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PERIODIC SOLUTIONS FOR DUFFING TYPE p-LAPLACIAN EQUATION WITH MULTIPLE DEVIATING ARGUMENTS†

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Abstract. In this paper, we consider the Duffing type p-Laplacian equation with multiple deviating arguments of the form

$$(\varphi_p(x'(t)))' + Cx'(t) + g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k(t))) = e(t).$$

By using the coincidence degree theory, we establish new results on the existence and uniqueness of periodic solutions for the above equation. Moreover, an example is given to illustrate the effectiveness of our results.

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1. Introduction

The dynamic behaviors of Duffing equation and Duffing type equations have been widely investigated and are still being investigated due to their applications in many fields such as physics, mechanics, the engineering technique fields and so on. In recent years, the existence of periodic solutions for Duffing equation and Duffing type equations with and without delays have been discussed by various researchers (see, for example [1,3,4,6-10] and the references given therein). However, to the best of our knowledge, the existence and uniqueness of periodic solutions of Duffing type p-Laplacian equation whose delays more than two have not been sufficiently researched. Motivated by this, we shall consider the Duffing type p-Laplacian equations with multiple deviating arguments of the form

$$(\varphi_p(x'(t)))' + Cx'(t) + g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k(t))) = e(t), \tag{1.1}$$

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where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $C \in R$ is a constant, e, $\tau_k : R \to R$ and g_0, $g_k : R \times R \to R$ are continuous functions, e and τ_k are T-periodic, g_0 and g_k are T-periodic in the first argument, $T > 0$ and $k = 1, 2, \ldots, n$. The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.1). The results of this paper are new and complement previously known results. Moreover, we give an example to illustrate the results.

The organization of this paper is as follows. In Section 2, some necessary lemmas are given. In Section 3, by using Mawhin-Manásevich continuation theorem, some sufficient conditions for the uniqueness of periodic solutions of Eq. (1.1) are obtained. In the last section, an example is given to show the feasibility of the main results of this paper, and finally, some remarks are given to illustrate the main results.

2. Preliminary Results

For convenience, let us denote

$$C_T^1 := \{x \in C^1(R) : x \ \text{ is } \ \text{T-periodic}\},$$

$$|x|_k = (\int_0^T |x(t)|^k dt)^{1/k} (k > 0), \qquad |x|_\infty = \max_{t \in [0,T]} |x(t)|.$$

We also assume that $\tau_k \in C_T^1$, $1 - \tau_k' > 0$ and $k = 1, 2, \ldots, n$.

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = \widetilde{f}(t, x, x'), x(0) = x(T), x'(0) = x'(T). \tag{2.1}$$

where $\widetilde{f} \in C(R^3, R)$ is T$-$periodic in the first variable, we have the following lemma:

Lemma 2.1 ([5]). *Let Ω be an open bounded set in C_T^1, if the following conditions hold.*

(i) *For each $\lambda \in (0, 1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda \widetilde{f}(t, x, x'), \ x(0) = x(T), \ \ x'(0) = x'(T)$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) := \frac{1}{T} \int_0^T \widetilde{f}(t, a, 0)\, dt = 0,$$

has no solution on $\partial\Omega \bigcap R$.

(iii) *The Brouwer degree of F*

$$\deg(F,\, \Omega \bigcap R,\, 0) \neq 0.$$

Then the periodic boundary value problem (2.1) has at least one $T-$periodic solution on $\overline{\Omega}$.

We can easily obtain the homotopic equation of Eq.(1.1) as following:

$$(\varphi_p(x'(t)))' + \lambda Cx'(t) + \lambda[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t)))] = \lambda e(t), \quad \lambda \in (0,1). \tag{2.2}$$

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.2. *Assume that the following conditions are satisfied.*

(A_1) *there exists a constant* $d > 0$ *such that*

(1) $\sum\limits_{k=0}^{n} g_k(t, x_k) - e(t) < 0$ *for* $x_k > d, t \in R, k = 0, 1, 2, \cdots, n;$

(2) $\sum\limits_{k=0}^{n} g_k(t, x_k) - e(t) > 0$ *for* $x_k < -d, t \in R, k = 0, 1, 2, \cdots, n.$

Moveover, if $x(t)$ *is a* T*-periodic solution of (2.2), then*

$$|x|\infty \le d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.3}$$

Proof. Let $x(t)$ be a T-periodic solution of (2.2). Then, integrating (2.2) over $[0,\ T]$, we have

$$\int_0^T [g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) - e(t)]dt = 0.$$

Using the integral mean-value theorem, it follows that there exists $t_1 \in [0,\ T]$ such that

$$g_0(t_1, x(t_1)) + \sum_{k=1}^{n} g_k(t_1, x(t_1 - \tau_k(t_1))) - e(t_1) = 0. \tag{2.4}$$

We first claim that there exists a constant $t_2 \in R$ such that

$$|x(t_2)| \le d. \tag{2.5}$$

Assume, on the contrary, that (2.5) does not hold. Then

$$|x(t)| > d \text{ for all } t \in R. \tag{2.6}$$

Let $\tau_0 \equiv 0$ and $t_1 \in [0,\ T]$ be the constant prescribed in (2.4). Using (A_1), (2.4) and (2.6), we see that there exist $0 \le i, j \le n$ such that

$$x(t_1 - \tau_i(t_1)) = \max_{0\le k\le n} x(t_1 - \tau_k(t_1)) \ge \min_{0\le k\le n} x(t_1 - \tau_k(t_1)) = x(t_1 - \tau_j(t_1)),$$

which, together with (2.6), implies the fact that

$$-d > x(t_1-\tau_i(t_1)) = \max_{0\le k\le n} x(t_1-\tau_k(t_1)) \text{ or } x(t_1-\tau_j(t_1)) = \min_{0\le k\le n} x(t_1-\tau_k(t_1)) > d.$$

Without loss of generality, we may assume that $x(t_1 - \tau_j(t_1)) = \min\limits_{0\le k\le n} x(t_1 - \tau_k(t_1)) > d$ (The situation is analogous for $-d > x(t_1 - \tau_i(t_1)) = \max\limits_{0\le k\le n} x(t_1 - \tau_k(t_1)))$. Then, we have

$$x(t_1 - \tau_i(t_1)) \ge x(t_1 - \tau_k(t_1)) \ge x(t_1 - \tau_j(t_1)) > d, k = 0, 1, 2, \cdots, n. \tag{2.7}$$

According to (2.7) and (A_1), we obtain

$$0 > \sum_{k=0}^{n} g_k(t_1, x(t_1 - \tau_k(t_1))) - e(t_1),$$

this contradicts the fact (2.4), thus (2.5) is true.

Let $t_2 = mT + t_0$ where $t_0 \in [0, T]$ and m is an integer. Then

$$|x(t_2)| = |x(t_0)| \leq d,$$

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \leq d + \int_{t_0}^{t} |x'(s)|ds, \quad t \in [t_0, t_0 + T],$$

and

$$|x(t)| = |x(t-T)| = |x(t_0) - \int_{t-T}^{t_0} x'(s)ds| \leq d + \int_{t-T}^{t_0} |x'(s)|ds, \quad t \in [t_0, t_0 + T].$$

Combining the above two inequalities and using Schwarz inequality, for any T-periodic solution $x(t)$ of (2.2), we have

$$|x|_\infty = \max_{t\in[t_0,t_0+T]} |x(t)| \leq \max_{t\in[t_0,t_0+T]} \{d + \frac{1}{2}(\int_{t_0}^{t} |x'(s)|ds + \int_{t-T}^{t_0} |x'(s)|ds)\} \leq d + \frac{1}{2}\sqrt{T}|x'|_2.$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let (A_1) hold. Assume that the following condition is satisfied:*

(A_2) there exist nonnegative constants $\overline{b_0}, b_0, b_1, b_2, \ldots, b_n$ such that

$$\overline{b_0}|x_1 - x_2|^2 \leq -(g_0(t, x_1) - g_0(t, x_2))(x_1 - x_2),$$

and

$$\overline{b_0} > \sum_{k=1}^{n} b_k \max_{t\in R}(\frac{1}{1 - \tau_k'(t)})^{\frac{1}{2}}, \text{ and } |g_k(t, x_1) - g_k(t, x_2)| \leq b_k|x_1 - x_2|,$$

for all $t, x_1, x_2 \in R, k = 0, 1, 2, \ldots, n$.

Then (1.1) has at most one T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then, we obtain

$$\begin{aligned}(\varphi_p(x_1'(t)) - \varphi_p(x_2'(t)))' + C(x_1'(t) - x_2'(t)) + [g_0(t, x_1(t)) - g_0(t, x_2(t))] \\ + \sum_{k=1}^{n} [g_k(t, x_1(t - \tau_k(t))) - g_k(t, x_2(t - \tau_k(t)))] = 0. \qquad (2.8)\end{aligned}$$

Multiplying $Z(t)$ and (2.8) and then integrating it from 0 to T, from (A_2) and Schwarz inequality, we get

$$\begin{aligned}\overline{b_0}|Z|_2^2 &= \overline{b_0}\int_0^T |Z(t)|^2 dt \\ &\leq -\int_0^T (x_1(t) - x_2(t))[g_0(t, x_1(t)) - g_0(t, x_2(t))]dt\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^T (\varphi_p(x_1'(t)) - \varphi_p(x_2'(t)))(x_1'(t) - x_2'(t))dt \\
&\quad +\sum_{k=1}^{n}\int_0^T [g_k(t, x_1(t-\tau_k(t))) - g_k(t, x_2(t-\tau_k(t)))]Z(t)dt \\
&\le \sum_{k=1}^{n} b_k \int_0^T |x_1(t-\tau_k(t)) - x_2(t-\tau_k(t))||Z(t)|dt \\
&\le \sum_{k=1}^{n} b_k (\int_0^T |Z(t-\tau_k(t))|^2 dt)^{\frac{1}{2}} |Z|_2 \\
&= \sum_{k=1}^{n} b_k (\int_{-\tau_k(0)}^{T-\tau_k(0)} |Z(s)|^2 \frac{1}{1-\tau_k'(t)} ds)^{\frac{1}{2}} |Z|_2 \\
&= \sum_{k=1}^{n} b_k \int_0^T |Z(s)|^2 \frac{1}{1-\tau_k'(t)} ds)^{\frac{1}{2}} |Z|_2 \\
&\le \sum_{k=1}^{n} b_k \max_{t\in R}(\frac{1}{1-\tau_k'(t)})^{\frac{1}{2}} |Z|_2^2. \ (2.9)
\end{aligned}
$$

Since $\overline{b_0} > \sum_{k=1}^{n} b_k \max_{t\in R}(\frac{1}{1-\tau_k'(t)})^{\frac{1}{2}}$, we have

$$Z(t) \equiv 0 \text{ for all } t \in R.$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in R$. Therefore, (1.1) has at most one T-periodic solution. The proof of Lemma 2.3 is now complete. □

3. Main results

Theorem 3.1. *Let* (A_1) *and* (A_2) *hold. Then equation (1.1) has a unique T-periodic solution in* C_T^1.

Proof. By Lemma 2.3, it is easy to see that equation (1.1) has at most one T-periodic solution in C_T^1. Thus, in order to prove Theorem 3.1, it suffices to show that equation (1.1) has at least one T-periodic solution in C_T^1. To do this, we are going to apply Lemma 2.1. Firstly, we claim that the set of all possible T-periodic solutions of equation (2.2) in C_T^1 is bounded.

Let $x(t) \in C_T^1$ be a T-periodic solution of equation (2.2). Multiplying $x(t)$ and (2.2) and then integrating it from 0 to T, we have

$$-\int_0^T \varphi_p(x'(t))x'(t)dt + \lambda\int_0^T x(t)[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) - e(t)]dt = 0. \quad (3.1)$$

Since $x(0) = x(T)$, then there exists $t_0 \in [0, T]$ such that $x'(t_0) = 0$. And since $\varphi_p(0) = 0$, integrating (2.2) from 0 to T, we have

$$|\varphi_p(x'(t))| = |\int_{t_0}^{t} (\varphi_p(x'(s)))'\, ds| \le \lambda \int_{t_0}^{t_0+T} |g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) - e(t)|dt, \quad (3.2)$$

where $t \in [t_0,\ t_0+T]$.

In view of (3.1), (A_2) and Schwarz inequality, we get

$$\begin{aligned}
\overline{b_0}|x|_2^2 &= \overline{b_0}\int_0^T |x(t)|^2 dt \\
&\le -\int_0^T (x(t)-0)(g_0(t,x(t))-g_0(t,0))dt \\
&= -\int_0^T \tfrac{1}{\lambda}(\varphi_p(x'(t))x'(t)dt + \sum_{k=1}^{n}\int_0^T [g_k(t,x(t-\tau_k(t)))-g_k(t,0)]x(t)dt \\
&\quad + \sum_{k=0}^{n}\int_0^T g_k(t,0)x(t)dt - \int_0^T x(t)e(t)dt \\
&\le \sum_{k=1}^{n} b_k\int_0^T |x(t-\tau_k(t))||x(t)|dt + \sum_{k=0}^{n}\int_0^T |g_k(t,0)||x(t)|dt \\
&\quad + \sqrt{T}|e|_\infty |x|_2 \\
&\le \sum_{k=1}^{n} b_k(\int_0^T |x(t-\tau_k(t))|^2 dt)^{\frac{1}{2}}|x|_2 + \sqrt{T}\sum_{k=0}^{n}|g_k(t,0)|_\infty |x|_2 \\
&\quad + \sqrt{T}|e|_\infty |x|_2 \\
&= \sum_{k=1}^{n} b_k \max_{t\in R}(\tfrac{1}{1-\tau_k'(t)})^{\frac{1}{2}}|x|_2^2 + \sqrt{T}\sum_{k=0}^{n}|g_k(t,0)|_\infty |x|_2 + \sqrt{T}|e|_\infty |x|_2. \quad (3.3)
\end{aligned}$$

It follows that

$$|x|_2 \le \frac{\sqrt{T}\sum\limits_{k=0}^{n}|g_k(t,0)|_\infty + \sqrt{T}|e|_\infty}{\overline{b_0} - \sum\limits_{k=1}^{n} b_k \max\limits_{t\in R}(\frac{1}{1-\tau_k'(t)})^{\frac{1}{2}}} := \theta. \tag{3.4}$$

Again from (A_2) and Schwarz inequality, (3.2) and (3.4) yield

$$\begin{aligned}
|x'|_\infty^{p-1} &= \max_{t\in[t_0,\ t_0+T]}\{|\varphi_p(x'(t))|\} = \max_{t\in[t_0,\ t_0+T]}\{|\int_{t_0}^{t}(\varphi_p(x'(s)))'\,ds|\} \\
&\le \int_{t_0}^{t_0+T} |g_0(t,x(t)) + \sum_{k=1}^{n} g_k(t,x(t-\tau_k(t))) - e(t)|dt \\
&= \int_0^T |g_0(t,x(t)) + \sum_{k=1}^{n} g_k(t,x(t-\tau_k(t))) - e(t)|dt \\
&\le \int_0^T |g_0(t,x(t)) - g_0(t,0)|dt + \sum_{k=1}^{n}\int_0^T |g_k(t,x(t-\tau_k(t))) - g_k(t,0)|dt \\
&\quad + \sum_{k=0}^{n}\int_0^T |g_k(t,0)|dt + T|e|_\infty \\
&\le b_0\int_0^T |x(t)|dt + \sum_{k=1}^{n}\int_0^T b_k|x(t-\tau_k(t))|dt + \sum_{k=0}^{n} T|g_k(t,0)|_\infty + T|e|_\infty \\
&\le b_0\sqrt{T}|x|_2 + \sum_{k=1}^{n} b_k\max_{t\in R}(\frac{1}{1-\tau_k'(t)})^{\frac{1}{2}}\sqrt{T}|x|_2 + \sum_{k=0}^{n} T|g_k(t,0)|_\infty + T|e|_\infty \\
&\le b_0\sqrt{T}\theta + \sum_{k=1}^{n} b_k\max_{t\in R}(\frac{1}{1-\tau_k'(t)})^{\frac{1}{2}}\sqrt{T}\theta + \sum_{k=0}^{n} T|g_k(t,0)|_\infty + T|e|_\infty \\
&:= \overline{\eta},
\end{aligned}$$

which, together with (2.3), implies that there exists a positive constant $M > 1 + (\overline{\eta})^{\frac{1}{p-1}}$ such that for all $t \in R$,

$$|x'|_\infty < M, \quad |x|_\infty \le d + \frac{1}{2}\sqrt{T}|x'|_2 \le d + \frac{1}{2}T|x'|_\infty < M.$$

Set

$$\Omega = \{x \in C_T^1 : |x|_\infty < M, |x'|_\infty < M\},$$

then we know that equation(2.2) has no T-periodic solution on $\partial\Omega$ as $\lambda \in (0,1)$ and when $x(t) \in \partial\Omega \bigcap R$, $x(t) = M$ or $x(t) = -M$, from (A_1) , we can see that

$$\begin{aligned} \frac{1}{T}\int_0^T \{-g_0(t,M) - \sum_{k=1}^n g_k(t,M) + e(t)\}\,dt &> 0, \\ \frac{1}{T}\int_0^T \{-g_0(t,-M) - \sum_{k=1}^n g_k(t,-M) + e(t)\}\,dt &< 0, \end{aligned}$$

so condition (ii) of Lemma 2.1 is also satisfied. Set

$$H(x,\mu) = \mu x - (1-\mu)\frac{1}{T}\int_0^T [g_0(t,x) + \sum_{k=1}^n g_k(t,x) - e(t)]\,dt,$$

and when $x \in \partial\Omega \bigcap R$, $\mu \in [0,1]$ we have

$$xH(x,\mu) = \mu x^2 - (1-\mu)x\frac{1}{T}\int_0^T [g_0(t,x) + \sum_{k=1}^n g_k(t,x) - e(t)]\,dt > 0.$$

Thus $H(x,\mu)$ is a homotopic transformation and

$$\begin{aligned} \deg\{F, \Omega\bigcap R, 0\} &= \deg\{-\frac{1}{T}\int_0^T [g_0(t,x) + \sum_{k=1}^n g_k(t,x) - e(t)]\,dt, \Omega\bigcap R, 0\} \\ &= \deg\{x, \Omega\bigcap R, 0\} \ne 0. \end{aligned}$$

so condition (iii) of Lemma 2.1 is satisfied. In view of the previous Lemma 2.1, equation (1.1) has at least one solution with period T. This completes the proof. □

4. Example and Remark

Example 4.1. Let $p = 4$, $g_0(t,x) = -10e^{20+\sin t}x$, $g_1(t,x) = -\frac{1}{200}e^{2+\sin t}\sin x$ and $g_2(t,x) = -\frac{1}{300}e^{3+\cos t}\cos x$ for all $t, x \in R$. Then, the following Duffing type p-Laplacian equation with two deviating arguments

$$(\varphi_p x'(t))' + 55x'(t) + g_0(t,x(t)) + g_1(t,x(t - \frac{1}{2}\sin t)) + g_2(t,x(t - \frac{1}{2}\cos t)) = \cos t \quad (4.1)$$

has a unique 2π-periodic solution since all the conditions needed in Theorem 3.1 are satisfied.

Remark 4.1. Notice that the following assumptions:

$g_k(t,x)$ are strict monotone in their second variable for any $t \in R, k = 1, 2, \cdots, n$, has been considered as fundamental for the considered existence and uniqueness of periodic solutions of second-order differential equations with multiple delays. We refer the reader to [1, 6, 7, 8] and the references cited therein. In view of the fact that $g_1(t,x) = -\frac{1}{200}e^{2+\sin t}\sin x$ and $g_2(t,x) = -\frac{1}{300}e^{3+\cos t}\cos x$ are not strict monotone in their second variable, so the results obtained in [3,4,6,10] and the references cited therein can not be applicable to equation (4.1). Moreover, the results in [1,7,8,9,11] obtained on Duffing type p-Laplacian equation without multiple deviating arguments also can not be applicable to equation (4.1). This implies that the results of this paper are essentially new.

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