

**ANALYTICAL SOLUTION OF SINGULAR FOURTH ORDER
PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS OF
VARIABLE COEFFICIENTS BY USING HOMOTOPY
PERTURBATION TRANSFORM METHOD**

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ABSTRACT. In this paper, we apply Homotopy perturbation transform method (HPTM) for solving singular fourth order parabolic partial differential equations with variable coefficients. This method is the combination of the Laplace transform method and Homotopy perturbation method. The nonlinear terms can be easily handled by the use of He's polynomials. The aim of using the Laplace transform is to overcome the deficiency that is mainly caused by unsatisfied conditions in other semi-analytical methods such as Homotopy perturbation method (HPM), Variational iteration method (VIM) and Adomain Decomposition method (ADM). The proposed scheme finds the solutions without any discretization or restrictive assumptions and avoids the round-off errors. The comparison shows a precise agreement between the results and introduces this method as an applicable one which it needs fewer computations and is much easier and more convenient than others, so it can be widely used in engineering too.

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1. Introduction

Analytical methods have made a comeback in research methodology after taking a backseat to the numerical techniques for the latter half of the preceding century. The advantages of analytical methods are manifolds, the main being that they give a much better insight than the numbers crunched by a computer using a purely numerical algorithm. Many such physical phenomena are modeled

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in terms of partial differential equations. For example, the parabolic equations with variable coefficients which are of the form:

$$\frac{\partial^2 u}{\partial t^2} + \mu(x, y, z) \frac{\partial^4 u}{\partial x^4} + \lambda(x, y, z) \frac{\partial^4 u}{\partial y^4} + \eta(x, y, z) \frac{\partial^4 u}{\partial z^4} = g(x, y, z, t), \quad a < x, y, z < b, \quad t > 0$$

where $\mu(x, y, z)$, $\lambda(x, y, z)$ and $\eta(x, y, z)$ are positive. subject to the following initial conditions

$$\begin{aligned} u(x, y, z, 0) &= f_0(x, y, z), \quad \frac{\partial u}{\partial t}(x, y, z, 0) = f_1(x, y, z) \\ u(a, y, z, t) &= g_0(y, z, t), \quad u(b, y, z, t) = g_1(y, z, t), \\ u(x, a, z, t) &= k_0(x, z, t), \quad u(x, b, z, t) = k_1(x, z, t), \\ u(x, y, a, t) &= h_0(y, z, t), \quad u(x, y, b, t) = h_1(y, z, t), \\ \frac{\partial^2 u}{\partial x^2}(a, y, z, t) &= \bar{g}_0(y, z, t), \quad \frac{\partial^2 u}{\partial x^2}(b, y, z, t) = \bar{g}_1(y, z, t) \\ \frac{\partial^2 u}{\partial y^2}(x, a, z, t) &= \bar{k}_0(x, z, t), \quad \frac{\partial^2 u}{\partial y^2}(x, b, z, t) = \bar{k}_1(x, z, t) \\ \frac{\partial^2 u}{\partial z^2}(x, y, a, t) &= \bar{h}_0(x, y, t), \quad \frac{\partial^2 u}{\partial z^2}(x, y, b, t) = \bar{h}_1(x, y, t) \end{aligned}$$

where $k_i, g_i, h_i, \bar{k}_i, \bar{g}_i, \bar{h}_i, i = 0, 1$ are continuous, $\mu(x, y, z) > 0$ is the ratio of flexural rigidity [21-22] of the beam to its mass per unit length, arise in various fields of physics, engineering and applied sciences. The importance of obtaining the exact or approximate solutions of linear and nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions. Most new nonlinear equations do not have a precise analytic solution; so numerical methods have largely been used to handle these equations. In recent years, many authors have paid attention to studying the solutions of nonlinear partial differential equations by various methods. Among these are Adomian Decomposition method [1-4], the tanh-method [5], the sine-cosine method [6-7], the differential transform method [8-9], the Variational iteration method [10-15] and the Laplace decomposition method [16-20]. Singular fourth-order parabolic partial differential equations govern the transverse vibrations of a homogeneous beam. Such types of equations arise in mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory. The studies of such problems have attracted the attention of many mathematicians and physicists [21-32]. Several techniques including the Adomian Decomposition method, the variational iteration method and Laplace decomposition method can be used to solve the nonhomogeneous variable coefficients partial differential equations with accurate approximation, but this approximation is acceptable only for a small range, because, boundary conditions in one dimension are satisfied via these methods. Therefore, unsatisfied conditions play no role in the final results. Consequently, this shows that most of these semi-analytical methods encounter the inherent deficiencies like the calculation of Adomian's polynomials, involve huge

computational work and divergent results. One of the analytical methods of recent vintage, namely the homotopy perturbation method (HPM), first proposed by He [33-42] by combining the standard homotopy and classical perturbation technique for solving various linear, nonlinear initial and boundary value problems [43-53]. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal these such nonlinearities such as the Adomian decomposition method [54], Homotopy perturbation method [55] and Homotopy perturbation method with Variational iteration method [56] to produce highly effective techniques for solving many nonlinear problems.

The basic motivation of this paper is to propose a new modification of HPM to overcome the deficiency. The suggested HPTM provides the solution in a rapid convergent series which may leads the solution in closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solution for nonlinear equations. The use of He's polynomials in nonlinear terms first proposed by Ghorbani [57-58]. The HPTM method has been successfully introduced by Y. Khan and Q. Wu [59] to homogeneous and nonhomogeneous advection equations. It is worth mentioning that the HPTM is applied without any discretization or restrictive assumptions or transformations and free from round-off errors. Unlike the method of separation of variables that require initial or boundary conditions, The HPTM provides an analytical solution by using the initial conditions only. The boundary conditions can be used only to justify the obtained results. The proposed method work efficiently and the results so far are very encouraging and reliable. We would like to emphasize that the HPTM may be considered as an important and significant refinement of the previously developed techniques and can be viewed as an alternative to the recently developed methods such as Adomian's decomposition method, Variational iteration method and Homotopy perturbation method. Several examples are given to verify the reliability and efficiency of the homotopy perturbation transform method. In this paper we have considered the effectiveness of the homotopy perturbation transform method (HPTM) for solving fourth order parabolic partial differential equations with variable coefficients.

2. Homotopy perturbation transform method

This method was introduced by Y. Khan and Q. Wu [59] by combining the Homotopy Perturbation Method and Laplace Transform Method for solving various types of linear and nonlinear systems of partial differential equations. To illustrate the basic idea of HPTM, we consider a general nonlinear partial differential equation with the initial conditions of the form [59].

$$Du(x, t) + Ru(x, t) + Nu(x, t) = g(x, t) \quad (1)$$

where D is the second order linear differential operator $D = \partial^2/\partial t^2$, R is the linear differential operator of less order than D ; N represents the general

nonlinear differential operator and $g(x, t)$ is the source term. Taking the Laplace transform (denoted in this paper by L) on both sides of Eq. (1):

$$L[Du(x, t)] + L[Ru(x, t)] + L[Nu(x, t)] = L[g(x, t)] \quad (2)$$

Using the differentiation property of the Laplace transform, we have

$$L[u(x, t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{L[Ru(x, t)]}{s^2} + \frac{L[g(x, t)]}{s^2} - \frac{L[Nu(x, t)]}{s^2} \quad (3)$$

Operating with the Laplace inverse on both sides of Eq. (3) gives

$$u(x, t) = G(x, t) - L^{-1} \left[\frac{L[Ru(x, t)] + L[Nu(x, t)]}{s^2} \right] \quad (4)$$

where $G(x, t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method.

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \quad (5)$$

and the nonlinear term can be decomposed as

$$Nu(x, t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (6)$$

for some He's polynomials $H_n(u)$ (see [57-58]) that are given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots \quad (7)$$

Substituting Eq.(5), Eq.(6) and Eq.(7) into Eq.(4), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p \left(L^{-1} \left[\frac{1}{s^2} L \left[R \sum_{n=0}^{\infty} p^n u_n(x, t) + N \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right) \quad (8)$$

which is the coupling of the Laplace transform method and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p , the following approximations are obtained.

$$\begin{aligned} p^0 : u_0(x, t) &= G(x, t), \\ p^1 : u_1(x, t) &= -\frac{1}{s^2} L [Ru_0(x, t) + H_0(u)], \\ p^2 : u_2(x, t) &= -\frac{1}{s^2} L [Ru_1(x, t) + H_1(u)], \\ p^3 : u_3(x, t) &= -\frac{1}{s^2} L [Ru_2(x, t) + H_2(u)], \\ &\vdots \end{aligned} \quad (9)$$

and so on.

3. Application

In this section we will present some examples to access the efficiency of the Homotopy perturbation transform method (HPTM).

Example 3.1. First we consider the following parabolic partial differential equation [21-32].

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right) \frac{\partial^4 u}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, t > 0 \tag{10}$$

with the initial conditions:

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 1 + \frac{x^5}{120}$$

and the boundary conditions:

$$u\left(\frac{1}{2}, t\right) = \left(1 + \frac{(0.5)^5}{120}\right) \sin(t), \quad u(1, t) = \left(1 + \frac{121}{120}\right) \sin(t)$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, t\right) = \frac{1}{6} \left(\frac{1}{2}\right)^3 \sin(t), \quad \frac{\partial^2 u}{\partial x^2}(1, t) = \frac{1}{6} \sin(t)$$

Taking Laplace Transform both of sides of Eq. (10) subject to the initial conditions, we have

$$L[u(x, t)] = \frac{1}{s^2} \left(1 + \frac{x^5}{120}\right) - \frac{1}{s^2} L \left[\left(\frac{1}{x} + \frac{x^4}{120}\right) u_{xxxx} \right] \tag{11}$$

Taking Inverse Laplace transform of Eq.(11), we have

$$[u(x, t)] = \left(1 + \frac{x^5}{120}\right) - L^{-1} \left[\frac{1}{s^2} L \left[\left(\frac{1}{x} + \frac{x^4}{120}\right) u_{xxxx} \right] \right] \tag{12}$$

By homotopy perturbation method, we have

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \tag{13}$$

Substituting Eq.(13) into Eq.(12), we have

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = \left(1 + \frac{x^5}{120}\right) - p.L^{-1} \left[\frac{1}{s^2} L \left[\left(\frac{1}{x} + \frac{x^4}{120}\right) \left(\sum_{n=0}^{\infty} p^n u_n(x, t)\right)_{xxxx} \right] \right] \tag{14}$$

Comparing the coefficient of like powers of p

$$p^0 : u_0(x, t) = \left(1 + \frac{x^5}{120}\right),$$

$$p^1 : u_1(x, t) = \left(1 + \frac{x^5}{120}\right) \left(\frac{-t^3}{6}\right),$$

$$p^2 : u_2(x, t) = \left(1 + \frac{x^5}{120}\right) \left(\frac{t^5}{120}\right),$$

$$p^3 : u_3(x, t) = \left(1 + \frac{x^5}{120}\right) \left(\frac{-t^7}{5040}\right),$$

$$\vdots$$

and so on. Therefore the approximate solution is given by

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

$$u(x, t) = \left(1 + \frac{x^5}{120}\right) \sin(t) \quad (15)$$

which is the exact solution. The results of the above example shows that our method is capable of reducing the huge computational work and generates the modification of homotopy perturbation method in the convergence rate and is same as obtained by the implicit and explicit methods [21, 23-27], the Variational iteration method [28, 29], the Adomian decomposition method [31] and Homotopy perturbation method [32].

Example 3.2. We now consider the following fourth-order parabolic equation [21-32].

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{x}{\sin(x)} - 1\right) \frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < x < 1, t > 0 \quad (16)$$

with the initial conditions:

$$u(x, 0) = x - \sin(x), \quad \frac{\partial u}{\partial t}(x, 0) = (x - \sin(x))$$

and the boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = e^{-t}(1 - \sin(1)),$$

$$\frac{\partial^2 u}{\partial x^2}(0, t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1, t) = e^{-t} \sin(1)$$

By applying aforesaid method, we have

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = (x - \sin(x))(1-t) - pL^{-1} \left[\left(\frac{x}{\sin(x)} - 1\right) \frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right]_{xxxx} \quad (17)$$

Comparing the coefficients of various powers of p , we have

$$p^0 : u_0(x, t) = (x - \sin(x))(1 - t),$$

$$p^1 : u_1(x, t) = (x - \sin(x)) \left(\frac{t^2}{2!} - \frac{t^3}{3!}\right),$$

$$p^2 : u_2(x, t) = (x - \sin(x)) \left(\frac{t^4}{4!} - \frac{t^5}{5!}\right),$$

$$p^3 : u_3(x, t) = (x - \sin(x)) \left(\frac{t^6}{6!} - \frac{t^7}{7!}\right),$$

$$\vdots$$

Therefore the approximate solution is given by

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

$$u(x, t) = (x - \sin(x)) e^{-t} \tag{18}$$

This is the exact solution and is same as obtained by the implicit and explicit methods [21, 23-27], the Variational iteration method [28, 29], the Adomain decomposition method [31] and Homotopy perturbation method [32].

Example 3.3. Let us consider the following fourth-order parabolic partial differential equation [21-32].

$$\frac{\partial^2 u}{\partial t^2} + (1+x) \frac{\partial^4 u}{\partial x^4} = \left(x^4 + x^3 - \frac{6 \cdot x^7}{7!} \right) \cos(t) \tag{19}$$

with the initial conditions:

$$u(x, 0) = \frac{6 \cdot x^7}{7!}, \quad u_t(x, 0) = 0$$

and the boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = \frac{6 \cdot \cos(t)}{7!}$$

$$\frac{\partial^2 u}{\partial x^2}(0, t) = 0, \quad \frac{\partial^2 u}{\partial t^2}(1, t) = \frac{\cos(t)}{20}$$

By applying aforesaid method, we have

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = \frac{6 \cdot x^7}{7!} + \left(x^4 + x^3 - \frac{6 \cdot x^7}{7!} \right) (1 - \cos(t)) - p L^{-1} \left[\frac{1}{s^2} (1+x) L \left(\sum_{n=0}^{\infty} p^n u_n(x, t)_{xxxx} \right) \right] \tag{20}$$

Comparing the coefficients of various powers of p , we have

$$p^0 : u_0(x, t) = \frac{6 \cdot x^7 \cos(t)}{7!} + (x^4 + x^3)(1 - \cos(t)),$$

$$p^1 : u_1(x, t) = -(x^4 + x^3)(1 - \cos(t)) - 12(1+x)t^2 + 24(1 - \cos(t))(1+x)$$

$$p^2 : u_2(x, t) = 12(1+x)t^2 - 24(1 - \cos(t))(1+x)$$

$$p^3 : u_3(x, t) = 0$$

$$\vdots$$

The noise term appearing between the components are cancelled out and remaining terms will satisfies the equations Therefore the approximate solution is given by

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

$$u(x, t) = \frac{6 \cdot x^7 \cos(t)}{7!} \tag{21}$$

The solution is same as obtained by the implicit and explicit methods [21, 23-27], the Variational iteration method [28, 29], the Adomain decomposition method [31] and Homotopy perturbation method [32].

Example 3.4. Consider the following fourth-order parabolic differential equation [21-32].

$$\frac{\partial^2 u}{\partial t^2} + 2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} + 2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} = 0, \quad \frac{1}{2} < x < 1, t > 0 \quad (22)$$

with the initial conditions:

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}$$

and the boundary conditions:

$$\begin{aligned} u\left(\frac{1}{2}, y, t\right) &= \left(2 + \frac{(0.5)^6}{6!} + \frac{y^6}{6!}\right) \sin(t), \quad u(1, y, t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right) \sin(t) \\ \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, y, t\right) &= \frac{(0.5)^4}{24} \sin(t), \quad \frac{\partial^2 u}{\partial x^2}(1, y, t) = \frac{1}{24} \sin(t) \\ \frac{\partial^2 u}{\partial y^2}\left(x, \frac{1}{2}, t\right) &= \frac{(0.5)^4}{24} \sin(t), \quad \frac{\partial^2 u}{\partial y^2}(x, 1, t) = \frac{1}{24} \sin(t) \end{aligned}$$

By applying aforesaid method, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) t - pL^{-1} \left[\frac{1}{s^2} \left(2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) L \left(\sum_{n=0}^{\infty} p^n u_n(x, y, t) \right)_{xxxx} \right) \right] \\ &\quad - pL^{-1} \left[\frac{1}{s^2} \left(2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) L \left(\sum_{n=0}^{\infty} p^n u_n(x, y, t) \right)_{yyyy} \right) \right] \end{aligned}$$

Comparing the coefficients of various powers of p , we have

$$\begin{aligned} p^0 : u_0(x, y, t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) t, \\ p^1 : u_1(x, y, t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \left(-\frac{t^3}{3!}\right) \\ p^2 : u_2(x, y, t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \left(\frac{t^5}{5!}\right) \\ p^3 : u_3(x, y, t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \left(-\frac{t^7}{7!}\right) \\ &\quad \vdots \end{aligned}$$

Therefore the approximate solution is given by

$$u(x, y, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \sin(t) \quad (23)$$

The solution is same as obtained by the implicit and explicit methods [21, 23-27], the Variational iteration method [28, 29], the Adomian decomposition method [31] and Homotopy perturbation method [32].

Example 3.5. Consider the following fourth-order parabolic equation [21-32].

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{y+z}{2\cos(x)} - 1\right) \frac{\partial^4 u}{\partial x^4} + \left(\frac{x+z}{2\cos(y)} - 1\right) \frac{\partial^4 u}{\partial y^4} + \left(\frac{y+x}{2\cos(z)} - 1\right) \frac{\partial^4 u}{\partial z^4} = 0 \quad (24)$$

with the initial conditions:

$$u(x, y, z, 0) = x + y + z - (\cos(x) + \cos(y) + \cos(z))$$

$$u_t(x, y, z, 0) = -(x + y + z) + (\cos(x) + \cos(y) + \cos(z))$$

and the boundary conditions:

$$u(0, y, z, t) = e^{-t}(-1 + y + z - \cos(y) - \cos(z))$$

$$u\left(\frac{\pi}{3}, y, z, t\right) = e^{-t} \left(\frac{2\pi - 3}{6} + y + z - \cos(y) - \cos(z)\right)$$

$$u(x, 0, z, t) = e^{-t}(-1 + x + z - \cos(x) - \cos(z))$$

$$u\left(x, \frac{\pi}{3}, z, t\right) = e^{-t} \left(\frac{2\pi - 3}{6} + x + z - \cos(x) - \cos(z)\right)$$

$$u(x, y, 0, t) = e^{-t}(-1 + y + x - \cos(y) - \cos(x))$$

$$u\left(x, y, \frac{\pi}{3}, t\right) = e^{-t} \left(\frac{2\pi - 3}{6} + y + x - \cos(y) - \cos(x)\right)$$

$$u_x(0, y, z, t) = u_y(x, 0, z, t) = u_z(x, y, 0, t) = e^{-t}$$

$$u_x\left(\frac{\pi}{3}, y, z, t\right) = u_y\left(x, \frac{\pi}{3}, z, t\right) = u_z\left(x, y, \frac{\pi}{3}, t\right) = \frac{3^{1/2} + 2}{2} e^{-t}$$

By applying aforesaid method and comparing the coefficients of various powers of p , we get the following solution

$$u(x, y, z, t) = [(x + y + z) - (\cos(x) + \cos(y) + \cos(z))] e^{-t} \quad (25)$$

which is an exact solution and is same as obtained by the implicit and explicit methods [21, 23-27], the Variational iteration method [28, 29], the Adomain decomposition method [31] and Homotopy perturbation method [32].

Example 3.6. Consider the following fourth-order parabolic equation [21-32].

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{4!z}\right) \frac{\partial^4 u}{\partial x^4} + \left(\frac{1}{4!x}\right) \frac{\partial^4 u}{\partial y^4} + \left(\frac{1}{4!y}\right) \frac{\partial^4 u}{\partial z^4} = \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5}\right] \cos(t) \quad (26)$$

with the initial conditions:

$$u(x, y, z, 0) = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right), \quad u_t(x, y, z, 0) = 0$$

and the boundary conditions:

$$u\left(\frac{1}{2}, y, z, t\right) = \left(\frac{1}{2y} + \frac{y}{z} + 2z\right) \cos(t), \quad u(1, y, z, t) = \left(\frac{1}{y} + \frac{y}{z} + z\right) \cos(t),$$

$$u\left(x, \frac{1}{2}, z, t\right) = \left(\frac{1}{2z} + \frac{z}{x} + 2x\right) \cos(t), \quad u(x, 1, z, t) = \left(\frac{1}{z} + \frac{z}{x} + x\right) \cos(t),$$

$$\begin{aligned}
u\left(x, y, \frac{1}{2}, t\right) &= \left(\frac{1}{2x} + \frac{x}{y} + 2y\right) \cos(t), & u(x, y, 1, t) &= \left(\frac{1}{x} + \frac{x}{y} + y\right) \cos(t), \\
u_x\left(\frac{1}{2}, y, z, t\right) &= \left(\frac{1}{y} - 4z\right) \cos(t), & u_x(1, y, z, t) &= \left(\frac{1}{y} - z\right) \cos(t), \\
u_y\left(x, \frac{1}{2}, z, t\right) &= \left(\frac{1}{z} - 4x\right) \cos(t), & u_y(x, 1, z, t) &= \left(\frac{1}{z} - x\right) \cos(t), \\
u_z\left(x, y, \frac{1}{2}, t\right) &= \left(\frac{1}{x} - 4y\right) \cos(t), & u_z(x, y, 1, t) &= \left(\frac{1}{x} - y\right) \cos(t),
\end{aligned}$$

By applying aforesaid method and the noise term appearing between the components are cancelled out and remaining terms will satisfies the equations and we get the following solution

$$u(x, y, z, t) = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \cos(t) \quad (27)$$

which is an exact solution .The results of the above example shows that our method is capable of reducing the huge computational work and generates the modification of homotopy perturbation method in the good convergence rate and is same as obtained by the implicit and explicit methods [21, 23-27], the Variational iteration method [28, 29], the Adomain decomposition method [31].

4. Conclusions

The main concern of this article is to construct an analytic solution for fourth-order parabolic partial differential equations of variable coefficients. We have achieved this goal by applying homotopy perturbation transform method (HPTM). The main advantage of this algorithm is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. Analytical solutions enable researchers to study the effect of different variables under study easily. Its small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence show that the method is reliable and introduces a significant improvement in solving partial differential equations over existing methods. The solution procedure by using He's polynomials is simple, but the calculation of Adomain's polynomials is complex. The fact that the HPTM solves nonlinear problems without using the Adomain's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. Also the proposed scheme exploits full advantage of Variational iteration method (VIM), Adomain's decomposition method (ADM) and Variational iteration decomposition method (VIDM). Finally, we conclude that HPTM can be considered as a nice refinement in existing numerical technique and might find wide applications in different fields of Sciences. Numerical computation has been done by Maple-13 software package.

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