J. Appl. Math. & Informatics Vol. 31(2013), No. 1 - 2, pp. 247 - 262
Website: http://www.kcam.biz

ALMOST PERIODIC SOLUTION FOR A n-SPECIES COMPETITION MODEL WITH FEEDBACK CONTROLS ON TIME SCALES†

YONGKUN LI* AND XIAOFANG HAN

ABSTRACT. In this paper, using the time scale calculus theory, we first discuss the permanence of a n-species competition system with feedback control on time scales. Based on the permanence result, by the Lyapunov functional method, we establish sufficient conditions for the existence and uniformly asymptotical stability of almost periodic solutions of the considered model. The results of this paper is completely new. An example is employed to show the feasibility of our main result.

AMS Mathematics Subject Classification : 34N05, 34C27, 92D25.
Key words and phrases : Competition system, permanence, almost periodic solution, feedback controls, time scales.

1. Introduction

The traditional Lotka-Volterra competitive system is a rudimentary model on mathematical ecology which can be expressed as follows:

$$x(t) = x_i(t)\left[r_i(t) - \sum_{i=1}^{n} a_{ij}(t)x_j(t)\right],\ i = 1, \ldots, n. \tag{1.1}$$

Many excellent results concerned with the permanence, extinction and global attractivity of periodic solutions or almost periodic solutions of system (1.1) were obtained. In a Lotka-Volterra model, the per capita rate of change of the density of each species is a linear function of densities of the interacting species. However, the growth rate of some competitive species does not correspond with that of the Lotka-Volterra model.

Moreover, as we know, ecosystems in the real world are often disturbed by unpredictable forces which can result in changes in biological parameters such

Received April 13, 2012. Revised June 22, 2012. Accepted July 2, 2012. *Corresponding author. †This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 10971183.

as survival rates. Of particular interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control, we call the disturbance functions as control variables. For more discussion on this direction, we refer to [1-6].

Considering the above Xia et al.[7] studied the following n-species competitive system with feedback controls:

$$\begin{cases} \dot{y}_i(t) = y_i(t)\Big[b_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t)y_j(t) - \sum\limits_{j=1, j\neq i}^{n} c_{ij}(t)y_i(t)y_j(t) - d_i(t)u_i(t)\Big], \\ \dot{u}_i(t) = r_i(t) - e_i(t)u_i(t) + f_i(t)y_i(t), \ i = 1, 2, \ldots, n, \end{cases} \tag{1.2}$$

where $y_i(t)(i = 1, \ldots, n)$ are the density of competitive species, $u_i(t)(i = 1, \ldots, n)$ are the control variables. They obtained some sufficient conditions for the existence of a unique almost periodic solution of system (1.2) by using the comparison theorem and constructing suitable Lyapunov function.

However, many authors have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations[8, 9]. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Therefore, lots have been done on discrete Lotka-Volterra systems, we refer to [10-15].

As a discrete analogues of system (1.2), Liao et al.[15] considered the following discrete n-species competition system with feedback controls

$$\begin{cases} y_i(t+1) = y_i(t)\exp\Big\{b_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t)y_j(t) \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^{n} c_{ij}(t)y_i(t)y_j(t) - d_i(t)u_i(t)\Big\}, \\ \Delta u_i(t) = r_i(t) - e_i(t)u_i(t) + f_i(t)y_i(t), \ i = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

where Δ is the first-order forward difference operator, that is, $\Delta u_i(k) = u_i(k+1) - u_i(k)$, $e_i(\cdot) : \mathbb{Z}^+ \to (0, 1); r_i(\cdot), d_i(\cdot), b_i(\cdot), a_{ij}(\cdot), c_{ij}(\cdot)$ and $f_i(\cdot) : \mathbb{Z}^+ \to \mathbb{R}^+$ are bounded sequences. They indicated that system (1.3) reflected the effect of toxic substances and age structures simultaneously and investigated the permanence and global stability of system (1.3).

In fact, both continuous and discrete systems are very important in implementing and applications. But it is troublesome to study the dynamics behavior for continuous and discrete systems, respectively. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations.

Motivated by the above, in this paper, we are concerned with the following n-species competition system with feedback controls on time scales:

$$\begin{cases} x_i^{\Delta}(t) = b_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t)\exp\{x_j(t)\} \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^{n} c_{ij}(t)\exp\{x_i(t)+x_j(t)\} - d_i(t)u_i(t), \\ u_i^{\Delta}(t) = r_i(t) - e_i(t)u_i(t) + f_i(t)\exp\{x_i(t)\},\ i = 1,2,\ldots,n, \end{cases} \tag{1.4}$$

where $u_i(t)$ is the control variables of species $x_i, b_i(t), a_{ij}(t), c_{ij}(t), d_i(t), e_i(t), f_i(t)$ and $r_i(t)$ are nonnegative almost periodic functions, and $e_i(\cdot) : \mathbb{T} \to (0,1)$, $i, j = 1,2,\ldots,n$.

Remark 1.1. Let $y_i(t) = \exp\{x_i(t)\}, i = 1,2,\ldots,n$. If $\mathbb{T} = \mathbb{R}$, then (1.4) is reduced to (1.2) and If $\mathbb{T} = \mathbb{Z}$, then (1.4) is reduced to (1.3).

Our main purpose of this paper is to discuss the permanence of (1.4) and based on the permanence result, to establish sufficient conditions for the existence and uniformly asymptotical stability of almost periodic solutions of (1.4).

For an almost periodic function $f : \mathbb{T} \to \mathbb{R}$, we denote $f^M = \sup_{t\in\mathbb{T}} f(t), f^m = \inf_{t\in\mathbb{T}} f(t)$. Throughout this paper, we assume that

(H_1) $a_{ii}^m > 0$ and $e_i^m > 0, i, j = 1,2,\ldots,n$.

The organization of the rest of this paper is as follows: In Section 2, we introduce some notations and definitions and state some preliminary results which are needed in later sections. In Section 3, we establish some sufficient conditions for the permanence of (1.4). In Section 4, we establish some sufficient conditions for the existence of a unique almost periodic solution of (1.4). In Section 5, we give an example to illustrate the feasibility of our results obtained in previous sections.

2. Main results

In this section, we introduce some definitions and state some preliminary results.

Definition 2.1 ([16])**.** Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. For any subset $\mathbf{I}$ of $\mathbb{R}$, we denote $\mathbf{I}_{\mathbb{T}} = \mathbf{I} \cap \mathbb{T}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}_+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

Lemma 2.1 ([16])**.** *The following holds:*

(i) $(\nu_1 f + \nu_2 g)^{\Delta} = \nu_1 f^{\Delta} + \nu_2 g^{\Delta}$, *for any constants* ν_1, ν_2;

(ii) $(fg)^{\Delta} = (t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$;

(iii) *if* $f^{\Delta} \geq 0$, *then* f *is nondecreasing.*

Definition 2.2 ([16])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is positively regressive if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$.

Denote $\mathcal{R}^+$ is the set of positively regressive functions from $\mathbb{T}$ to $\mathbb{R}$.

Lemma 2.2 ([16]). *Suppose that* $p \in \mathcal{R}^+$, *then*

(*i*) $e_p(t,s) > 0$, *for all* $t, s \in \mathbb{T}$;

(*ii*) *if* $p(t) \leq q(t)$ *for all* $t \geq s, t, s \in \mathbb{T}$, *then* $e_p(t,s) \leq e_q(t,s)$ *for all* $t \geq s$.

Definition 2.3 ([17]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \Big\{\tau \in \mathbb{R} : t + \tau \in \mathbb{T}, \forall t \in \mathbb{T}\Big\} \neq \{0\}.$$

Throughout this paper, we restrict our discussion on almost periodic time scales.

Definition 2.4 ([17]). Let $\mathbb{T}$ be an almost periodic time scale. A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be almost periodic on $\mathbb{T}$, if for any $\varepsilon > 0$, the set

$$E(\varepsilon, f) = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$, that is, for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau \in E(\varepsilon, f)$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

The set $E(\varepsilon, f)$ is called the ε-translation set of $f(t)$, τ is called the ε-translation number of $f(t)$, and $l(\varepsilon)$ is called the inclusion of $E(\varepsilon, f)$.

Lemma 2.3 ([17]). *If* $f \in C(\mathbb{T}, \mathbb{R})$ *is an almost periodic function, then* f *is bounded on* $\mathbb{T}$.

Lemma 2.4 ([17]). *If* $f, g \in C(\mathbb{T}, \mathbb{R})$ *are almost periodic functions, then* $f+g, fg$ *are also almost periodic.*

Definition 2.5. System (1.4) is said to be permanent if for any solution $(x(t), u(t))$ of (1.4), there exist positive constants m_1, m_2, M_1, M_2 such that

$$m_1 \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M_1,\ m_2 \leq \liminf_{t\to\infty} u(t) \leq \limsup_{t\to\infty} u(t) \leq M_2.$$

Lemma 2.5 ([18]). *Let* $-a \in \mathcal{R}^+$.

(*i*) *If* $x^\Delta(t) \leq b - ax(t)$, *then for* $t > t_0$

$$x(t) \leq x(t_0)e_{(-a)}(t, t_0) + \frac{b}{a}(1 - e_{(-a)}(t, t_0)).$$

In particular, if $a > 0, b > 0$, *we have* $\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}$.

(*ii*) *If* $x^\Delta(t) \geq b - ax(t)$, *then for* $t > t_0$

$$x(t) \geq x(t_0)e_{(-a)}(t, t_0) + \frac{b}{a}(1 - e_{(-a)}(t, t_0)).$$

In particular, if $a > 0, b > 0$, *we have* $\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}$.

3. Permanence of solutions

In this section, we will state and prove the sufficient conditions for the permanence of (1.4).

Theorem 3.1. *Assume that* (H_1) *holds. Then every solution* $(X(t), U(t))$ *of system* (1.4) *satisfies*

$$\lim \sup_{t\to+\infty} x_i(t) \le x_i^*, \ \lim \sup_{t\to+\infty} u_i(t) \le u_i^*,$$

where

$$x_i^* = \frac{b_i^M - a_{ii}^m}{a_{ii}^m}, \quad u_i^* = \frac{r_i^M + f_i^M \exp\{x_i^*\}}{e_i^m}, \ i = 1, 2, \ldots, n.$$

Proof. From the first equation of (1.4), we have

$$\begin{aligned} x_i^{\Delta}(t) &\le b_i(t) - a_{ii}(t)\exp\{x_i(t)\} \\ &\le b_i(t) - a_{ii}(t)(x_i(t) + 1) \\ &\le b_i^M - a_{ii}^m - a_{ii}^m x_i(t), \ i = 1, 2, \ldots, n. \end{aligned}$$

It follows from Lemma 2.5 that $\lim \sup_{t\to+\infty} x_i(t) \le x_i^*, \ i = 1, 2, \ldots, n.$

Now, for any $\epsilon > 0$, there exists a $t_0 \in \mathbb{T}$ such that $x_i(t) \le x_i^* + \epsilon$ for all $t \ge t_0, \ i = 1, 2, \ldots, n$. Then, from the second equation of (1.4), we have

$$\begin{aligned} u_i^{\Delta}(t) &\le r_i(t) + f_i(t)\exp\{x_i^* + \epsilon\} - e_i(t)u_i(t) \\ &\le r_i^M + f_i^M \exp\{x_i^* + \epsilon\} - e_i^m u_i(t), \ i = 1, 2, \ldots, n. \end{aligned}$$

It follows from Lemma 2.5 that $\lim \sup_{t\to+\infty} u_i(t) \le \frac{r_i^M + f_i^M \exp\{x_i^* + \epsilon\}}{e_i^m}, \ i = 1, 2, \ldots, n.$ Letting $\epsilon \to 0$, we get $\lim \sup_{t\to+\infty} u_i(t) \le u_i^*, \ i = 1, 2, \ldots, n$. The proof is complete. □

Theorem 3.2. *Assume that* (H_1) *holds. Assume further that*

(H_2) $b_i^m - d_i^M u_i^* - \sum_{j=1, j\ne i}^{n} a_{ij}^M \exp\{x_j^*\} > 0, \ i = 1, 2, \ldots, n.$

Then every solution $(X(t), U(t)$ *of system* (1.4) *satisfies*

$$\lim \inf_{t\to+\infty} x_i(t) \ge x_{i*}, \ \lim \inf_{t\to+\infty} u_i(t) \ge u_{i*},$$

where

$$x_{i*} = \ln \frac{b_i^m - d_i^M u_i^* - \sum_{j=1, j\ne i}^{n} a_{ij}^M \exp\{x_j^*\}}{a_{ii}^M + \sum_{j=1, j\ne i}^{n} c_{ij}^M \exp\{x_j^*\}}, \ u_{i*} = \frac{r_i^m + f_i^m \exp\{x_{i*}\}}{e_i^M}, \ i = 1, 2, \ldots, n.$$

Proof. For any $\epsilon > 0$, according to Theorem 3.1, there exists a $t_1 \in \mathbb{T}$ such that $x_i(t) \leq x_i^* + \epsilon, u_i(t) \leq u_i^* + \epsilon$ for all $t \geq t_1,\ i = 1, 2, \ldots, n$. Then for $t \geq t_1$, from the first equation of (1.4), we have

$$\begin{aligned} x_i^{\Delta}(t) \geq\ & b_i(t) - d_i(t)(u_i^* + \epsilon) - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \exp\{x_j^* + \epsilon\} \\ & -\left(a_{ii}(t) + \sum_{j=1, j\neq i}^{n} c_{ij}(t) \exp\{x_j^* + \epsilon\}\right) \exp\{x_i(t)\} \\ \geq\ & b_i^m - d_i^M(u_i^* + \epsilon) - \sum_{j=1, j\neq i}^{n} a_{ij}^M \exp\{x_j^* + \epsilon\} \\ & -\left(a_{ii}^M + \sum_{j=1, j\neq i}^{n} c_{ij}^M \exp\{x_j^* + \epsilon\}\right) \exp\{x_i(t)\},\ i = 1, 2, \ldots, n. \end{aligned}$$

We claim that for $t \geq t_1, i = 1, 2, \ldots, n$,

$$\begin{aligned} & b_i^m - d_i^M(u_i^* + \epsilon) - \sum_{j=1, j\neq i}^{n} a_{ij}^M \exp\{x_j^* + \epsilon\} \\ & -\left(a_{ii}^M + \sum_{j=1, j\neq i}^{n} c_{ij}^M \exp\{x_j^* + \epsilon\}\right) \exp\{x_i(t)\} \leq 0. \end{aligned} \tag{3.5}$$

Otherwise, assume that there exists $\tilde{t} \geq t_1, i_0 \in \{1, 2, \ldots, n\}$ such that

$$\begin{aligned} & b_{i_0}^m - d_{i_0}^M(u_{i_0}^* + \epsilon) - \sum_{j=1, j\neq i_0}^{n} a_{i_0 j}^M \exp\{x_j^* + \epsilon\} \\ & -\left(a_{i_0 i_0}^M + \sum_{j=1, j\neq i_0}^{n} c_{i_0 j}^M \exp\{x_j^* + \epsilon\}\right) \exp\{x_{i_0}(\tilde{t})\} > 0 \end{aligned}$$

and for any $t \in [t_1, \tilde{t})_{\mathbb{T}}$,

$$\begin{aligned} & b_{i_0}^m - d_{i_0}^M(u_{i_0}^* + \epsilon) - \sum_{j=1, j\neq i_0}^{n} a_{i_0 j}^M \exp\{x_j^* + \epsilon\} \\ & -\left(a_{i_0 i_0}^M + \sum_{j=1, j\neq i_0}^{n} c_{i_0 j}^M \exp\{x_j^* + \epsilon\}\right) \exp\{x_{i_0}(t)\} \leq 0. \end{aligned}$$

Hence

$$x_{i_0}(\tilde{t}) < \ln \frac{b_{i_0}^m - d_{i_0}^M(u_{i_0}^* + \epsilon) - \sum\limits_{j=1, j\neq i_0}^{n} a_{i_0 j}^M \exp\{x_j^* + \epsilon\}}{a_{i_0 i_0}^M + \sum\limits_{j=1, j\neq i_0}^{n} c_{i_0 j}^M \exp\{x_j^* + \epsilon\}},\ i_0 \in \{1, 2, \ldots, n\}$$

and for any $t \in [t_1, \tilde{t})_{\mathbb{T}}$,

$$x_{i_0}(t) \geq \ln \frac{b_{i_0}^m - d_{i_0}^M (u_{i_0}^* + \epsilon) - \sum\limits_{j=1, j\neq i_0}^{n} a_{i_0 j}^M \exp\{x_j^* + \epsilon\}}{a_{i_0 i_0}^M + \sum\limits_{j=1, j\neq i_0}^{n} c_{i_0 j}^M \exp\{x_j^* + \epsilon\}},\ i_0 \in \{1, 2, \ldots, n\},$$

which imply $x_{i0}^{\Delta}(\tilde{t}) < 0, i_0 \in \{1, 2, \ldots, n\}$. It is a contraction. Therefore, (3.5) holds, for $t \geq t_1$. Consequently, for $t \geq t_1, i = 1, 2, \ldots, n$,

$$x_i(t) \geq \ln \frac{b_i^m - d_i^M (u_i^* + \epsilon) - \sum\limits_{j=1, j\neq i}^{n} a_{ij}^M \exp\{x_j^* + \epsilon\}}{a_{ii}^M + \sum\limits_{j=1, j\neq i}^{n} c_{ij}^M \exp\{x_j^* + \epsilon\}}, \tag{3.6}$$

then

$$\lim \inf_{t\to+\infty} x_i(t) \geq \ln \frac{b_i^m - d_i^M (u_i^* + \epsilon) - \sum\limits_{j=1, j\neq i}^{n} a_{ij}^M \exp\{x_j^* + \epsilon\}}{a_{ii}^M + \sum\limits_{j=1, j\neq i}^{n} c_{ij}^M \exp\{x_j^* + \epsilon\}},\ i = 1, 2, \ldots, n.$$

Letting $\epsilon \to 0$, we get $\lim \inf_{t\to+\infty} x_i(t) \geq x_{i*},\ i = 1, 2, \ldots, n$. Now, for any small enough $\eta > 0$, there exists a $t_2 \in \mathbb{T}$ such that $x_i(t) \geq x_{i*} - \eta$ for all $t \geq t_2,\ i = 1, 2, \ldots, n$. From the second equation of system (1.4), we have

$$\begin{aligned} u_i^{\Delta}(t) &\geq r_i(t) - e_i(t) u_i(t) + f_i(t) \exp\{x_{i*} - \eta\} \\ &\geq r_i^m - e_i^M u_i(t) + f_i^m \exp\{x_{i*} - \eta\},\ i = 1, 2, \ldots, n. \end{aligned}$$

It follows from Lemma 2.5 that

$$\lim \inf_{t\to+\infty} u_i(t) \geq \frac{r_i^m + f_i^m \exp\{x_{i*} - \eta\}}{e_i^M},\ i = 1, 2, \ldots, n.$$

Letting $\eta \to 0$, we get $\lim \inf_{t\to+\infty} u_i(t) \geq u_{i*},\ i = 1, 2, \ldots, n$. The proof is complete. □

From Theorem 3.1 and Theorem 3.2 it follows that

Theorem 3.3. *Assume* (H_1) *and* (H_2) *hold. Then system* (1.4) *is permanent.*

4. Existence of almost periodic solutions

In this section, we will study the existence of almost periodic solutions of (1.4). Consider the following equation

$$x^{\Delta}(t) = f(t, x),\ t \in \mathbb{T}^+, \tag{4.7}$$

where $f : \mathbb{T} \times \mathbb{S}_B \to \mathbb{R}, \mathbb{S}_B = \{x \in \mathbb{R} : ||x||_0 < B\}$, $||x||_0 = \sup_{t\in\mathbb{T}} |x(t)|$, $f(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{S}_B$ and is continuous in x. To find the solution of (4.1), we consider the product system of (4.1) as follows

$$x^{\Delta}(t) = f(t,x),\ y^{\Delta}(t) = f(t,y).$$

Lemma 4.1 ([18])**.** *Suppose that there exists a Lyapunov functional $V(t,x,y)$ defined on $\mathbb{T}^+ \times \mathbb{S}_B \times \mathbb{S}_B$ satisfying the following conditions*

- *(i)* $a(||x-y||_0) \le V(t,x,y) \le b(||x-y||_0)$, *where* $a,b \in \kappa, \kappa = \{a \in C(\mathbb{R}^+,\mathbb{R}^+) : a(0) = 0$ *and* a *is increasing*$\}$.
- *(ii)* $|V(t,x,y) - V(t,x_1,y_1)| \le L(||x-x_1||_0 + ||y-y_1||_0)$, *where* $L > 0$ *is a constant.*
- *(iii)* $D^+V^{\Delta}_{(4.1)}(t,x,y) \le -cV(t,x,y)$, *where* $c > 0, -c \in \mathcal{R}^+$.

Moreover, if there exists a solution $x(t) \in \mathbb{S}$ of (4.1) for $t \in \mathbb{T}^+$, where $\mathbb{S} \subset \mathbb{S}_B$ is a compact set, then there exists a unique almost periodic solution $p(t) \in \mathbb{S}$ of (4.1), which is uniformly asymptotically stable. In particular, if $f(t,x)$ is periodic in t uniformly for $x \in \mathbb{S}_B$, then $p(t)$ is also periodic.

Denote $\Omega = \{(x(t),u(t)) : (x(t),u(t))$ is the solution of (1.4) and $0 < x_* \le x(t) \le x^*, 0 < u_* \le u(t) \le u^*\}$. It is easy to verify that under the conditions of Theorem 3.3, Ω is an invariant set of (1.4).

Lemma 4.2. *Let (H_1) and (H_2) hold. Then $\Omega \neq \phi$.*

Proof. By the almost periodicity of $b_i(t), a_{ij}(t), c_{ij}(t), d_i(t), e_i(t), f_i(t)$ and $r_i(t),\ i,j = 1,2,\dots,n$, there exists a sequence $\tau = \{\tau_p\} \subseteq \mathbb{T}$ with $\tau_p \to +\infty$ as $p \to +\infty$ such that for $i,j = 1,2,\dots,n$,

$$\begin{aligned} &b_i(t+\tau_p) \to b_i(t), a_{ij}(t+\tau_p) \to a_{ij}(t), c_{ij}(t+\tau_p) \to c_{ij}(t), d_i(t+\tau_p) \to d_i(t),\\ &e_i(t+\tau_p) \to e_i(t), f_i(t+\tau_p) \to f_i(t), r_i(t+\tau_p) \to r_i(t), as\ p \to +\infty. \end{aligned}$$

Let ϵ be an arbitrary small positive number. It follows from Theorem 3.1 and Theorem 3.2 that there exists a $t_0 \in \mathbb{T}$ such that

$$x_{i*} - \epsilon \le x_i(t) \le x_i^* + \epsilon,\ u_{i*} - \epsilon \le u_i(t) \le u_i^* + \epsilon,\ \text{for } t \ge t_0,\ i = 1,2,\dots,n.$$

Write $x_{ip}(t) = x_i(t+\tau_p)$ and $u_{ip}(t) = u_i(t+\tau_p)$ for $t \ge t_0 - \tau_p, p = 1,2,\dots$. For any positive integer q, it is easy to see that there exist sequences $\{x_{ip}(t) : p \ge q\}$ and $\{u_{ip}(t) : p \ge q\}$ such that the sequences $\{x_{ip}(t)\}$ and $\{u_{ip}(t)\}$ have subsequences, denoted by $\{x_{ip}(t)\}$ and $\{u_{ip}(t)\}$ again, converging on any finite interval of $\mathbb{T}$ as $p \to +\infty$, respectively. Thus we have sequences $\{y_i(t)\}$ and $\{v_i(t)\}$ such that

$$x_{ip}(t) \to y_i(t),\ u_{ip}(t) \to v_i(t),\ \text{for } t \in \mathbb{T},\ \text{as } p \to +\infty,\ i = 1,2,\dots,n.$$

Combined with

$$\begin{cases} x_{ip}^{\Delta}(t) = b_i(t+\tau p) - \sum\limits_{j=1}^{n} a_{ij}(t+\tau p)\exp\{x_{jp}(t)\} \\ \qquad\qquad - \sum\limits_{j=1,j\neq i}^{n} c_{ij}(t+\tau p)\exp\{x_{ip}(t)+x_{jp}(t)\} - d_i(t+\tau p)u_{ip}(t), \\ u_{ip}^{\Delta}(t) = r_i(t+\tau p) - e_i(t+\tau p)u_{ip}(t) + f_i(t+\tau p)\exp\{x_{ip}(t)\},\ i=1,2,\ldots,n, \end{cases}$$

gives

$$\begin{cases} y_i^{\Delta}(t) = b_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t)\exp\{y_j(t)\} \\ \qquad\qquad - \sum\limits_{j=1,j\neq i}^{n} c_{ij}(t)\exp\{y_i(t)+y_j(t)\} - d_i(t)v_i(t), \\ v_i^{\Delta}(t) = r_i(t) - e_i(t)v_i(t) + f_i(t)\exp\{y_i(t)\},\ i=1,2,\ldots,n. \end{cases}$$

We can easily see that $(Y(t),V(t))$ is a solution of system (1.4) and $x_{i*}-\epsilon \le y_i(t) \le x_i^*+\epsilon,\quad u_{i*}-\epsilon \le v_i(t) \le u_i^*+\epsilon$, for $t\in\mathbb{T},\ i=1,2,\ldots,n$. Since ϵ is an arbitrary small positive number, it follows that $x_{i*} \le y_i(t) \le x_i^*, u_{i*} \le v_i(t) \le u_i^*$ for $t\in\mathbb{T},\ i=1,2,\ldots,n$. □

Theorem 4.3. *Assume that* (H_1) *and* (H_2) *hold. Suppose further that*

(H_3) $\Theta > 0$ *and* $-\Theta \in \mathcal{R}^+$, *where* $\Theta = \min\limits_{1\le i\le n}\{B_i-\mu D_i, E_i-\mu F_i\}$, *for* $i=1,2,\ldots,n$,

$$\begin{aligned} B_i &= 2e_i^m - f_i^M\exp\{x_i^*\} - d_i^{M2}, D_i = e_i^{M2}+3d_i^{M2},\ \mu=\sup_{t\in\mathbb{T}}\{\mu(t)\}, \\ E_i &= 2a_{ii}^m\exp\{x_{i*}\} - f_i^M\exp\{x_i^*\} - 3 \\ &\quad -4\sum_{j=1,j\neq i}^{n}\left(a_{ij}^{M2}+c_{ij}^{M2}\right)\exp\{2x_i^*+2x_j^*\}, \\ F_i &= 6a_{ii}^{M2}\exp\{2x_i^*\} + f_i^{M2}\exp\{2x_i^*\} \\ &\quad +24\sum_{j=1,j\neq i}^{n}\left(a_{ij}^{M2}+c_{ij}^{M2}\right)\exp\{2x_i^*+2x_j^*\}. \end{aligned}$$

Then there exists a unique uniformly asymptotically stable almost periodic solution $(X(t),U(t))$ *of system* (1.4), *and* $(X(t),U(t))\in\Omega$.

Proof. From Lemma 4.2, there exists $(X(t),U(t))$ such that $x_{i*} \le x_i(t) \le x_i^*, u_{i*} \le u_i(t) \le u_i^*$ for $t\in\mathbb{T},\ i=1,2,\ldots,n$. Hence, $|x_i(t)|<A_i, |u_i(t)|<B_i$, where $A_i=\max\{|x_{i*}|,|x_i^*|\}, B_i=\max\{|u_{i*}|,|u_i^*|\},\ i=1,2,\ldots,n$.

Define$\|(X,U)\| = \sup\limits_{t\in\mathbb{T}}\sum\limits_{i=1}^{n}|x_i(t)| + \sup\limits_{t\in\mathbb{T}}\sum\limits_{i=1}^{n}|u_i(t)|,\ (X,U)\in\mathbb{R}^{2n}$. Suppose that $X_1=(X(t),U(t)), X_2=(Y(t),V(t))$ are any two positive solutions of system (1.4), then $\|X_1\|\le C, \|X_2\|\le C$, where $C=\sum\limits_{i=1}^{n}(A_i+B_i)$. In view of system

(1.4), we have

$$\begin{cases} x_i^{\Delta}(t) = b_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t)\exp\{x_j(t)\} \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^{n} c_{ij}(t)\exp\{x_i(t)+x_j(t)\} - d_i(t)u_i(t), \\ u_i^{\Delta}(t) = r_i(t) - e_i(t)u_i(t) + f_i(t)\exp\{x_i(t)\}, \\ y_i^{\Delta}(t) = b_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t)\exp\{y_j(t)\} \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^{n} c_{ij}(t)\exp\{y_i(t)+y_j(t)\} - d_i(t)v_i(t), \\ v_i^{\Delta}(t) = r_i(t) - e_i(t)v_i(t) + f_i(t)\exp\{y_i(t)\},\ i = 1, 2, \ldots, n. \end{cases} \tag{4.8}$$

Consider the Lyapunov function $V(t, X_1, X_2)$ on $\mathbb{T}^+ \times \Omega \times \Omega$ defined by

$$V(t, X_1, X_2) = \sum_{i=1}^{n}(x_i(t) - y_i(t))^2 + \sum_{i=1}^{n}(u_i(t) - v_i(t))^2.$$

It is easy to see that the norm $\|X_1 - X_2\| = \sup\limits_{t\in\mathbb{T}} \sum\limits_{i=1}^{n} |x_i(t) - y_i(t)| + \sup\limits_{t\in\mathbb{T}} \sum\limits_{i=1}^{n} |u_i(t) - v_i(t)|$ and the norm $\|X_1 - X_2\|_* = \sup\limits_{t\in\mathbb{T}} \left[\sum\limits_{i=1}^{n}(x_i(t) - y_i(t))^2 + (u_i(t) - v_i(t))^2\right]^{\frac{1}{2}}$ are equivalent, that is, there exist two constants $C_1 > 0, C_2 > 0$ such that $C_1\|X_1 - X_2\| \le \|X_1 - X_2\|_* \le C_2\|X_1 - X_2\|$. Hence, $(C_1\|X_1 - X_2\|)^2 \le V(t, X_1, X_2) \le (C_2\|X_1 - X_2\|)^2$. Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+), a(x) = C_1^2x^2, b(x) = C_2^2x^2$, so the condition (i) of Lemma 4.1 is satisfied. Besides,

$$\begin{aligned}
&\Big|V(t, X_1, X_2) - V(t, X_1^*, V_2^*)\Big| \\
=\ & \Big|\sum_{i=1}^{n}\left[(x_i(t) - y_i(t))^2 + (u_i(t) - v_i(t))^2\right] - \sum_{i=1}^{n}\left[(x_i^*(t) - y_i^*(t))^2 + (u_i^*(t) - v_i^*(t))^2\right]\Big| \\
\le\ & \Big|\sum_{i=1}^{n}\left[(x_i(t) - y_i(t))^2 - (x_i^*(t) - y_i^*(t))^2\right]\Big| + \Big|\sum_{i=1}^{n}\left[(u_i(t) - v_i(t))^2 - (u_i^*(t) - v_i^*(t))^2\right]\Big| \\
\le\ & \sum_{i=1}^{n}\Big|(x_i(t) - y_i(t)) - (x_i^*(t) - y_i^*(t))\Big|\Big|(x_i(t) - y_i(t)) + (x_i^*(t) - y_i^*(t))\Big| \\
& + \sum_{i=1}^{n}\Big|(u_i(t) - v_i(t)) - (u_i^*(t) - v_i^*(t))\Big|\Big|(u_i(t) - v_i(t)) + (u_i^*(t) - v_i^*(t))\Big| \\
\le\ & \sum_{i=1}^{n}\Big|(x_i(t) - y_i(t)) - (x_i^*(t) - y_i^*(t))\Big|\Big(|x_i(t)| + |y_i(t)| + |x_i^*(t)| + |y_i^*(t)|\Big) \\
& + \sum_{i=1}^{n}\Big|(u_i(t) - v_i(t)) - (u_i^*(t) - v_i^*(t))\Big|\Big(|u_i(t)| + |v_i(t)| + |u_i^*(t)| + |v_i^*(t)|\Big) \\
\le\ & L\sum_{i=1}^{n}\Big\{|x_i(t) - x_i^*(t)| + |u_i(t) - u_i^*(t)| + |y_i(t) - y_i^*(t)| + |v_i(t) - v_i^*(t)|\Big\} \\
\le\ & L\Big(\|X_1 - X_1^*\| + \|X_2 - X_2^*\|\Big),
\end{aligned}$$

where $(X_1^*, X_2^*) = (x_1^*(t), \ldots, x_n^*(t), u_1^*(t), \ldots, u_n^*(t)), L = 4\max\{A_i, B_i : i = 1, 2, \ldots, n\}$. So condition (ii) of Lemma 4.1 is also satisfied. Calculating the right derivative D^+V^Δ of V along the solution of (4.8)

$$\begin{aligned} &D^+V^\Delta(t, X_1, X_2) \\ &= \sum_{i=1}^n [2(x_i(t) - y_i(t)) + \mu(t)(x_i(t) - y_i(t))^\Delta](x_i(t) - y_i(t))^\Delta \\ &\quad + \sum_{i=1}^n [2(u_i(t) - v_i(t)) + \mu(t)(u_i(t) - v_i(t))^\Delta](u_i(t) - v_i(t))^\Delta \\ &= V_1 + V_2, \end{aligned}$$

where

$$V_1 = \sum_{i=1}^n [2(x_i(t) - y_i(t)) + \mu(t)(x_i(t) - y_i(t))^\Delta](x_i(t) - y_i(t))^\Delta,$$

$$V_2 = \sum_{i=1}^n [2(u_i(t) - v_i(t)) + \mu(t)(u_i(t) - v_i(t))^\Delta](u_i(t) - v_i(t))^\Delta.$$

In view of system (4.8), we have for $i = 1, 2, \ldots, n$,

$$\begin{cases} (x_i(t) - y_i(t))^\Delta = -\sum\limits_{j=1}^n a_{ij}(t)(\exp\{x_j(t)\} - \exp\{y_j(t)\}) \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^n c_{ij}(t)(\exp\{x_i(t) + x_j(t)\} \\ \qquad\qquad - \exp\{y_i(t) + y_j(t)\}) - d_i(t)(u_i(t) - v_i(t)), \\ (u_i(t) - v_i(t))^\Delta = -e_i(t)(u_i(t) - v_i(t)) + f_i(t)(\exp\{x_i(t)\} - \exp\{y_i(t)\}). \end{cases} \tag{4.9}$$

Using the mean value theorem we get

$$\exp\{x_j(t)\} - \exp\{y_j(t)\} = \exp\{\xi_j(t)\}(x_j(t) - y_j(t)),$$

$$\exp\{x_i(t) + x_j(t)\} - \exp\{y_i(t) + y_j(t)\} = \exp\{\eta_{ij}(t)\}(x_i(t) + x_j(t) - y_i(t) - y_j(t)),$$

where $\xi_j(t), \eta_{ij}(t)$ lie between $x_j(t)$ and $y_j(t), x_i(t) + x_j(t)$ and $y_i(t) + y_j(t)$, respectively, $i, j = 1, 2, \ldots, n$. Then, (4.9) can be written as for $i, j = 1, 2, \ldots, n$,

$$\begin{cases} (x_i(t) - y_i(t))^\Delta = -\sum\limits_{j=1}^n a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t) - y_j(t))) \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^n c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t) + x_j(t) - y_i(t) - y_j(t))) \\ \qquad\qquad - d_i(t)(u_i(t) - v_i(t)), \\ (u_i(t) - v_i(t))^\Delta = -e_i(t)(u_i(t) - v_i(t)) + f_i(t)(\exp\{\xi_i(t)\}(x_i(t) - y_i(t))). \end{cases}$$

Hence,

$$V_1 = -\sum_{i=1}^n \Bigg\{ 2(x_i(t) - y_i(t)) - \mu(t)\Bigg[\sum_{j=1}^n a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t) - y_j(t)))$$

$$
\begin{aligned}
&+\sum_{j=1,j\neq i}^{n} c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)-y_i(t)-y_j(t)))\\
&+d_i(t)(u_i(t)-v_i(t))\bigg]\Bigg\}\bigg[\sum_{j=1}^{n} a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t)-y_j(t)))\\
&+\sum_{j=1,j\neq i}^{n} c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)-y_i(t)-y_j(t)))\\
&+d_i(t)(u_i(t)-v_i(t))\bigg]\\
=&\sum_{i=1}^{n}\Bigg\{-2(x_i(t)-y_i(t))\sum_{j=1}^{n} a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t)-y_j(t)))\\
&-2(x_i(t)-y_i(t))d_i(t)(u_i(t)-v_i(t))\\
&-2(x_i(t)-y_i(t))\sum_{j=1,j\neq i}^{n} c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)-y_i(t)\\
&-y_j(t)))+\mu(t)\bigg[\sum_{j=1}^{n} a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t)-y_j(t)))\bigg]^2\\
&+\mu(t)\bigg[\sum_{j=1,j\neq i}^{n} c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)-y_i(t)-y_j(t)))\bigg]^2\\
&+\mu(t)d_i^2(t)(u_i(t)-v_i(t))^2+2\mu(t)\sum_{j=1,j\neq i}^{n} c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)\\
&-y_i(t)-y_j(t)))\sum_{j=1}^{n} a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t)-y_j(t)))\\
&+2\mu(t)\sum_{j=1,j\neq i}^{n} c_{ij}(t)(\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)-y_i(t)-y_j(t)))d_i(t)(u_i(t)\\
&-v_i(t))+2\mu(t)\sum_{j=1}^{n} a_{ij}(t)(\exp\{\xi_j(t)\}(x_j(t)-y_j(t)))d_i(t)(u_i(t)-v_i(t))\Bigg\}\\
\leq&\sum_{i=1}^{n}\Bigg\{3(x_i(t)-y_i(t))^2+(1+6\mu(t))\\
&\times\bigg[\sum_{j=1,j\neq i}^{n} a_{ij}(t)\exp\{\xi_j(t)\}(x_j(t)-y_j(t))\bigg]^2\\
&+(1+3\mu(t))\times\bigg[\sum_{j=1,j\neq i}^{n} c_{ij}(t)\exp\{\eta_{ij}(t)\}(x_i(t)+x_j(t)-y_i(t)-y_j(t))\bigg]^2
\end{aligned}
$$

$$
\begin{aligned}
&+(1+3\mu(t))d_i^2(t)(u_i(t)-v_i(t))^2-2a_{ii}(t)\exp\{\xi_i(t)\}(x_i(t)-y_i(t))^2\\
&+6\mu(t)a_{ii}^2(t)\exp\{2\xi_i(t)\}(x_i(t)-y_i(t))^2\Bigg\}\\
\leq&\sum_{i=1}^n\Bigg\{\bigg[3+6\mu(t)a_{ii}^{M2}\exp\{2x_i^*\}-2a_{ii}^m\exp\{x_{i*}\}\bigg](x_i(t)-y_i(t))^2\\
&+(1+3\mu(t))d_i^{M2}(u_i(t)-v_i(t))^2+2(1+6\mu(t))\\
&\times\sum_{j=1,j\neq i}^n a_{ij}^{M2}\exp\{2x_j^*\}(x_j(t)-y_j(t))^2+(2+6\mu(t))\\
&\times\bigg(\sum_{j=1,j\neq i}^n c_{ij}^{M2}\exp\{2x_i^*+2x_j^*\}(x_i(t)-y_i(t))^2\\
&+\sum_{j=1,j\neq i}^n c_{ij}^{M2}\exp\{2x_i^*+2x_j^*\}(x_j(t)-y_j(t))^2\bigg)\Bigg\}\\
\leq&\sum_{i=1}^n\Bigg\{\bigg[3+6\mu a_{ii}^{M2}\exp\{2x_i^*\}-2a_{ii}^m\exp\{x_{i*}\}\\
&+(2+6\mu)\sum_{j=1,j\neq i}^n c_{ij}^{M2}\exp\{2x_i^*+2x_j^*\}\bigg](x_i(t)-y_i(t))^2\\
&+(1+3\mu)d_i^{M2}(u_i(t)-v_i(t))^2+(2+12\mu)\sum_{j=1,j\neq i}^n(a_{ij}^{M2}+c_{ij}^{M2})\\
&\times\exp\{2x_i^*+2x_j^*\}(x_j(t)-y_j(t))^2\Bigg\},
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
V_2=&\sum_{i=1}^n[2(u_i(t)-v_i(t))+\mu(t)(u_i(t)-v_i(t))^\Delta](u_i(t)-v_i(t))^\Delta\\
=&\sum_{i=1}^n\bigg\{2(u_i(t)-v_i(t))+\mu(t)[-e_i(t)(u_i(t)-v_i(t))\\
&+f_i(t)(\exp\{\xi_i(t)\}(x_i(t)-y_i(t)))]\bigg\}[-e_i(t)(u_i(t)-v_i(t))\\
&+f_i(t)(\exp\{\xi_i(t)\}(x_i(t)-y_i(t)))]\\
=&\sum_{i=1}^n\bigg\{-2e_i(t)(u_i(t)-v_i(t))^2\\
&+2f_i(t)(\exp\{\xi_i(t)\}(x_i(t)-y_i(t)))(u_i(t)-v_i(t))\\
&+\mu(t)e_i^2(t)(u_i(t)-v_i(t))^2+\mu(t)f_i^2(t)\exp\{2\xi_i(t)\}(x_i(t)-y_i(t))^2
\end{aligned}
$$

$$
\begin{aligned}
&-2\mu(t)e_i(t)f_i(t)\exp\{\xi_i(t)\}(u_i(t)-v_i(t))(x_i(t)-y_i(t))\Bigg\}\\
\le&\sum_{i=1}^{n}\Bigg\{(\mu(t)e_i^2(t)-2e_i(t))(u_i(t)-v_i(t))^2\\
&+\mu(t)f_i^2(t)\exp\{2\xi_i(t)\}(x_i(t)-y_i(t))^2\\
&+f_i(t)\exp\{\xi_i(t)\}[(u_i(t)-v_i(t))^2+(x_i(t)-y_i(t))^2]\Bigg\}\\
\le&\sum_{i=1}^{n}\Bigg\{\Big[\mu e_i^{M2}-2e_i^m+f_i^M\exp\{x_i^*\}\Big](u_i(t)-v_i(t))^2\\
&+\Big[\mu f_i^{M2}\exp\{2x_i^*\}+f_i^M\exp\{x_i^*\}\Big](x_i(t)-y_i(t))^2\Bigg\}. \qquad (4.11)
\end{aligned}
$$

In view of (4.4) and (4.5), we obtain

$$
\begin{aligned}
&D^+V^{\Delta}(t,X_1,X_2)\\
=&\ V_1+V_2\\
\le&\sum_{i=1}^{n}\Big[\mu e_i^{M2}-2e_i^m+f_i^M\exp\{x_i^*\}+(1+3\mu)d_i^{M2}\Big](u_i(t)-v_i(t))^2\\
&+\sum_{i=1}^{n}\Big[\mu f_i^{M2}\exp\{2x_i^*\}+f_i^M\exp\{x_i^*\}+3+6\mu a_{ii}^{M2}\exp\{2x_i^*\}\\
&-2a_{ii}^m\exp\{x_{i*}\}+(4+24\mu)\\
&\times\sum_{j=1,j\neq i}^{n}(a_{ij}^{M2}+c_{ij}^{M2})\exp\{2x_i^*+2x_j^*\}\Big](x_i(t)-y_i(t))^2\\
\le&-\sum_{i=1}^{n}\Big[2e_i^m-f_i^M\exp\{x_i^*\}-d_i^{M2}-\mu(e_i^{M2}+3d_i^{M2})\Big](u_i(t)-v_i(t))^2\\
&-\sum_{i=1}^{n}\Big[2a_{ii}^m\exp\{x_{i*}\}-f_i^M\exp\{x_i^*\}-3\\
&-4\sum_{j=1,j\neq i}^{n}(a_{ij}^{M2}+c_{ij}^{M2})\exp\{2x_i^*+2x_j^*\}-\mu(f_i^{M2}\exp\{2x_i^*\}+6a_{ii}^{M2}\exp\{2x_i^*\}\\
&+24\sum_{j=1,j\neq i}^{n}a_{ij}^{M2}+c_{ij}^{M2}\exp\{2x_i^*+2x_j^*\})\Big]\times(x_i(t)-y_i(t))^2\\
=&-\sum_{i=1}^{n}(B_i-\mu D_i)(u_i(t)-v_i(t))^2-\sum_{i=1}^{n}(E_i-\mu F_i)(x_i(t)-y_i(t))^2\\
\le&-\Theta V(t,X_1,X_2).
\end{aligned}
$$

By the condition (H_3), we see that condition (iii) of Lemma 4.1 holds. Hence, according to Lemma 4.1, there exists a unique uniformly asymptotically stable almost periodic solution $(X(t),U(t))$ of system (1.4), and $(X(t),U(t))\in\Omega$. □

5. An Example

Example 5.1. Consider the following system:

$$\begin{cases} x_1^{\Delta}(t) = 0.035 + 0.017\sin 2t - 0.01\exp\{x_1(t)\} - 0.000002\exp\{x_2(t)\} \\ \qquad -0.000002\exp\{x_1(t)+x_2(t)\} - 0.001(1+\sin\sqrt{2}t)u_1(t), \\ x_2^{\Delta}(t) = 0.035 + 0.017\cos 2t - 0.01\exp\{x_2(t)\} - 0.000002\exp\{x_1(t)\} \\ \qquad -0.000002\exp\{x_1(t)+x_2(t)\} - 0.001(1+\cos\sqrt{2}t)u_2(t), \\ u_1^{\Delta}(t) = 0.007 + 0.003\sin 2t - 0.2u_1(t) + 0.00125(1+\cos t)\exp\{x_1(t)\}, \\ u_2^{\Delta}(t) = 0.007 + 0.003\cos 2t - 0.2u_2(t) + 0.00125(1+\sin t)\exp\{x_2(t)\}. \end{cases} \tag{5.12}$$

One can calculate that

$$\begin{aligned} &x_1^* = x_2^* = 4.2, u_1^* = u_2^* = 8.3860, x_{1*} = x_{2*} = 2.7, \\ &u_{1*} = u_{2*} = 0.02, B_1 = B_2 = 0.2333, \\ &D_1 = D_2 \approx 0.040012, E_1 = E_2 \approx 0.7329, F_1 = F_2 \approx 0.2779. \end{aligned}$$

It is easy to see that, for $\mu \in [0,1]$, system (5.12) satisfies the condition of Theorem 4.1. Therefore, there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(t), x_2(t), u_1(t), u_2(t))$ of system (5.12).

References

1. J.D. Murray, Mathematical biology I: An introduction, Springer-Verlag, Berlin, 2002.
2. R.H. Tan, Z.J. Liu and R.A. Cheke, *Periodicity and stability in a single-species model governed by impulsive differential equation*, Appl. Math. Modelling, **36** (2012) 1085-1094.
3. Y.K. Li., P. Liu and L.F. Zhu, *Positive periodic solutions of a class of functional differential systems with feedback controls*, Nonlinear Anal. **57** (2004) 655-666.
4. P. Weng, *Existence and global stability of positive periodic solution of periodic integro-differential systems with feedback controls*, Comput. Math. Appl. **40** (2000) 747-759.
5. Y.K. Li and T.W. Zhang, *Permanence of a discrete n-species cooperation system with time-varying delays and feedback controls*, Math. Comput. Modelling **53** (2011) 1320-1330.
6. F.D. Chen, F.X. Lin and X.X. Chen, *Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models with feedback control*, Appl. Math. Comput. **158** (2004) 45-68.
7. Y. Xia, J. Cao, H. Zhang and F. Chen, *Almost periodic solutions of n-species competitive system with feedback controls*, J. Math. Anal. Appl. **294** (2004) 503-522.
8. R.P. Agarwal, Difference Equations and Inequalities: Theory Methods and Applications Monographs and Textbooks in Pure and Applied Mathematics, 228, Marcel Dekker, New York, 2000.
9. J.D. Murray, Mathematical Biology, Springer-Verlag, New York, 1989.
10. X.Y. Liao and S.S. Cheng, *Convergent and divergent solutions of a discrete nonautonomous Lotka-Volterra model*, Tamkang J. Math. **36** (2005) 337-344.
11. X.Y. Liao, W.T. Li and Y.N. Raffoul, *Boundedness in nonlinear functional difference equation via non-negative definite Lyapunov functionals with applications to Volterra discrete systems*, Nonlinear Stud. **13** (2006) 1-13.
12. X.Y. Liao, S.F. Zhou, Z. Ouyang, On a stoichiometric two predators on one prey discrete model, Appl. Math. Lett. 20 (2007) 272-278.
13. Y.K. Li and T.W. Zhang, *Permanence and almost periodic sequence solution for a discrete delay logistic equation with feedback control*, Nonlinear Anal. Real World Appl. **12** (2011) 1850-1864.

14. Y.K. Li, T.W. Zhang and Y. Ye, *On the existence and stability of a unique almost periodic sequence solution in discrete predator-prey models with time delays*, Appl. Math. Model. **35** (2011) 5448-5459.
15. X. Liao, S. Zhou and Y. Chen, *Permanence and global stability in a discrete n-species competition system with feedback controls*, Nonlinear Anal. Real World Appl. **9** (2008) 1661-1671.
16. M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Boston: Birkhäuser; 2001.
17. Y.K. Li and C. Wang, *Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales*, Abs. Appl. Anal. **2011** (2011) [Article ID 341520, 22 pp.]
18. Y.K. Li, L. Yang and H.T. Zhang, *Permanence and uniformly asymptotical stability of almost periodic solutions for a single-species model with feedback control on time scales*, Asian-European Journal of Mathematics, in press.

Yongkun Li received M.Sc. from Yunnan University and Ph.D at Sichuan University. Since 1990 he has been at Yunnan University. His research interests include nonlinear differential equations and biological mathematics.

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China.
e-mail: yklie@ynu.edu.cn

Xiaofang Han is currently a graduate student at Yunnan University since 2010. Her research interests are nonlinear differential equations and biological mathematics.

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China.
e-mail: hanxiaofang_1987@126.com