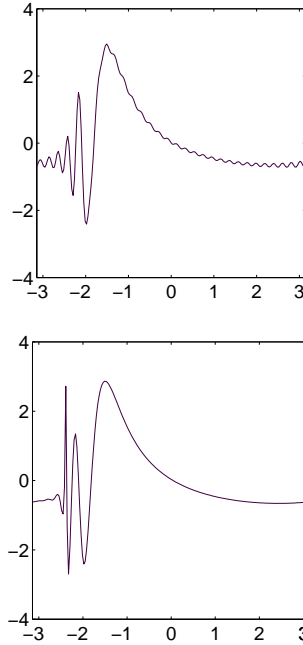


FIGURE 1. Real parts of $S_{30}f$ and $\varepsilon_2(S_{29}f)$ for $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^{75}} e^{i(nx+n^{1.2})}$

using the notation of that example, $\gamma_{30} = 30 \log(30) - 29 \log(29) \approx 4.38$, whereas $\gamma_{31} = \frac{\theta_{32} + \theta_{30}}{2} - \theta_{30} = (32 \log(32) - 30 \log(30))/2 \approx 4.43$. Thus, if $x \in (1.85, 1.90)$, (roughly) $x + \gamma_{30} \leq 2\pi$ and $x + \gamma_{31} > 2\pi$, so as computed in the example, the quantity $|S_{30}f(x) - S_{31}^*f(x)|$ is bounded below.

4. Conclusions

Both the δ^2 process and Lubkin transform are known to accelerate many types of numerical sequences, but little is known about their behavior when applied to Fourier series. The work of Beckermann, Matos, and Wielonsky [2] shows that for a certain type of function, the δ^2 process can be used to accelerate convergence. In this paper, we have shown that for a wide variety of functions, the δ^2 process and Lubkin transform do not destroy the convergence of the partial sums, but we haven't proved they actually accelerate convergence.

These transforms destroy convergence in the case of functions which are smooth with a single jump discontinuity, and, as we have shown here, there are functions which are analytic on the unit disk, whose Fourier series are Hölder continuous functions, for which these methods destroy convergence at every point.

Naturally, it would be interesting to know necessary and sufficient conditions on a function so that either of these methods accelerate the convergence of its

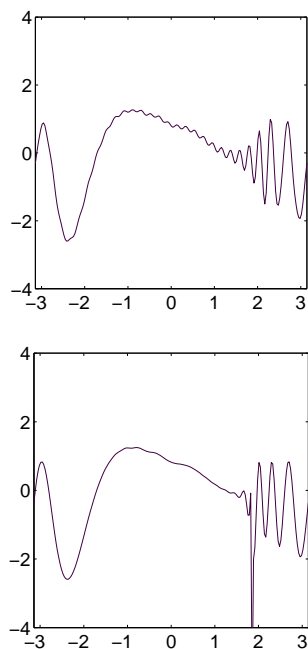


FIGURE 2. Real parts of $S_{30}f$ and S_{31}^*f for $f(x)$ of Example 2, with $\beta = .75$

Fourier series, but given what we have just noted, it is not at all evident what such conditions may be. It would also be interesting to develop a transform which could accelerate the convergence of the Fourier series of functions which need it the most, e.g. functions with jump discontinuities.

REFERENCES

1. E. Abebe, J. Graber, and C. N. Moore, *Fourier series and the δ^2 process*, J. Comput. Appl. Math. **224**, no.1 (2009), 146-151.
2. B. Beckermann, A. Matos, and F. Wielonsky, *Reduction of the Gibbs phenomenon for smooth functions with jumps by the ϵ -algorithm*, J. Comput. Appl. Math. **219** (2008), 329-349.
3. J. Boggess, E. Bunch, and C. N. Moore, *Fourier series and the Lubkin W-transform*, Numer. Algorithms **47** (2008), 133-142.
4. C. Brezinski, *Extrapolation algorithms for filtering series of functions, and treating the Gibbs phenomenon*, Numer. Algorithms **36** (2004), 309-329.
5. C. Brezinski and M. Redivo-Zaglia, *Extrapolation Methods. Theory and Practice*. North-Holland, Amsterdam, 1991.
6. G. H. Hardy and J. E. Littlewood, *Some problems of Diophantine approximation: A remarkable trigonometric series*, Proc. Natl. Acad. Sci. USA **2** (1916), 583-586.

7. S. Lubkin, A method for summing infinite series, J. Res. Natl. Bur. Stand. **48** (1952), 228-254.
8. D. Shanks, Non-linear transformations of divergent and slowly convergent sequences, J. Math. Phys. **34** (1955), 1-42.
9. A. Sidi, *Practical Extrapolation Methods*, Cambridge Monographs on Applied and Computational Mathematics 10, Cambridge University Press, Cambridge, 2003.
10. P. Wynn, *On a Device for Computing the $e_m(S_n)$ Transformation*, Math. Tables Aids Comput. **10** (1956), 91-96.
11. P. Wynn, *Transformations to accelerate the convergence of Fourier series*, in: Gertrude Blanch Anniversary Volume (Wright Patterson Air Force Base, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, 1967), 339-379.
12. A. Zygmund, *Trigonometric Series, Second Edition*, Cambridge University Press, Cambridge, 1959.

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