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SYMMETRY PROPERTIES FOR A UNIFIED CLASS OF POLYNOMIALS ATTACHED TO χ

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Abstract. In this paper, we obtain some generalized symmetry identities involving a unified class of polynomials related to the generalized Bernoulli, Euler and Genocchi polynomials of higher-order attached to a Dirichlet character. In particular, we prove a relation between a generalized χ version of the power sum polynomials and this unified class of polynomials.

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1. Introduction, Definitions and Notations

In the last years, Q.-M. Luo and Srivastava [10, 11] introduced the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x)$ of order α, Q.-M. Luo [9] investigated the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x)$ of order α and the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x)$ of order α.

The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined respectively by the following generating functions

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \qquad (|t+\log\lambda| < 2\pi; 1^{\alpha} := 1) \tag{1}$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \qquad (|t+\log\lambda| < \pi; 1^{\alpha} := 1) \tag{2}$$

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and

$$\left(\frac{2t}{\lambda \mathrm{e}^t+1}\right)^{\alpha} \mathrm{e}^{xt}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \qquad (|t+\log\lambda|<\pi; 1^{\alpha}:=1). \tag{3}$$

It is obvious that the case $\lambda=1$ and $\alpha=1$ in the above relations gives respectively the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$.

Recently, Ozden [13, 14] and Ozden et al. [15] provides an interesting unification of the Apostol-Bernoulli, Euler and Genocchi polynomials respectively (1), (2) and (3). Explicitly, Ozden studied the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b \mathrm{e}^t-a^b}\right)^{\alpha} \mathrm{e}^{xt}=\sum_{n=0}^{\infty}\mathcal{Y}_{n,\,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!} \tag{4}$$

$$\left(\left|t+b\log\left(\frac{\beta}{a}\right)\right|<2\pi,\ x\in\mathbb{R}; 1^{\alpha}:=1;\ k\in\mathbb{N}_0;\ a,b\in\mathbb{R}\setminus\{0\};\ \alpha,\beta\in\mathbb{C}\right).$$

Shorthly after Özarslan [12] gives the following precise conditions of convergence of the series involved in (4)

(i) if $a^b>0$ and $k\in\mathbb{N}$, then $\left|t+b\log\left(\frac{\beta}{a}\right)\right|<2\pi;\ 1^{\alpha}:=1,\ x\in\mathbb{R},\ \alpha,\beta\in\mathbb{C}$.

(ii) if $a^b>0$ and $k=0$, then $0<\mathrm{Im}\left(t+b\log\left(\frac{\beta}{a}\right)\right)<2\pi;\ 1^{\alpha}:=1,$ $x\in\mathbb{R},\ \alpha,\beta\in\mathbb{C}$.

(iii) if $a^b<0$ and $k\in\mathbb{N}_0$, then $\left|t+b\log\left(\frac{\beta}{a}\right)\right|<\pi;\ 1^{\alpha}:=1,\ x\in\mathbb{R},\ \alpha,\beta\in\mathbb{C}$.

This family of polynomials includes the above mentioned well known polynomials. We can see that

$$\mathcal{Y}_{n,\,\lambda}^{(\alpha)}(x;1,1,1)=\mathcal{B}_n^{(\alpha)}(x;\lambda),\ \mathcal{Y}_{n,\,\lambda}^{(\alpha)}(x;0,-1,1)=\mathcal{E}_n^{(\alpha)}(x;\lambda)$$

and

$$\mathcal{Y}_{n,\,\lambda}^{(\alpha)}(x;1,-1,1)=\frac{1}{2}\mathcal{G}_n^{(\alpha)}(x;\lambda).$$

Moreover, Ozden et al. in [15] have extended and investigated the generating function (4) in terms of a Dirichlet character χ of conductor $f\in\mathbb{N}$ since these polynomials are very important in several fields of mathematics and physics. They proposed the following χ-extension of the generating function for the generalized Apostol-Bernoulli, Euler and Genocchi polynomials. Let χ be a Dirichlet character of conductor $f\in\mathbb{N}$ then

$$2^{1-k}t^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jt}}{\beta^{bf}\mathrm{e}^{ft}-a^{bf}}\ \mathrm{e}^{xt}=\sum_{n=0}^{\infty}\mathcal{Y}_{n,\,\chi,\,\beta}(x;k,a,b)\frac{t^n}{n!} \tag{5}$$

$$\left(\left|ft+bf\log\left(\frac{\beta}{a}\right)\right|<2\pi,\ x\in\mathbb{R}; k\in\mathbb{N}_0;\ f\in\mathbb{N},\ a,b\in\mathbb{R}\setminus\{0\};\ \beta\in\mathbb{C}\right).$$

In this paper, we study the generating function for the χ-extended unified polynomials of higher order denoted by $\mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;k,a,b)$ and defined as follows:

Definition 1.1. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}_0$, $a, b \in \mathbb{R} \setminus \{0\}$ and $\alpha, \beta \in \mathbb{C}$. Then the χ-extended unified polynomials of higher order $\mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;k,a,b)$ are given by

$$\left(2^{1-k}t^k \sum_{j=0}^{f-1} \frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jt}}{\beta^{bf}\mathrm{e}^{ft} - a^{bf}}\right)^{\alpha} \mathrm{e}^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;k,a,b)\frac{t^n}{n!} \tag{6}$$

$$\left(\left|ft + bf\log\left(\frac{\beta}{a}\right)\right| < 2\pi,\ x \in \mathbb{R}; k \in \mathbb{N}_0;\ f \in \mathbb{N},\ a, b \in \mathbb{R}\setminus\{0\};\ \beta \in \mathbb{C}\right).$$

The case $x = 0$, $\mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(0;k,a,b) = \mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(k,a,b)$ gives the χ-extended unified numbers of higher order.

Remark 1.1. If we set $\chi \equiv 1$ in (6), we obtain the generating function (4). The χ-extended versions $\mathcal{B}^{(\alpha)}_{n,\,\chi}(x,\beta)$ of the Apostol-Bernoulli polynomials of higher order, $\mathcal{E}^{(\alpha)}_{n,\,\chi}(x,\beta)$ of the Apostol-Euler polynomials of higher order and $\mathcal{G}^{(\alpha)}_{n,\,\chi}(x,\beta)$ of the Apostol-Genocchi polynomials of higher order are given respectively by

$$\mathcal{B}^{(\alpha)}_{n,\,\chi}(x,\beta) = \mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;\ 1,1,1) \tag{7}$$

$$\mathcal{E}^{(\alpha)}_{n,\,\chi}(x,\beta) = \mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;\ 0,-1,1) \tag{8}$$

$$\mathcal{G}^{(\alpha)}_{n,\,\chi}(x,\beta) = 2\mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;\ 1,-1,1). \tag{9}$$

Moreover, if we put $\alpha = 1$ and $\beta = 1$ in (7)-(9), we get the following χ-extended version of the classical Bernoulli, Euler and Genocchi polynomials [18, 19]:

$$B_{n,\,\chi}(x) = \mathcal{Y}^{(1)}_{n,\,\chi,\,1}(x;\ 1,1,1) \tag{10}$$

$$E_{n,\,\chi}(x) = \mathcal{Y}^{(1)}_{n,\,\chi,\,1}(x;\ 0,-1,1) \tag{11}$$

$$G_{n,\,\chi}(x) = 2\mathcal{Y}^{(1)}_{n,\,\chi,\,1}(x;\ 1,-1,1) \tag{12}$$

respectively.

Now, making use of this definition and the Cauchy product, we find the next theorem.

Theorem 1.2. *Let $k \in \mathbb{N}_0$, $a, b \in \mathbb{R} \setminus \{0\}$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(x;k,a,b) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}\, \mathcal{Y}^{(\alpha)}_{n,\,\chi,\,\beta}(k,a,b). \tag{13}$$

Recently, many authors have studied the symmetry properties for the Bernoulli, Euler and Genocchi polynomials [1, 2, 3, 4, 6, 7, 16, 17]. The purpose of this paper is to prove some identities of symmetry for this χ-extended unified polynomials of higher order. It points out the recurrence relation and multiplication theorem for the unified polynomials of higher order attached to χ.

2. Symmetry identities related to the generalized unified polynomials of higher order

In this section, we aim to obtain several symmetry identities involving the χ-extended unified polynomials of higher order defined by (6). In particular, we obtain an identity related to a generalized χ version of the power sum. Some special cases are also given for each identities. At this point, we need the following definition for the generalized χ version of the power sum.

Definition 2.1. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}_0$; $l, n \in \mathbb{N}$; $a, b \in \mathbb{R} \setminus \{0\}$ and $\alpha, \beta \in \mathbb{C}$, then we set

$$\widetilde{T}_{l,\,\chi}(n;\ k,a,\beta,b) = 2^{1-k}k!\binom{l}{k}\sum_{j=0}^{f-1}\sum_{i=0}^{n-1}\chi(j)\left(\frac{\beta^b}{a^b}\right)^{j+if}(j+if)^{l-k} \qquad (l \geq k). \tag{14}$$

The next theorem holds for the χ-extended unified polynomials $\mathcal{Y}^{(1)}_{n,\,\chi,\,\beta}(x;k,a,b)$:

Theorem 2.2. *Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}_0$; $l\ (l \geq k), n \in \mathbb{N}$; $a, b \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{C}$, then*

$$\left(\frac{\beta^b}{a^b}\right)^{fn}\mathcal{Y}^{(1)}_{l,\,\chi,\,\beta}(nf;k,a,b) - \mathcal{Y}^{(1)}_{l,\,\chi,\,\beta}(0;k,a,b) = \frac{\widetilde{T}_{l,\,\chi}(n;\ k,a,\beta,b)}{a^{bf}}. \tag{15}$$

Proof. Consider the generating function (6) with $\alpha = 1$, we have

$$\begin{aligned}
&\sum_{l=0}^{\infty}\left[\left(\frac{\beta^b}{a^b}\right)^{fn}\mathcal{Y}^{(1)}_{l,\,\chi,\,\beta}(nf;k,a,b) - \mathcal{Y}^{(1)}_{l,\,\chi,\,\beta}(0;k,a,b)\right]\frac{t^l}{l!}\\
&= \frac{\left(\frac{\beta^b}{a^b}\right)^{fn} 2^{1-k}t^k \sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jt}\,\mathrm{e}^{nft}}{a^{bf}\left[\left(\frac{\beta}{a}\right)^{bf}\mathrm{e}^{ft}-1\right]} - \frac{2^{1-k}t^k\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jt}}{a^{bf}\left[\left(\frac{\beta}{a}\right)^{bf}\mathrm{e}^{ft}-1\right]}\\
&= \frac{2^{1-k}t^k\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jt}}{a^{bf}}\left(\frac{\left(\frac{\beta^b}{a^b}\right)^{fn}\mathrm{e}^{nft}-1}{\left(\frac{\beta}{a}\right)^{bf}\mathrm{e}^{ft}-1}\right)\\
&= \frac{2^{1-k}t^k\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jt}}{a^{bf}}\sum_{i=0}^{n-1}\left(\frac{\beta}{a}\right)^{bfi}\mathrm{e}^{ift}\\
&= \frac{2^{1-k}t^k}{a^{bf}}\sum_{j=0}^{f-1}\sum_{i=0}^{n-1}\chi(j)\left(\frac{\beta}{a}\right)^{bfi+bj}\mathrm{e}^{(j+if)t}\\
&= \sum_{l=0}^{\infty}\frac{2^{1-k}}{a^{bf}}\sum_{j=0}^{f-1}\sum_{i=0}^{n-1}\chi(j)\left(\frac{\beta}{a}\right)^{bfi+bj}(j+if)^l\,\frac{t^{l+k}}{l!}.
\end{aligned} \tag{16}$$

Substituting $l \to l-k$ $(l \geq k)$ and comparing the coefficient of $\frac{t^l}{l!}$ yields the desired result. □

Remark 2.1. If we set $k=0$, $a=-1$, $\beta=b=1$ and $f \equiv 1 \pmod 2$ in (15), we recover a relation for the χ-extended version of the classical Euler polynomials [1]

$$(-1)^{n-1}E_{l,\,\chi}(nf)+E_{l,\,\chi}(0)=2\,\widetilde{T}_{l,\,\chi}(n;\;0,-1,1,1). \tag{17}$$

Remark 2.2. Putting $k=1$, $a=b=1$ in (15), we recover a relation for the χ-extended version of the Apostol-Bernoulli polynomials [4]

$$\beta^{fn}\,\mathcal{B}_{l,\,\chi}(fn,\beta)-\mathcal{B}_{l,\,\chi}(0,\beta)=\widetilde{T}_{l,\,\chi}(n;\;1,1,\beta,1). \tag{18}$$

Theorem 2.3. *Let χ be a Dirichlet character of conductor $f\in\mathbb{N}$. Let $k\in\mathbb{N}_0$; l $(l\geq k), m,\ n\in\mathbb{N}$; $a,b\in\mathbb{R}\setminus\{0\}$ and $\beta\in\mathbb{C}$. Let w_1 and $w_2\in\mathbb{N}$. Then, we have*

$$\begin{aligned}
&a^{bw_2f(w_1-1)}\sum_{s=0}^{n}\binom{n}{s}w_1^{n-s}w_2^{s+k}\mathcal{Y}^{(m)}_{n-s,\,\chi,\,\beta^{w_1}}(w_2x;k,a^{w_1},b)\\
&\times\sum_{l=0}^{s}\binom{s}{l}\widetilde{T}_{s-l,\,\chi}(w_1;\;0,a^{w_2},\beta^{w_2},b)\mathcal{Y}^{(m-1)}_{l,\,\chi,\,\beta^{w_2}}(w_1y;k,a^{w_2},b)\\
&=a^{bw_1f(w_2-1)}\sum_{s=0}^{n}\binom{n}{s}w_1^{s+k}w_2^{n-s}\mathcal{Y}^{(m)}_{n-s,\,\chi,\,\beta^{w_2}}(w_1x;k,a^{w_2},b)\\
&\times\sum_{l=0}^{s}\binom{s}{l}\widetilde{T}_{s-k,\,\chi}(w_2;\;0,a^{w_1},\beta^{w_1},b)\mathcal{Y}^{(m-1)}_{l,\,\chi,\,\beta^{w_1}}(w_2y;k,a^{w_1},b).
\end{aligned} \tag{19}$$

Proof. Let

$$\begin{aligned}
G(t):=&\frac{2^{(1-k)(2m-1)}t^{2km-k}\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}\right)^m\mathrm{e}^{w_1w_2xt}}{(\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf})^m}\\
&\times\frac{\left(\beta^{bw_1w_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_1w_2f}\right)\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}\right)^m\mathrm{e}^{w_1w_2yt}}{(\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf})^m}.
\end{aligned} \tag{20}$$

Expanding $G(t)$ into a series, we get

$$\begin{aligned}
G(t)=&\frac{1}{w_1^{km}w_2^{k(m-1)}}\left(2^{1-k}w_1^kt^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)^m\mathrm{e}^{w_1w_2xt}\\
&\times\left(\frac{\beta^{bw_1w_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_1w_2f}}{\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf}}\right)\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}\right)
\end{aligned}$$

$$\times\left(2^{1-k}w_2^k t^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta}{a}\right)^{bw_2j}\mathrm{e}^{jw_2t}}{\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf}}\right)^{m-1}\mathrm{e}^{w_1w_2yt}$$

$$=\frac{a^{bw_2f(w_1-1)}}{w_1^{km}w_2^{k(m-1)}}\sum_{n=0}^{\infty}\mathcal{Y}_{n,\,\chi,\,\beta^{w_1}}^{(m)}(w_2x;k,a^{w_1},b)\frac{(w_1t)^n}{n!}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}$$

$$\times\sum_{i=0}^{w_1-1}\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bfi}\mathrm{e}^{w_2ift}\sum_{l=0}^{\infty}\mathcal{Y}_{l,\,\chi,\,\beta^{w_2}}^{(m-1)}(w_1y;k,a^{w_2},b)\frac{(w_2t)^l}{l!}$$

$$=\frac{a^{bw_2f(w_1-1)}}{2w_1^{km}w_2^{k(m-1)}}\sum_{n=0}^{\infty}\mathcal{Y}_{n,\,\chi,\,\beta^{w_1}}^{(m)}(w_2x;k,a^{w_1},b)\frac{(w_1t)^n}{n!}$$

$$\times\sum_{s=0}^{\infty}2\sum_{j=0}^{f-1}\sum_{i=0}^{w_1-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{b(j+fi)}(j+fi)^s\frac{(w_2t)^s}{s!}$$

$$\times\sum_{l=0}^{\infty}\mathcal{Y}_{l,\,\chi,\,\beta^{w_2}}^{(m-1)}(w_1y;k,a^{w_2},b)\frac{(w_2t)^l}{l!}$$

$$=\frac{a^{bw_2f(w_1-1)}}{2w_1^{km}w_2^{k(m-1)}}\sum_{n=0}^{\infty}\mathcal{Y}_{n,\,\chi,\,\beta^{w_1}}^{(m)}(w_2x;k,a^{w_1},b)\frac{(w_1t)^n}{n!}$$

$$\times\sum_{s=0}^{\infty}\widetilde{T}_{s,\,\chi}(w_1;\ 0,a^{w_2},\beta^{w_2},b)\frac{(w_2t)^s}{s!}\sum_{l=0}^{\infty}\mathcal{Y}_{l,\,\chi,\,\beta^{w_2}}^{(m-1)}(w_1y;k,a^{w_2},b)\frac{(w_2t)^l}{l!}$$

$$=\frac{a^{bw_2f(w_1-1)}}{2w_1^{km}w_2^{km}}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}w_1^{n-s}w_2^{s+k}\mathcal{Y}_{n-s,\,\chi,\,\beta^{w_1}}^{(m)}(w_2x;k,a^{w_1},b)\right.$$

$$\left.\times\sum_{l=0}^{s}\binom{s}{l}\widetilde{T}_{s-l,\,\chi}(w_1;\ 0,a^{w_2},\beta^{w_2},b)\mathcal{Y}_{l,\,\chi,\,\beta^{w_2}}^{(m-1)}(w_1y;k,a^{w_2},b)\right]\frac{t^n}{n!}. \qquad (21)$$

In a similar way, we have

$$G(t)=\frac{1}{w_1^{k(m-1)}w_2^{km}}\left(2^{1-k}w_2^k t^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}}{\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf}}\right)^m\mathrm{e}^{w_1w_2xt}$$

$$\times\left(\frac{\beta^{bw_1w_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_1w_2f}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}\right)$$

$$\times\left(2^{1-k}w_1^k t^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)^{m-1}\mathrm{e}^{w_1w_2yt}$$

$$=\frac{a^{bw_1f(w_2-1)}}{2w_1^{km}w_2^{km}}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}w_1^{s+k}w_2^{n-s}\mathcal{Y}_{n-s,\,\chi,\,\beta^{w_2}}^{(m)}(w_1x;k,a^{w_2},b)\right.$$

$$\left.\times\sum_{l=0}^{s}\binom{s}{l}\widetilde{T}_{s-k,\,\chi}(w_2;\ 0,a^{w_1},\beta^{w_1},b)\mathcal{Y}_{l,\,\chi,\,\beta^{w_1}}^{(m-1)}(w_2y;k,a^{w_1},b)\right]\frac{t^l}{l!}. \qquad (22)$$

□

If we set $k=0$, $a=-1$ and $b=1$ in Theorem 2.5, we obtain a result exhibited by Kim et al. [5, Eq.(2.13)].

Corollary 2.4. *Let $m,\ n\in\mathbb{N}$ and $\beta\in\mathbb{C}$. Let w_1, w_2 and $f\in\mathbb{N}$ with $w_1\equiv 1\ (mod\ 2)$, $w_2\equiv 1\ (mod\ 2)$ and $f\equiv 1\ (mod\ 2)$, we have*

$$\begin{aligned}&\sum_{s=0}^{n}\binom{n}{s}w_1^{n-s}w_2^{s}\,\mathcal{E}_{n-s,\,\chi}^{(m)}(w_2x;\ \beta^{w_1})\\ &\times\sum_{l=0}^{s}\binom{s}{l}\widetilde{T}_{s-l,\,\chi}(w_1;\ 0,-1,\beta^{w_2},1)\,\mathcal{E}_{l,\,\chi}^{(m-1)}(w_1y;\ \beta^{w_2})\\ =&\sum_{s=0}^{n}\binom{n}{s}w_1^{s}w_2^{n-s}\mathcal{E}_{n-s,\,\chi}^{(m)}(w_1x;\ \beta^{w_2})\\ &\times\sum_{l=0}^{s}\binom{s}{l}\widetilde{T}_{s-k,\,\chi}(w_2;\ 0,-1,\beta^{w_1},1)\mathcal{E}_{l,\,\chi}^{(m-1)}(w_2y;\ \beta^{w_1}).\end{aligned}\tag{23}$$

Theorem 2.5. *Let χ be a Dirichlet character of conductor $f\in\mathbb{N}$. Let $k\in\mathbb{N}_0$, $m,\ n\in\mathbb{N}$; $a,b\in\mathbb{R}\setminus\{0\}$ and $\beta\in\mathbb{C}$. Let w_1 and w_2 be two natural numbers. Then, we have*

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{w_1-1}\sum_{l=0}^{w_2-1}\left(\frac{\beta}{a}\right)^{bf(i+l)}w_1^{n-r}w_2^{r}\,\mathcal{Y}_{n-r,\,\chi,\,\beta}^{(m)}\left(w_2x+\frac{w_2f}{w_1}i;k,a,b\right)\\ &\times\mathcal{Y}_{r,\,\chi,\,\beta}^{(m)}\left(w_1y+\frac{w_1f}{w_2}l;k,a,b\right)\\ =&\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{w_2-1}\sum_{l=0}^{w_1-1}\left(\frac{\beta}{a}\right)^{bf(i+l)}w_1^{r}w_2^{n-r}\,\mathcal{Y}_{n-r,\,\chi,\,\beta}^{(m)}\left(w_1x+\frac{w_1f}{w_2}i;k,a,b\right)\\ &\times\mathcal{Y}_{r,\,\chi,\,\beta}^{(m)}\left(w_2y+\frac{w_2f}{w_1}l;k,a,b\right).\end{aligned}\tag{24}$$

Proof. Let

$$\begin{aligned}H(t):=&\frac{2^{2m(1-k))}t^{2km}\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jw_1t}\right)^m\mathrm{e}^{w_1w_2xt}\left(\beta^{bw_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_2f}\right)}{(\beta^{bf}\mathrm{e}^{w_1ft}-a^{bf})^{m+1}}\\ &\frac{\left(\beta^{bw_1f}\mathrm{e}^{w_1w_2ft}-a^{bw_1f}\right)\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jw_2t}\right)^m\mathrm{e}^{w_1w_2yt}}{(\beta^{bf}\mathrm{e}^{w_2ft}-a^{bf})^{m+1}}.\end{aligned}\tag{25}$$

Expanding $H(t)$ into a series, we get

$$\begin{aligned}H(t)=&\frac{1}{(w_1w_2)^{km}}\left(2^{1-k}w_1^kt^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jw_1t}}{\beta^{bf}\mathrm{e}^{w_1ft}-a^{bf}}\right)^m\mathrm{e}^{w_1w_2xt}\\ &\times\left(\frac{\beta^{bw_1f}\mathrm{e}^{w_1w_2ft}-a^{bw_1f}}{\beta^{bf}\mathrm{e}^{w_2ft}-a^{bf}}\right)\left(2^{1-k}w_2^kt^k\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathrm{e}^{jw_2t}}{\beta^{bf}\mathrm{e}^{w_2ft}-a^{bf}}\right)^m\\ &\times\mathrm{e}^{w_1w_2yt}\left(\frac{\beta^{bw_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_2f}}{\beta^{bf}\mathrm{e}^{w_1ft}-a^{bf}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{bf(w_1+w_2-2)}}{(w_1w_2)^{km}} \left(2^{1-k} w_1^k t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jw_1t}}{\beta^{bf}\mathrm{e}^{w_1ft} - a^{bf}} \right)^m \mathrm{e}^{w_1w_2xt} \\
&\quad \times \left(\sum_{i=0}^{w_1-1} \left(\frac{\beta}{a}\right)^{bfi} \mathrm{e}^{w_2ift} \right) \left(2^{1-k} w_2^k t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jw_2t}}{\beta^{bf}\mathrm{e}^{w_2ft} - a^{bf}} \right)^m \\
&\quad \times \mathrm{e}^{w_1w_2yt} \left(\sum_{l=0}^{w_2-1} \left(\frac{\beta}{a}\right)^{bfl} \mathrm{e}^{w_1lft} \right) \\
&= \frac{a^{bf(w_1+w_2-2)}}{(w_1w_2)^{km}} \sum_{i=0}^{w_1-1} \left(\frac{\beta}{a}\right)^{bfi} \left(2^{1-k} w_1^k t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jw_1t}}{\beta^{bf}\mathrm{e}^{w_1ft} - a^{bf}} \right)^m \\
&\quad \times \mathrm{e}^{w_1t\left(w_2x + \frac{w_2f}{w_1}i\right)} \sum_{l=0}^{w_2-1} \left(\frac{\beta}{a}\right)^{bfl} \left(2^{1-k} w_2^k t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jw_2t}}{\beta^{bf}\mathrm{e}^{w_2ft} - a^{bf}} \right)^m \\
&\quad \times \mathrm{e}^{w_2t\left(w_1y + \frac{w_1f}{w_2}l\right)} \\
&= \frac{a^{bf(w_1+w_2-2)}}{(w_1w_2)^{km}} \left[\sum_{i=0}^{w_1-1} \left(\frac{\beta}{a}\right)^{bfi} \sum_{n=0}^{\infty} \mathcal{Y}_{n,\,\chi,\,\beta}^{(m)} \left(w_2x + \frac{w_2f}{w_1}i; k, a, b \right) \frac{(w_1t)^n}{n!} \right] \\
&\quad \times \left[\sum_{l=0}^{w_2-1} \left(\frac{\beta}{a}\right)^{bfl} \sum_{r=0}^{\infty} \mathcal{Y}_{r,\,\chi,\,\beta}^{(m)} \left(w_1y + \frac{w_1f}{w_2}l; k, a, b \right) \frac{(w_2t)^r}{r!} \right] \\
&= \frac{a^{bf(w_1+w_2-2)}}{(w_1w_2)^{km}} \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2-1} \left(\frac{\beta}{a}\right)^{bf(i+l)} w_1^{n-r} w_2^r \right. \\
&\quad \left. \times \mathcal{Y}_{n-r,\,\chi,\,\beta}^{(m)} \left(w_2x + \frac{w_2f}{w_1}i; k, a, b \right) \mathcal{Y}_{r,\,\chi,\,\beta}^{(m)} \left(w_1y + \frac{w_1f}{w_2}l; k, a, b \right) \right] \frac{t^n}{n!}. \qquad (26)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
H(t) &= \frac{1}{(w_1w_2)^{km}} \left(2^{1-k} w_2^k t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jw_2t}}{\beta^{bf}\mathrm{e}^{w_2ft} - a^{bf}} \right)^m \mathrm{e}^{w_1w_2xt} \\
&\quad \times \left(\frac{\beta^{bw_2f}\mathrm{e}^{w_1w_2ft} - a^{bw_2f}}{\beta^{bf}\mathrm{e}^{w_1ft} - a^{bf}} \right) \left(2^{1-k} w_1^k t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathrm{e}^{jw_1t}}{\beta^{bf}\mathrm{e}^{w_1ft} - a^{bf}} \right)^m \\
&\quad \times \mathrm{e}^{w_1w_2yt} \left(\frac{\beta^{bw_1f}\mathrm{e}^{w_1w_2ft} - a^{bw_1f}}{\beta^{bf}\mathrm{e}^{w_2ft} - a^{bf}} \right) \\
&= \frac{a^{bf(w_1+w_2-2)}}{(w_1w_2)^{km}} \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1-1} \left(\frac{\beta}{a}\right)^{bf(i+l)} w_1^{r} w_2^{n-r} \right. \\
&\quad \left. \times \mathcal{Y}_{n-r,\,\chi,\,\beta}^{(m)} \left(w_1x + \frac{w_1f}{w_2}i; k, a, b \right) \mathcal{Y}_{r,\,\chi,\,\beta}^{(m)} \left(w_2y + \frac{w_2f}{w_1}l; k, a, b \right) \right] \frac{t^n}{n!}. \qquad (27)
\end{aligned}
$$

□

Now, putting $\chi \equiv 1$ and replacing k, a, b by 1 in Theorem 2.7, we find a result of Zhang and Yang [20, Eq.18].

Corollary 2.6. *Let $m,\ n \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Let w_1 and w_2 be two natural numbers, then we have*

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{w_1-1}\sum_{l=0}^{w_2-1}\beta^{i+l}w_1^{n-r}w_2^{r}\mathcal{B}_{n-r}^{(m)}\left(w_2x+\frac{w_2}{w_1}i;\beta\right)\mathcal{B}_{r}^{(m)}\left(w_1y+\frac{w_1}{w_2}l;\beta\right)\\&=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{w_2-1}\sum_{l=0}^{w_1-1}\beta^{i+l}w_1^{r}w_2^{n-r}\mathcal{B}_{n-r}^{(m)}\left(w_1x+\frac{w_1}{w_2}i;\beta\right)\mathcal{B}_{r}^{(m)}\left(w_2y+\frac{w_2}{w_1}l;\beta\right).\end{aligned}\tag{28}$$

Theorem 2.7. *Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}_0,\ m,\ n \in \mathbb{N}$; $a, b \in \mathbb{R}\setminus\{0\}$ and $\beta \in \mathbb{C}$. Let w_1 and w_2 be two natural numbers. Then, we have*

$$\begin{aligned}&a^{bw_2f(w_1-1)}\sum_{r=0}^{n}\binom{n}{r}w_1^{r}w_2^{n-r+k}\,\mathcal{Y}_{n-r,\,\chi,\,\beta^{w_2}}^{(m-1)}(w_1y;k,a^{w_2},b)\\&\times\sum_{j=0}^{f-1}\sum_{i=0}^{w_1-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{b(j+fi)}\mathcal{Y}_{r,\,\chi,\,\beta^{w_1}}^{(m)}(w_2x+\frac{w_2f}{w_1}i+\frac{w_2}{w_1}j;k,a^{w_1},b)\\&=a^{bw_1f(w_2-1)}\sum_{r=0}^{n}\binom{n}{r}w_1^{n-r+k}w_2^{r}\,\mathcal{Y}_{n-r,\,\chi,\,\beta^{w_1}}^{(m-1)}(w_2y;k,a^{w_1},b)\\&\times\sum_{j=0}^{f-1}\sum_{i=0}^{w_2-1}\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{b(j+fi)}\mathcal{Y}_{r,\,\chi,\,\beta^{w_2}}^{(m)}(w_1x+\frac{w_1f}{w_2}i+\frac{w_1}{w_2}j;k,a^{w_2},b).\end{aligned}\tag{29}$$

Proof. Consider

$$\begin{aligned}M(t):=&\frac{2^{(1-k)(2m-1)}t^{2km-k}\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}\right)^{m}\mathrm{e}^{w_1w_2xt}}{(\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf})^{m}}\\&\times\frac{\left(\beta^{bw_1w_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_1w_2f}\right)\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}\right)^{m}\mathrm{e}^{w_1w_2yt}}{(\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf})^{m}}.\end{aligned}\tag{30}$$

Expanding $M(t)$ into a series, we obtain

$$\begin{aligned}M(t)=&\frac{1}{w_1^{km}w_2^{k(m-1)}}\left(2^{1-k}w_2^{k}t^{k}\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}}{\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf}}\right)^{m-1}\mathrm{e}^{w_1w_2yt}\\&\times\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}\right)\left(\frac{\beta^{bw_1w_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_1w_2f}}{\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf}}\right)\\&\times\left(2^{1-k}w_1^{k}t^{k}\sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)^{m}\mathrm{e}^{w_1w_2xt}\\=&\frac{a^{bw_2f(w_1-1)}}{w_1^{km}w_2^{k(m-1)}}\sum_{n=0}^{\infty}\mathcal{Y}_{n,\,\chi,\,\beta^{w_2}}^{(m-1)}(w_1y;k,a^{w_2},b)\frac{(w_2t)^{n}}{n!}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}\end{aligned}$$

$$\times \sum_{i=0}^{w_1-1}\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bfi} \mathrm{e}^{w_2 ift}\left(2^{1-k}w_1^k t^k \sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)^m \mathrm{e}^{w_1w_2xt}$$

$$= \frac{a^{bw_2f(w_1-1)}}{w_1^{km}w_2^{k(m-1)}}\sum_{n=0}^{\infty}\mathcal{Y}^{(m-1)}_{n,\,\chi,\,\beta^{w_2}}(w_1y;k,a^{w_2},b)\frac{(w_2t)^n}{n!}\sum_{r=0}^{\infty}\sum_{j=0}^{f-1}\sum_{i=0}^{w_1-1}\chi(j)$$

$$\times\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{b(j+fi)}\mathcal{Y}^{(m)}_{r,\,\chi,\,\beta^{w_1}}(w_2x+\frac{w_2f}{w_1}i+\frac{w_2}{w_1}j;k,a^{w_1},b)\frac{(w_1t)^r}{r!}$$

$$= \frac{a^{bw_2f(w_1-1)}}{w_1^{km}w_2^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}w_1^r w_2^{n-r+k}\mathcal{Y}^{(m-1)}_{n-r,\,\chi,\,\beta^{w_2}}(w_1y;k,a^{w_2},b)\right.$$

$$\left.\times\sum_{j=0}^{f-1}\sum_{i=0}^{w_1-1}\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{b(j+fi)}\mathcal{Y}^{(m)}_{r,\,\chi,\,\beta^{w_1}}(w_2x+\frac{w_2f}{w_1}i+\frac{w_2}{w_1}j;k,a^{w_1},b)\right]\frac{t^n}{n!}. \tag{31}$$

Similarly,

$$M(t) = \frac{1}{w_1^{k(m-1)}w_2^{km}}\left(2^{1-k}w_1^k t^k \sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)^{m-1}\mathrm{e}^{w_1w_2yt}$$

$$\times\left(\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{bj}\mathrm{e}^{jw_1t}\right)\left(\frac{\beta^{bw_1w_2f}\mathrm{e}^{w_1w_2ft}-a^{bw_1w_2f}}{\beta^{w_1bf}\mathrm{e}^{w_1ft}-a^{w_1bf}}\right)$$

$$\times\left(2^{1-k}w_2^k t^k \sum_{j=0}^{f-1}\frac{\chi(j)\left(\frac{\beta^{w_2}}{a^{w_2}}\right)^{bj}\mathrm{e}^{jw_2t}}{\beta^{w_2bf}\mathrm{e}^{w_2ft}-a^{w_2bf}}\right)^m \mathrm{e}^{w_1w_2xt}$$

$$= \frac{a^{bw_1f(w_2-1)}}{w_1^{km}w_2^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}w_1^{n-r+k}w_2^r\,\mathcal{Y}^{(m-1)}_{n-r,\,\chi,\,\beta^{w_1}}(w_2y;k,a^{w_1},b)\right.$$

$$\left.\times\sum_{j=0}^{f-1}\sum_{i=0}^{w_2-1}\chi(j)\left(\frac{\beta^{w_1}}{a^{w_1}}\right)^{b(j+fi)}\mathcal{Y}^{(m)}_{r,\,\chi,\,\beta^{w_2}}(w_1x+\frac{w_1f}{w_2}i+\frac{w_1}{w_2}j;k,a^{w_2},b)\right]\frac{t^n}{n!}. \tag{32}$$

□

Substituting k by 0, a by -1 and b by 1 in the last Theorem, we find the following symmetry relation for the χ-extended Apostol-Euler polynomials:

Corollary 2.8. *Let $m,\ n\in\mathbb{N}$ and $\beta\in\mathbb{C}$. Let w_1, w_2 and $f\in\mathbb{N}$ with $w_1\equiv 1\ (mod\ 2)$, $w_2\equiv 1\ (mod\ 2)$ and $f\equiv 1\ (mod\ 2)$, then*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}w_1^r w_2^{n-r+1}\ \mathcal{E}^{(m-1)}_{n-r,\,\chi}(w_1y;\ \beta^{w_2})\\
&\quad\times\sum_{j=0}^{f-1}\sum_{i=0}^{w_1-1}\chi(j)(-1)^{j+fi}\,(\beta^{w_2})^{j+fi}\,\mathcal{E}^{(m)}_{r,\,\chi}(w_2x+\frac{w_2f}{w_1}i+\frac{w_2}{w_1}j;\ \beta^{w_1})\\
&=\sum_{r=0}^{n}\binom{n}{r}w_1^{n-r+1}w_2^r\ \mathcal{E}^{(m-1)}_{n-r,\,\chi}(w_2y;\ \beta^{w_1})\\
&\quad\times\sum_{j=0}^{f-1}\sum_{i=0}^{w_2-1}\chi(j)(-1)^{j+fi}\,(\beta^{w_1})^{j+fi}\,\mathcal{E}^{(m)}_{r,\,\chi}(w_1x+\frac{w_1f}{w_2}i+\frac{w_1}{w_2}j;\ \beta^{w_2}).
\end{aligned}\tag{33}$$

Finally, if we set $\chi \equiv 1$ and $\beta = 1$ in (2.9), we recover a result of Liu et al. [8, Theorem 2.10] for the Euler polynomials of higher order:

Corollary 2.9. *Let m, $n \in \mathbb{N}$. Let w_1 and $w_2 \in \mathbb{N}$ with $w_1 \equiv 1$ $(mod\ 2)$ and $w_2 \equiv 1$ $(mod\ 2)$, then*

$$\sum_{r=0}^{n} \binom{n}{r} w_1^r w_2^{n-r+1}\, E_{n-r}^{(m-1)}(w_1 y) \sum_{i=0}^{w_1-1} (-1)^i E_r^{(m)}(w_2 x + \frac{w_2}{w_1} i)$$
$$= \sum_{r=0}^{n} \binom{n}{r} w_1^{n-r+1} w_2^r\, E_{n-r}^{(m-1)}(w_2 y) \sum_{i=0}^{w_2-1} (-1)^i E_r^{(m)}(w_1 x + \frac{w_1}{w_2} i).$$

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