

DRINKING AS AN EPIDEMIC: A MATHEMATICAL MODEL WITH DYNAMIC BEHAVIOUR

SWARNALI SHARMA AND G. P. SAMANTA*

ABSTRACT. In this paper we have developed a mathematical model of alcohol abuse. It consists of four compartments corresponding to four population classes, namely, moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class. Basic reproduction number R_0 has been determined. Sensitivity analysis of R_0 identifies β_1 , the transmission coefficient from moderate and occasional drinker to heavy drinker, as the most useful parameter to target for the reduction of R_0 . The model is locally asymptotically stable at disease free or problem free equilibrium (DFE) E_0 when $R_0 < 1$. It is found that, when $R_0 = 1$, a backward bifurcation can occur and when $R_0 > 1$, the endemic equilibrium E^* becomes stable. Further analysis gives the global asymptotic stability of DFE. Our aim of this analysis is to identify the parameters of interest for further study with a view for informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

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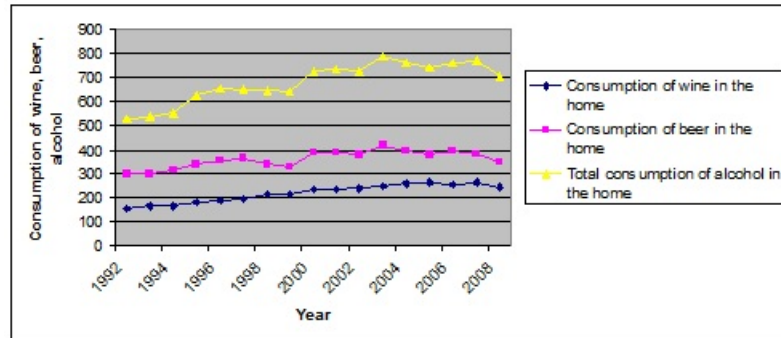
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1. Introduction

Alcohol abuse is a disease that is characterized by the sufferer having a pattern of drinking excessively despite the negative effects of alcohol on the individual's work, health, educational and social life. The National Institute on Alcohol Abuse and Alcoholism [23,24] estimates that 18 million Americans suffer from alcohol abuse or dependence. Alcohol related problems cost the United States (U.S.) nearly \$185 billion annually while alcohol abuse was responsible for nearly 80,000 fatalities per year during 2001-2005 and it is now the third leading cause of death in the U.S. [9,10,11,12,15,21,27]. The World Health Organization (WHO) estimates that there are about 2 billion people worldwide who consume alcoholic beverages and 76.3 million with diagnosable alcohol use disorders. Alcohol abuse

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affects about 20% of men and 10% of women in the U.S., most beginning by their mid teens [9,10,11,12,15,27,37]. Overall consumption of alcohol in the home (which was rising faster than outside the home) in the latest years [25] is given below:



Alcohol consumption has health and social consequences via intoxication (drunkenness), alcohol dependence and other biochemical effects of alcohol. Overall there is a casual relationship between alcohol consumption and more than 60 types of diseases and injury. Alcohol is estimated to cause about 20-30% of esophageal cancer, liver cancer, cirrhosis of the liver, homicide, epileptic seizures and motor vehicle accidents worldwide (WHO,2002). Alcohol is involved in nearly half of all violent deaths involving teens. Prevention and control efforts that include treatment and education programmes that target specific populations including children or adolescents [9,10,11,12] are in need of improvement. Among the many problems confronting these programmes are the very high rates of relapse after treatment that are observed. Up to 70% of treated alcohol abusers relapse after treatment [9,10,11,12,27,28]. Developing comprehensive, effective and sustainable strategies of prevention and management of alcohol abuse requires a multi-sectoral approach, involving health care professionals, policy makers, psychiatrists and researchers. The two major forms of intervention include: (i) prevention initiation into alcohol abuse and (ii) rehabilitation of alcohol abusers. Mathematical studies can be particularly effective as guides to the evaluation, testing and implementation strategies over short or long time scales. This is particularly true in the study of chronic relapsing diseases such as alcohol addiction. While social problems such as alcohol and drug use have been referred to in terms of epidemics, little has been published on Mathematical Modelling methods to such problems while there are many mathematical models for other epidemic problems [3,5,13,16,17,18,19,29,31,33].

Two very interesting models have recently been proposed. One for treating heroin users proposed by White and Comiskey [38], and a similar one for those with alcohol problems proposed by Sánchez et. al. [32]. Both models divide the mathematical problem into three classes, namely susceptible, heroin users or alcoholics and heroin users or alcoholics undergoing treatment. In fact, the

two types of model are very similar. The one of Sánchez et. al. [32] defers from that of White and Comiskey [38] only in that, she assumes the same death/removal rate for each of the three classes, whereas White and Comiskey [38] correctly allow the drug users and those in treatment to have enhanced death rates. After that the heroin epidemic model of White and Comiskey [38] was revisited and modified by many researchers like Mulone and Straughan [22], Liu and Zhang [20], Nyabadza and Hove-Musekwa [26], Samanta [30] etc. There are some discussions [4,14] based on the model of Sánchez et. al. [32]. There are also many useful mathematical epidemic models which have generated useful insights about the role of behaviour on the transmission dynamics of sexually transmitted diseases like gonorrhoea [7,18] or HIV [33], about the intensity and frequency of travel on the spread of communicable diseases such as SARS [13] etc.

In this paper, we have modified the alcohol abuse model proposed by Sánchez et. al. [32]. We have divided the mathematical problem into four classes, namely, moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class. Next we have found the basic reproduction number R_0 [34,35]. Then the stability analysis of the model is made using the basic reproduction number. We have found that the model is locally asymptotically stable at disease free equilibrium E_0 when $R_0 < 1$. When $R_0 = 1$, a backward bifurcation can occur although R_0 may be less than 1, an endemic equilibrium exists. This case shows that it is not enough to only reduce R_0 to less than one to eliminate the problem and that when R_0 crosses unity, hysteresis takes place. When $R_0 > 1$ endemic equilibrium exists and becomes stable. Next we have found the conditions of global asymptotic stability of E_0 and the conditions of local asymptotic stability of E^* by Routh-Hurwitz criterion. Next we have illustrated some of the key findings through numerical simulations followed by conclusions. Our aim of introducing this four compartmental alcohol abuse model is to identify the parameters of interest for further study, with a view for informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

2. Mathematical Model

In this section we formulate a mathematical model of alcohol abuse. The adult human population is divided into four different classes, namely, moderate and occasional drinkers with frequency $S(t)$, heavy drinkers with frequency $D(t)$, drinkers in treatment with frequency $T(t)$, temporarily recovered class with frequency $R(t)$ at time t .

We diagrammatically represent the flow of individuals from one class to the other in figure below.

The model can be presented by the following set of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S(t) \frac{D(t)}{N} - \mu S(t) + \beta_3 R(t) \frac{S(t)}{N} \\
\frac{dD}{dt} &= \beta_1 S(t) \frac{D(t)}{N} + \beta_2 T(t) \frac{D(t)}{N} - (\mu + \delta_1 + \phi) D(t) \\
\frac{dT}{dt} &= \phi D(t) - \beta_2 T(t) \frac{D(t)}{N} - (\mu + \delta_2 + \sigma) T(t) \\
\frac{dR}{dt} &= \sigma T(t) - \mu R(t) - \beta_3 R(t) \frac{S(t)}{N}
\end{aligned} \tag{1}$$

with initial densities

$$S(0) > 0, D(0) \geq 0, T(0) \geq 0, R(0) \geq 0. \tag{2}$$

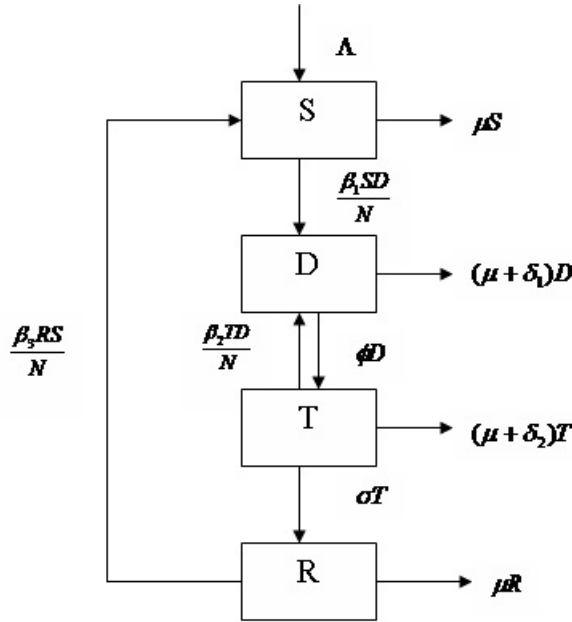


Fig.1. Transfer diagram of the alcohol abuse model

The model parameters are described below:

Λ : Recruitment rate of moderate and occasional drinkers,

β_1 : The transmission coefficient from moderate and occasional drinkers to heavy drinkers,

β_2 : The transmission coefficient from drinker in treatment to heavy drinker,

β_3 : The transmission coefficient from recovered class to moderate and occasional drinkers,

μ : Natural death rate of population,

δ_1 : Drinking related death rate of heavy drinkers,

δ_2 : Drinking related death rate of drinkers in treatment,

ϕ : The proportion of drinkers who enter treatment,

σ : Recovery rate of drinkers in treatment.

Clearly the model involves certain assumptions. These consist of the following:

(i) The population we are studying is isolated and closed. This results in the total population size remaining constant. Every number of the population can be assigned to one of the subgroups and these subgroups are mutually exclusive. This means that $N = S(t) + D(t) + T(t) + R(t)$, where N is the total size of the population, i.e., $\Lambda = \mu S + (\mu + \delta_1)D + (\mu + \delta_2)T + \mu R$, i.e., the population is assumed to be of constant size within the modelling time.

(ii) All members of the population mix homogeneously, so each individual has an equal chance of becoming a heavy drinker.

(iii) The heavy drinking is passed to moderate and occasional drinkers by adequate contact with Heavy drinkers not in treatment.

(iv) Heavy drinkers not in treatment are infectious to moderate and occasional drinkers and to drinkers in treatment.

(v) Drinkers in treatment are not infectious to moderate and occasional drinkers.

(vi) The drinkers in treatment most commonly relapse due to contact with heavy drinkers who are not in treatment.

(vii) Those who stop drinking alcohol enter to the temporarily recovered class and one part of which relapse to the moderate and occasional drinkers' class.

(viii) The population in temporarily recovered class relapse due to the contact with moderate and occasional drinkers.

3. Basic Properties

3.1. Invariant Region

Theorem 1. *The feasible region G defined by*

$$G = \{(S(t), D(t), T(t), R(t)) \in \mathbb{R}_+^4 : 0 < N \leq \frac{\Lambda}{\mu}\}$$

with initial conditions $S(0) > 0, D(0) \geq 0, T(0) \geq 0, R(0) \geq 0$, is positively invariant.

Proof. Adding the equations of the system (1) we obtain

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \mu N - \delta_1 D(t) - \delta_2 T(t) \\ &\leq \Lambda - \mu N \end{aligned} \quad (3)$$

The solution $N(t)$ of the differential equation (3) has the following property,

$$0 \leq N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t})$$

where $N(0)$ represents the sum of the initial values of the variables. As $t \rightarrow \infty, 0 \leq N \leq \frac{\Lambda}{\mu}$. So if $N(0) \leq \frac{\Lambda}{\mu}$ then $\lim_{t \rightarrow \infty} N(t) \leq \frac{\Lambda}{\mu}$. This means that $\frac{\Lambda}{\mu}$ is the upper bound of N . On the other hand, if $N(0) > \frac{\Lambda}{\mu}$, then $N(t)$ will decrease

to $\frac{\Lambda}{\mu}$. This means that if $N(0) > \frac{\Lambda}{\mu}$, then the solution $(S(t), D(t), T(t), R(t))$ enters G or approach it asymptotically. Hence it is positively invariant under the flow induced by the system (1). Thus in G , the model (1) is well-posed epidemiologically and mathematically. Hence it is sufficient to study the dynamics of the model in G . \square

3.2. Positivity of Solutions

Theorem 2. *Given the initial conditions of system (1) are $S(0) > 0, D(0) \geq 0, T(0) \geq 0$ and $R(0) \geq 0$. $S(t), D(t), T(t), R(t)$ are positive for all $t \geq \bar{t}$, where $\bar{t} = \inf\{t > 0 : S(t) > 0, D(t) > 0, T(t) > 0, R(t) > 0\}$.*

Proof. Here

$$\bar{t} = \inf\{t > 0 : S(t) > 0, D(t) > 0, T(t) > 0, R(t) > 0\}.$$

Thus $\bar{t} > 0$ and it follows from the 1st equation of the system (1) that,

$$\frac{dS}{dt} = \Lambda - \left[\beta_1 \frac{D(t)}{N} + \mu - \beta_3 \frac{R(t)}{N} \right] S(t).$$

We thus have,

$$\begin{aligned} & \frac{d}{dt} \left[S(t) \exp \left\{ \mu t + \frac{1}{N} \int_0^t (\beta_1 D(s) - \beta_3 R(s)) ds \right\} \right] \\ &= \Lambda \exp \left\{ \mu t + \frac{1}{N} \int_0^t (\beta_1 D(s) - \beta_3 R(s)) ds \right\}. \end{aligned}$$

Hence

$$\begin{aligned} & S(t) \exp \left\{ \mu t + \frac{1}{N} \int_0^t (\beta_1 D(s) - \beta_3 R(s)) ds \right\} \\ & - S(\bar{t}) \exp \left\{ \mu \bar{t} + \frac{1}{N} \int_0^{\bar{t}} (\beta_1 D(s) - \beta_3 R(s)) ds \right\} \\ &= \int_{\bar{t}}^t \Lambda \exp \left\{ \mu t + \frac{1}{N} \int_0^t (\beta_1 D(\omega) - \beta_3 R(\omega)) d\omega \right\} dt, \end{aligned}$$

so that

$$\begin{aligned} S(t) &= S(\bar{t}) \exp \left[- \left\{ \mu(t - \bar{t}) + \frac{1}{N} \int_{\bar{t}}^t (\beta_1 D(s) - \beta_3 R(s)) ds \right\} \right] \\ &+ \exp \left[- \left\{ \mu t + \frac{1}{N} \int_0^t (\beta_1 D(s) - \beta_3 R(s)) ds \right\} \right] \\ & \left[\int_{\bar{t}}^t \Lambda \exp \left\{ \mu t + \frac{1}{N} \int_0^t (\beta_1 D(\omega) - \beta_3 R(\omega)) d\omega \right\} dt \right] > 0. \end{aligned}$$

From the 2nd equation of (1), we have

$$\begin{aligned} \frac{dD}{dt} &\geq -(\mu + \delta_1 + \phi)D(t) \\ \Rightarrow D(t) &\geq D(\bar{t}) \exp[-(\mu + \delta_1 + \phi)(t - \bar{t})] > 0. \end{aligned}$$

Similarly, from the 3rd equation of (1), we have

$$\begin{aligned} \frac{dT}{dt} &\geq - \left[\beta_2 \frac{D(t)}{N} + (\mu + \delta_2 + \sigma) \right] T(t) \\ \Rightarrow T(t) &\geq T(\bar{t}) \exp \left[- \left\{ (\mu + \delta_2 + \sigma)(t - \bar{t}) + \frac{1}{N} \int_{\bar{t}}^t \beta_2 D(s) ds \right\} \right] > 0. \end{aligned}$$

Similarly, from the 4th equation of (1), we have

$$\begin{aligned} \frac{dR}{dt} &\geq - \left[\mu + \beta_3 \frac{S(t)}{N} \right] R(t) \\ \Rightarrow R(t) &\geq R(\bar{t}) \exp \left[- \left\{ \mu(t - \bar{t}) + \frac{1}{N} \int_{\bar{t}}^t \beta_3 S(s) ds \right\} \right] > 0. \end{aligned}$$

Therefore, we can see that $S(t) > 0, D(t) > 0, T(t) > 0, R(t) > 0, \forall t \geq \bar{t} > 0$. This completes the proof. \square

4. The Basic Reproduction Number R_0

Basic reproduction number R_0 is defined as the number of heavy drinkers produced when a single drinker is introduced into moderate and occasional drinkers' population [34,35]. In this model, the basic reproduction number is the transmission coefficient from moderate and occasional drinker to heavy drinker divided by the sum of the natural death rate of the population, the drinking related death rate of heavy drinkers who are not in treatment and the proportion of individuals who enter treatment, i.e.,

$$R_0 = \frac{\beta_1}{\mu + \delta_1 + \phi}. \quad (4)$$

5. Sensitivity Analysis of R_0

To examine the sensitivity of R_0 to each of its parameters, following Arriola and Hyman [1], the normalized forward sensitivity index with respect to each of the parameters is calculated.

$$A_{\beta_1} = \frac{\frac{\partial R_0}{\partial \beta_1}}{\frac{R_0}{\beta_1}} = \frac{\beta_1}{R_0} \frac{\partial R_0}{\partial \beta_1} = \beta_1 \left(\frac{\mu + \delta_1 + \phi}{\beta_1} \right) \left(\frac{1}{\mu + \delta_1 + \phi} \right) = 1 \quad (5)$$

$$A_{\mu} = \frac{\frac{\partial R_0}{\partial \mu}}{\frac{R_0}{\mu}} = \frac{\mu}{R_0} \frac{\partial R_0}{\partial \mu} = \left| \frac{-\mu}{\mu + \delta_1 + \phi} \right| < 1,$$

$$A_{\delta_1} = \frac{\frac{\partial R_0}{\partial \delta_1}}{\frac{R_0}{\delta_1}} = \frac{\delta_1}{R_0} \frac{\partial R_0}{\partial \delta_1} = \left| \frac{-\delta_1}{\mu + \delta_1 + \phi} \right| < 1.$$

$$A_\phi = \frac{\frac{\partial R_0}{R_0}}{\frac{\partial \phi}{\phi}} = \frac{\phi}{R_0} \frac{\partial R_0}{\partial \phi} = \left| \frac{-\phi}{\mu + \delta_1 + \phi} \right| < 1,$$

We conclude that the basic reproduction number (R_0) is most sensitive to changes in β_1 . An increase in β_1 will cause an increase in R_0 with same proportion and a decrease in β_1 will cause a decrease in R_0 in same proportion. μ , δ_1 and ϕ have an inversely proportional relationship with R_0 , so an increase in any of them will bring about a decrease in R_0 , however, the size of the decrease will be proportionally smaller. Recall that μ is the natural death rate of the population and δ_1 is the drinking related death rate of the heavy drinkers not in treatment. It is clear that increase in either of these rates is neither ethical nor practical. Thus we choose to focus on one of two parameters: either ϕ , the proportion of drinkers who enter treatment or β_1 , the transmission rate from moderate and occasional drinker to heavy drinker. Given R_0 's sensitivity to β_1 and in the knowledge that a treatment cycle exists (individuals who enter treatment are likely to relapse and re-enter treatment), it seems sensible to focus efforts on the reduction of β_1 . In other words, this sensitivity analysis tells us that **prevention is better than cure**. Efforts to increase prevention are more effective in controlling the spread of habitual drinkers than efforts to increase the numbers of individuals accessing treatment.

6. Stability of Drinking Free or Problem Free Equilibrium, E_0 when $R_0 < 1$

We now use R_0 to determine the existence of equilibria for the system. In this section we will study the local stability behavior of the model system (1) at the drug free equilibrium (DFE) $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$.

Now, the variational matrix of system (1) at $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$ is given by

$$V(E_0) = \begin{pmatrix} -\mu & -\beta_1 & 0 & \beta_3 \\ 0 & \beta_1 - (\mu + \delta_1 + \phi) & 0 & 0 \\ 0 & \phi & -(\mu + \delta_2 + \sigma) & 0 \\ 0 & 0 & \sigma & -(\mu + \beta_3) \end{pmatrix}$$

Therefore, eigen values of the characteristic equation of $V(E_0)$ are

$\lambda_1 = -\mu$, $\lambda_2 = \beta_1 - (\mu + \delta_1 + \phi)$, $\lambda_3 = -(\mu + \sigma + \delta_2)$, $\lambda_4 = -(\mu + \beta_3)$. Here, λ_1 , λ_3 and λ_4 are clearly real and negative. Also as $R_0 < 1$, then

$$\beta_1 < \mu + \delta_1 + \phi$$

and therefore λ_2 is also real and negative. Therefore the system (1) shows local asymptotic stability at $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$. So, we arrive to the following result:

Theorem 3. *The problem free equilibrium E_0 of the model system (1) is locally asymptotically stable if $R_0 < 1$.*

7. Analysis at $R_0 = 1$

In this section, we determine the stability of heavy drinking persistent equilibrium or problem persistent equilibrium and investigate the possibility of occurring backward bifurcation [2,16,33,34,35,36,39] due to existence of multiple equilibrium. To analyze it for the system (1), we use the center manifold theory [6]. To apply this method, we first change the variables of the model equation (1) so that $S = x_1, D = x_2, T = x_3, R = x_4$ with $\frac{dx_1}{dt} = f_1, \frac{dx_2}{dt} = f_2, \frac{dx_3}{dt} = f_3, \frac{dx_4}{dt} = f_4$.

Thus system (1) becomes,

$$\begin{aligned} f_1 &= \Lambda - \frac{\beta_1 x_1 x_2}{N} - \mu x_1 + \frac{\beta_3 x_4 x_1}{N} \\ f_2 &= \frac{\beta_1 x_1 x_2}{N} + \frac{\beta_2 x_3 x_2}{N} - (\mu + \delta_1 + \phi) x_2 \\ f_3 &= \phi x_2 - \frac{\beta_2 x_3 x_2}{N} - (\mu + \delta_2 + \sigma) x_3 \\ f_4 &= \sigma x_3 - \mu x_4 - \frac{\beta_3 x_4 x_1}{N} \end{aligned} \tag{6}$$

We choose $\beta_1^* = \beta_1$ as the bifurcation parameter, particularly as it has been shown in equation (5) that R_0 is more sensitive to change in β_1 than in its other parameters. If we consider $R_0 = 1$, then we obtain,

$$\beta_1^* = \mu + \delta_1 + \phi. \tag{7}$$

Now, the Jacobian of the linearized system (6) using identity (7) at problem free equilibrium E_0 when $\beta_1^* = \beta_1$ is given by,

$$J(\beta_1^*) = \begin{pmatrix} -\mu & -\beta_1^* & 0 & \beta_3 \\ 0 & 0 & 0 & 0 \\ 0 & \phi & -(\mu + \delta_2 + \sigma) & 0 \\ 0 & 0 & \sigma & -(\mu + \beta_3) \end{pmatrix} \tag{8}$$

The matrix (8) has eigenvalues $(0, -\mu, -(\mu + \delta_2 + \sigma), -(\mu + \beta_3))^T$, which meets the requirement of simple zero eigenvalue and the others having negative real part. We can thus use the center manifold theory [6] to analyze the dynamics of system (6). The right eigenvector associated with zero eigenvalue is given by, $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$, where

$$\begin{aligned}\omega_1 &= \frac{1}{\mu} \left\{ \beta_3 - \frac{\beta_1^*(\mu + \delta_2 + \sigma)(\mu + \beta_3)}{\phi\sigma} \right\} \omega_4 \\ \omega_2 &= \frac{(\mu + \delta_2 + \sigma)(\mu + \beta_3)}{\phi\sigma} \omega_4 \\ \omega_3 &= \frac{(\mu + \beta_3)}{\sigma} \omega_4 \\ \omega_4 &= 1\end{aligned}$$

with ω_4 free. Further, $J(\beta_1^*)$ has a corresponding left eigen vector $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$, where

$$\begin{aligned}\nu_1 &= 0 \\ \nu_2 &= 1 \\ \nu_3 &= 0 \\ \nu_4 &= 0\end{aligned}$$

with ν_2 free. In order to establish the local stability of E^* , we use the following theorem.

Theorem 4 ([8]). *Consider the following general system of ordinary differential equations with a parameter ϕ ,*

$$\frac{dx}{dt} = f(x, \phi), \quad f : R^n \times R \rightarrow R \quad \text{and} \quad f \in C^2(R^n \times R), \quad (9)$$

where 0 is an equilibrium of the system that is $f(0, \phi) = 0, \quad \forall \phi$ and assume:

A1. $A = D_x f(0, 0) = \left(\frac{\partial f_i}{\partial x_j}(0, 0) \right)$ is linearization matrix of the system (9) around the equilibrium 0 with ϕ evaluated at 0 . Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts;

A2. Matrix A has a right eigenvector u and a left eigenvector ν corresponding to the zero eigenvalue.

Let f_k be the k -th component of f and

$$\begin{aligned}a &= \sum_{k,i,j=1}^n \nu_k u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0), \\ b &= \sum_{k,j=1}^n \nu_k u_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0, 0).\end{aligned} \quad (10)$$

The local dynamics of (9) around 0 are totally governed by a and b .

(i) $a, b > 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \phi \ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium,

(ii) $a < 0, b < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; when $0 < \phi \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium,

(iii) $a > 0, b < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 1$, 0 is stable, and a positive unstable equilibrium appears,

(iv) $a < 0, b > 0$. When ϕ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

The computation of a and b are necessary to apply the Theorem 1.

In particular, since $\nu_1 = \nu_3 = \nu_4 = 0$,

$$a = \nu_2 \sum_{i,j=1}^4 \omega_i \omega_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}(0,0)$$

$$\text{and } b = \nu_2 \sum_{i=1}^4 \omega_i \frac{\partial^2 f_2}{\partial x_i \partial \beta_1}(0,0)$$

For the system (6), the associated non-zero partial derivatives at the problem free equilibrium are given by:

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \frac{\mu(\mu + \delta_1 + \phi)}{\Lambda},$$

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_2} = \frac{\beta_2 \mu}{\Lambda},$$

$$\frac{\partial^2 f_2}{\partial x_2 \partial \beta_1} = 1.$$

It thus follows that,

$$a = \frac{2\mu\omega_2}{\Lambda}(X - \Gamma), \quad (11)$$

where

$$X = \frac{\beta_3}{\mu}(\mu + \delta_1 + \phi) + \frac{\beta_2}{\sigma}(\mu + \beta_3), \quad (11a)$$

$$\Gamma = \frac{(\mu + \delta_1 + \phi)^2(\mu + \delta_2 + \sigma)(\mu + \beta_3)}{\phi\sigma\mu}, \quad (11b)$$

$$\text{and } b = \omega_2 > 0. \quad (11c)$$

Hence the sign of a depends on the values of X and Γ , so that if $X > \Gamma$, then $a > 0$ and if $X < \Gamma$, then $a < 0$ while $b > 0$ always. Thus, we have the following result:

Theorem 5. *If $X > \Gamma$, then the system (1) has a backward bifurcation at $R_0 = 1$, otherwise if $X < \Gamma$, then it undergoes forward bifurcation and the endemic equilibrium is locally asymptotically stable for $R_0 > 1$, but close to one.*

8. Existence of Endemic Equilibrium $E^*(S^*, D^*, T^*, R^*)$ when $R_0 > 1$

In this section, we analyze the existence of non-trivial endemic equilibrium of system (1). At an endemic equilibrium, disease is present and the followings hold:

$$\begin{aligned} S > 0, D > 0, T > 0, R > 0, \\ \frac{dS}{dt} = \frac{dD}{dt} = \frac{dT}{dt} = \frac{dR}{dt} = 0. \end{aligned}$$

Solving the equations of system (1) at equilibrium state we get,

$$\begin{aligned} S^* &= \frac{N\{b_1(b_2 + \beta_2 D^*) - \beta_2 \phi D^*\}}{\beta_1(b_2 + \beta_2 D^*)}, \\ &= \frac{N\{b_1 b_2 + (\mu + \delta_1)\beta_2 D^*\}}{\beta_1(b_2 + \beta_2 D^*)}, \\ T^* &= \frac{N\phi D^*}{(b_2 + \beta_2 D^*)}, \\ R^* &= \frac{\beta_1 N \sigma \phi D^*}{\mu \beta_1 (b_2 + \beta_2 D^*) + \beta_3 \{b_1(b_2 + \beta_2 D^*) - \beta_2 \phi D^*\}}, \\ &= \frac{\beta_1 N \sigma \phi D^*}{\mu \beta_1 (b_2 + \beta_2 D^*) + \beta_3 \{b_1 b_2 + (\mu + \delta_1)\beta_2 D^*\}}, \end{aligned}$$

where

$$\begin{aligned} b_1 &= \mu + \delta_1 + \phi, \\ b_2 &= N(\mu + \delta_2 + \sigma). \end{aligned}$$

Now, putting the values of S^* , T^* , R^* into the first equation of (1) and simplifying we obtain,

$$a_1(D^*)^3 + a_2(D^*)^2 + a_3 D^* + a_4 = 0, \quad (12)$$

where

$$\begin{aligned}
a_1 &= (b_1 - \phi)\beta_1\beta_2\{\beta_3\phi - \beta_2(\mu\beta_1 + b_1\beta_3)\}, \\
a_2 &= (b_1 - \phi)\beta_2\{(\mu\beta_1 + b_1\beta_3)(\beta_2\mu N - b_2\beta_1) - \beta_3N\phi(\beta_2\mu + \beta_1\sigma)\} \\
&\quad + \beta_1\beta_2(b_1b_2 - \Lambda\beta_2)\{\beta_3\phi - (\mu\beta_1 + b_1\beta_3)\}, \\
a_3 &= (\mu\beta_1 + b_1\beta_3)\{2\Lambda b_2\beta_1\beta_2 - b_1b_2^2\beta_1 + b_1b_2\mu N\beta_2 + (b_1 - \phi)b_2\beta_2\mu N\} \\
&\quad - b_2\beta_3(\beta_1\beta_2\Lambda\phi + b_1\beta_2\mu N + b_1\beta_1\sigma\phi N), \\
a_4 &= b_2^2(\mu\beta_1 + b_1\beta_3)(\beta_1\Lambda + b_1\mu N).
\end{aligned}$$

Obviously a_4 is positive. However, the signs of a_1, a_2, a_3 are not obvious although it is known that $\beta_1 > \mu + \delta_1 + \phi$ as $R_0 > 1$. Now using Descartes' rule of signs in equation (12) we obtain:

- (i) if $a_1 > 0, a_2 > 0, a_3 > 0$, then there is no change of sign, so there exists no positive root of equation (12),
- (ii) if $a_1 < 0, a_2 > 0, a_3 > 0$, there exists only one positive root of equation (12),
- (iii) if $a_1 < 0, a_2 < 0, a_3 > 0$, there exists only one positive root of equation (12),
- (iv) if $a_1 < 0, a_2 < 0, a_3 < 0$, there exists only one positive root of equation (12),
- (v) if $a_1 < 0, a_2 > 0, a_3 < 0$, there exists three or one positive root of equation (12),
- (vi) if $a_1 > 0, a_2 > 0, a_3 < 0$, there exists two or no positive root of equation (12),
- (vii) if $a_1 > 0, a_2 < 0, a_3 < 0$, there exists two or no positive root of equation (12),
- (viii) if $a_1 > 0, a_2 < 0, a_3 > 0$, there exists two or no positive root of equation (12).

Therefore, if a_1 is negative, then there exists at least one positive value of D^* , i.e., at least one non-trivial endemic equilibrium.

Summarizing the previous discussions we come to the following result:

Theorem 6. *If a_1 in equation (12) is negative, then there exists at least one non-trivial endemic equilibrium of the system (1).*

Observation: If we take $\beta_2 = 0$, then

$$\begin{aligned}
a_1 &= 0, \\
a_2 &= -\beta_1\beta_3N\phi\sigma < 0, \\
a_3 &= -b_1b_2^2\beta_1(\mu\beta_1 + b_1\beta_3) - b_1b_2\beta_1\beta_3N\phi\sigma < 0,
\end{aligned}$$

and a_4 is always positive.

Then the equation (12) becomes,

$$a_2(D^*)^2 + a_3D^* + a_4 = 0$$

where $a_2 < 0, a_3 < 0$ and $a_4 > 0$. So there is only one change in sign. Then by Descartes' rule of signs, there exists only one positive root of the above equation, which is given by

$$D^* = \frac{-a_3 - (a_3^2 - 4a_2a_4)^{\frac{1}{2}}}{2a_2}.$$

This implies that there is a unique endemic equilibrium point of the system (1) when $\beta_2 = 0$. Therefore in this case there is no existence of backward bifurcation as there is a unique endemic equilibrium of the system (1) when $\beta_2 = 0$.

Therefore, we can conclude that the backward bifurcation occurs because of the insufficient capacity for treatment policies. As a result the drinkers in treatment come to the direct contact of heavy drinkers and they re-enter into the heavy drinkers class.

9. Global Stability Analysis of Disease Free or Problem Free Equilibrium E_0

First, let us state the **Poincaré-Bendixson Theorem** [19]:

Statement: Consider an autonomous system of differential equations of the form

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y),$$

where F and G have continuous first order partial derivatives and the solutions of the system exists for all t . Suppose that a positive semi-orbit C^+ of this system enters and does not leave some closed bounded domain D and that there are no equilibrium points in D . Then $\omega(C^+)$ is a periodic orbit.

When system (1) has no endemic equilibrium we have the following results on the global stability of E_0 :

Theorem 7. *If $R_0 < 1$ and $a < 0$ in (11), the disease free or problem free equilibrium E_0 is globally asymptotically stable.*

Proof. As $G = \{(S(t), D(t), T(t), R(t)) \in \mathbb{R}_+^4 : 0 < N \leq \frac{\Lambda}{\mu}\}$ is an invariant set of system (1), it attracts all the positive solutions of system (1) in \mathbb{R}_+^4 . Since system (1) has no endemic equilibrium, but has only problem free equilibrium when $R_0 < 1$ and $a < 0$ where a is given in (11), it follows from the Poincaré-Bendixson's theorem that no periodic solution exist in G . Since G is a bounded positively invariant region of system (1) and E_0 is the only equilibrium point in G , the ω -limit set of every solution starting in G is nothing but E_0 . Hence the stable problem free equilibrium E_0 is globally asymptotically stable. \square

10. Stability Analysis of Endemic Equilibrium $E^*(S^*, D^*, T^*, R^*)$

using **Routh-Hurwitz Criterion**

The variational matrix of system (1) at E^* is given by,

$$V(E^*) = \begin{pmatrix} m_{11} & m_{12} & 0 & m_{14} \\ m_{21} & 0 & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ m_{41} & 0 & m_{43} & m_{44} \end{pmatrix} \quad (13)$$

where,

$$m_{11} = -\frac{\beta_1 D^*}{N} - \mu + \frac{\beta_3 R^*}{N}, \quad m_{12} = -\frac{\beta_1 S^*}{N}, \quad m_{14} = \frac{\beta_3 S^*}{N},$$

$$\begin{aligned}
m_{21} &= \frac{\beta_1 D^*}{N}, & m_{23} &= \frac{\beta_2 D^*}{N}, \\
m_{32} &= \phi - \frac{\beta_2 T^*}{N}, & m_{33} &= -\frac{\beta_2 D^*}{N} - (\mu + \delta_2 + \sigma), \\
m_{41} &= -\frac{\beta_3 R^*}{N}, & m_{43} &= \sigma, & m_{44} &= -(\mu + \frac{\beta_3 S^*}{N}).
\end{aligned}$$

Therefore, the characteristic equation of $V(E^*)$ is,

$$\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 = 0, \quad (14)$$

where,

$$\begin{aligned}
A_1 &= -m_{11} - m_{33} - m_{44}, \\
A_2 &= m_{11}m_{33} + m_{11}m_{44} + m_{33}m_{44} - m_{12}m_{21} - m_{23}m_{32} - m_{14}m_{41}, \\
A_3 &= m_{12}m_{21}m_{33} + m_{12}m_{21}m_{44} + m_{11}m_{23}m_{32} + m_{23}m_{32}m_{44} \\
&\quad + m_{14}m_{41}m_{33} - m_{11}m_{33}m_{44}, \\
A_4 &= m_{14}m_{41}m_{23}m_{32} - m_{12}m_{21}m_{33}m_{44} - m_{11}m_{23}m_{32}m_{44} \\
&\quad - m_{21}m_{14}m_{32}m_{43}.
\end{aligned}$$

By the Routh-Hurwitz criterion [19], it follows that, all eigenvalues of the characteristic equation (14) has negative real part if and only if

$$A_1 > 0, \quad A_4 > 0, \quad B = A_1 A_2 - A_3 > 0, \quad B A_3 - A_1^2 A_4 > 0. \quad (15)$$

Theorem 8. *E^* is locally asymptotically stable if and only if the above inequalities of (15) are satisfied.*

11. Numerical Simulation

Analytical studies can never be completed without numerical verification of the results. In this section we present computer simulation of some solutions of the system (1). Beside verification of our analytical findings, these numerical solutions are very important from practical point of view.

We first consider the case when $R_0 = 0.538462 < 1$ using the parameter values given in Table 1. For different initial conditions the dynamics of the model is presented in fig.2. The figure shows that only moderate and occasional drinkers' population exists ($S = 1.6$) and the populations of heavy drinkers, drinkers in treatment and temporarily recovered population declines to zero ($D = 0, T = 0, R = 0$), i.e., approaches the disease free or problem free equilibrium (DFE). It shows that DFE is locally asymptotically stable whenever $R_0 < 1$. This numerical verification supports the result stated in Theorem 3 (art.6) on the stability of DFE.

Further using the parameter values given in Table 2, we consider the case when $R_0 = 1.25 > 1$. For different initial conditions the dynamics of the model is presented in fig.3. The figure shows that moderate and occasional drinkers' population, heavy drinkers, drinkers in treatment and temporarily recovered population all exist $[(S^*, D^*, T^*, R^*) = (11.4913, 0.2395, 0.7492, 2.2854)]$, i.e., the

population of drinkers tends to drinking persistent equilibrium (DPE) or endemic equilibrium when $R_0 > 1$. This indicates that irrespective of the initial conditions the population of heavy drinkers eventually settles at endemic equilibrium with increasing time and the problem free equilibrium became unstable when $R_0 > 1$, which supports our analytical results. In this case $X = 0.329$ and $\Gamma = 1.36102$, i.e., $X < \Gamma$, so $a < 0$ (see 11,11a,11b, art.7). Therefore, it also shows the forward bifurcation of system (1) which is good agreement with Theorem 5.

Further using the parameter values given in Table 3, we consider the case when $R_0 = 0.75 < 1$. For different initial conditions the dynamics of the model is presented in fig.4. The figure shows that in this case, there exists three equilibria of the system (1), among them problem free equilibrium $E_0(36, 0, 0, 0)$ and an endemic equilibrium $E^*(17.5685, 5.7595, 1.9980, 0.2130)$ are stable and the other endemic equilibrium $E_1(25.0263, 2.8633, 1.7897, 0.1644)$ is unstable. Here $X = 13.015$ and $\Gamma = 8.32$, i.e., $X > \Gamma$, so $a > 0$ (see 11,11a,11b, art.7). Therefore, it also shows the backward bifurcation of system (1) which is also in good agreement with Theorem 5.

TABLE 1

Parameter	Values	Source
Λ	0.4	Estimated
β_1	0.7	Estimated
β_2	0.3	Estimated
β_3	0.1	Estimated
μ	0.25	Estimated
σ	0.09	Estimated
ϕ	0.7	Estimated
δ_1	0.35	Estimated
δ_2	0.3	Estimated

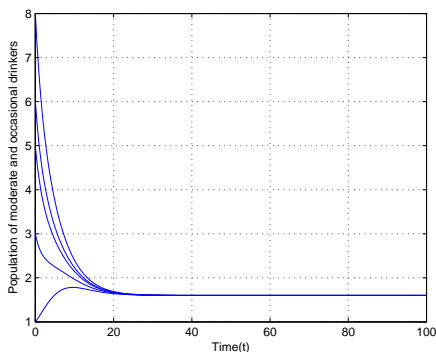


Fig.2a. Time series plot of the moderate and occasional drinkers for $R_0 = 0.538462 < 1$ with various initial conditions, parameter values are given in Table 1.

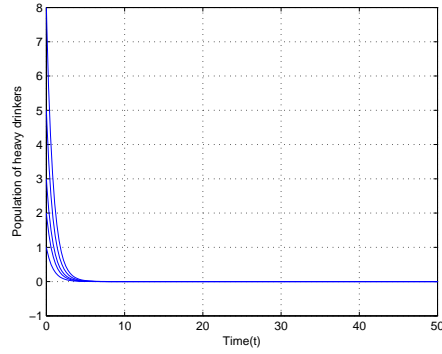


Fig.2b. Time series plot of heavy drinkers for $R_0 = 0.538462 < 1$ with various initial conditions, parameter values are given in Table 1.

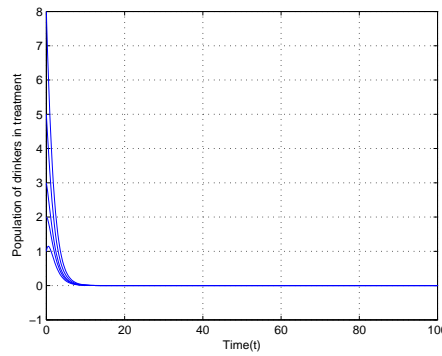


Fig.2c. Time series plot of drinkers in treatment for $R_0 = 0.538462 < 1$ with various initial conditions, parameter values are given in Table 1.

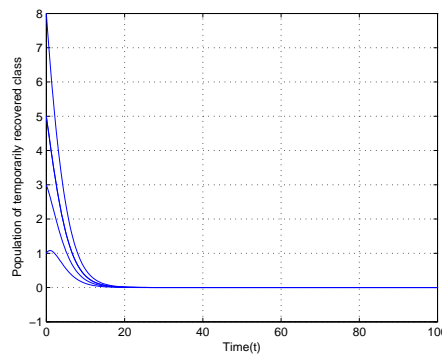


Fig.2d. Time series plot of temporarily recovered class for $R_0 = 0.538462 < 1$ with various initial conditions, parameter values are given in Table 1.

TABLE 2

Parameter	Values	Source
Λ	0.4	Estimated
β_1	0.7	Estimated
β_2	0.3	Estimated
β_3	0.01	Estimated
μ	0.025	Estimated
σ	0.1	Estimated
ϕ	0.5	Estimated
δ_1	0.035	Estimated
δ_2	0.03	Estimated

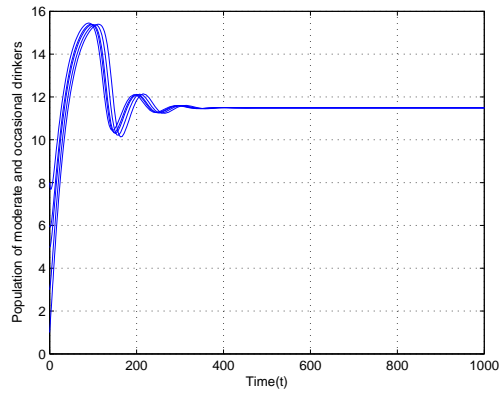


Fig.3a. Time series plot of moderate and occasional drinkers for $R_0 = 1.25 > 1$ with various initial conditions, parameter values are given in Table 2.

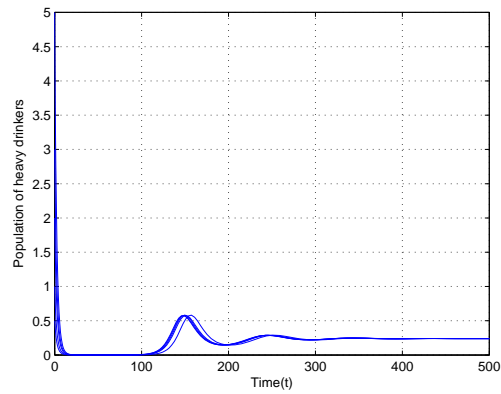


Fig.3b. Time series plot of heavy drinkers for $R_0 = 1.25 > 1$ with various initial conditions, parameter values are given in Table 2.

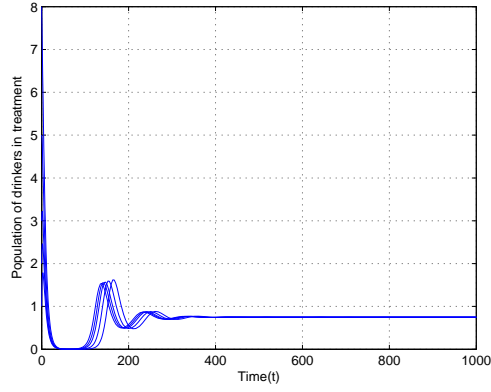


Fig.3c. Time series plot of the drinkers in treatment for $R_0 = 1.25 > 1$ with various initial conditions, parameter values are given in Table 2.

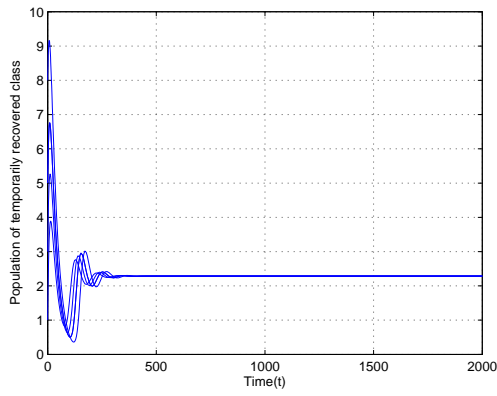


Fig.3d. Time series plot of temporarily recovered class for $R_0 = 1.25 > 1$ with various initial conditions, parameter values are given in Table 2.

TABLE 3

Parameter	Values	Source
Λ	0.9	Estimated
β_1	0.12	Estimated
β_2	0.99	Estimated
β_3	0.1	Estimated
μ	0.025	Estimated
σ	0.01	Estimated
ϕ	0.1	Estimated
δ_1	0.035	Estimated
δ_2	0.03	Estimated

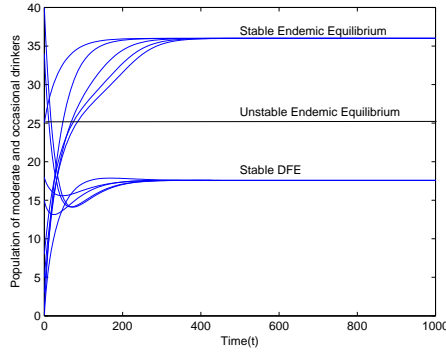


Fig.4a. Time series plot of moderate and occasional drinkers for $R_0 = 0.75 < 1$ with various initial conditions, parameter values are given in Table 3.

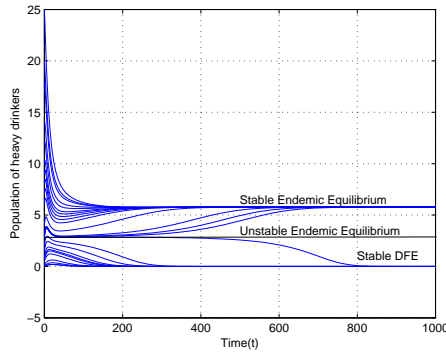


Fig.4b. Time series plot of heavy drinkers for $R_0 = 0.75 < 1$ with various initial conditions, parameter values are given in Table 3.

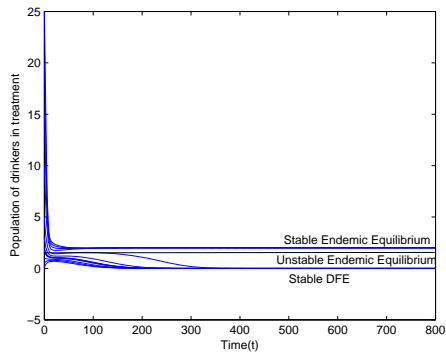


Fig.4c. Time series plot of the drinkers in treatment for $R_0 = 0.75 < 1$ with various initial conditions, parameter values are given in Table 3.

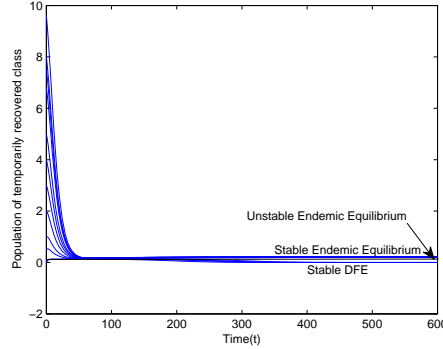


Fig.4d. Time series plot of temporarily recovered class for $R_0 = 0.75 < 1$ with various initial conditions, parameter values are given in Table 3.

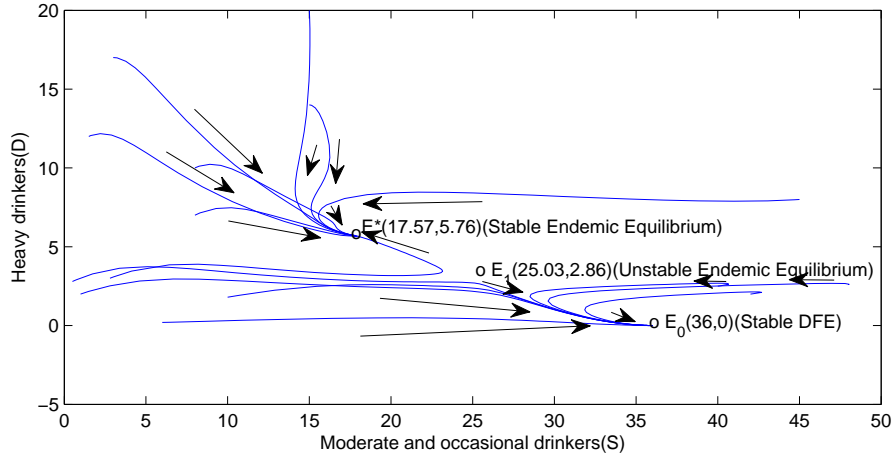


Fig.4e. Moderate and occasional drinkers and heavy drinkers' S-D plane projection of the solution for $R_0 = 0.75 < 1$ with various initial conditions, parameter values are given in Table 3.

12. Conclusions

In this paper, we have developed a mathematical model of alcohol abuse and introduced a four compartmental model with four population classes, namely, moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class. Here, we have found

$$R_0 = \frac{\beta_1}{\mu + \delta_1 + \phi}$$

as basic reproduction number of the model system (1). Sensitivity analysis of R_0 identifies β_1 , the transmission coefficient from moderate and occasional drinker

to heavy drinker, as the most useful parameter to target for the reduction of R_0 . Sensitivity analysis tells us that, **prevention is better than cure**; efforts to increase prevention are more effective in controlling the spread of habitual drinking than efforts to increase the numbers of drinkers undergoing treatment. The model (1) is locally asymptotically stable at disease free or problem free equilibrium (DFE) E_0 when $R_0 < 1$. For the most part, in epidemic models, there are two distinct bifurcations at $R_0 = 1$: (i) forward (supercritical) bifurcation and (ii) backward (subcritical) bifurcation [3,4,15,33,34,37,38,39]. A forward bifurcation happens when R_0 crosses unity from below; a small positive asymptotically stable equilibrium appears and the disease-free equilibrium loses its stability. On the other hand, a backward bifurcation happens when R_0 is less than unity; a small positive unstable equilibrium appears while the disease-free equilibrium and a large positive equilibrium are locally asymptotically stable. Epidemiologically, a backward bifurcation says that, it is not enough to only reduce the basic reproduction number R_0 to less than one to eliminate a disease and that when R_0 crosses unity, hysteresis takes place. In our model system (1), it is found that, when $R_0 = 1$, a backward bifurcation can occur if $X > \Gamma$ (see 11a,11b, art.7) and although R_0 may be less than 1, an endemic equilibrium exists. If this equilibrium is stable, substantial effort may be required to reduce prevalence and avoid an endemic. When $R_0 > 1$, analysis produces a cubic equation in D . The existence of an endemic equilibrium(or endemic equilibria) depends on the existence of at least one real positive value for D . The stability analysis of endemic equilibrium produces that, if the Routh-Hurwitz [19] criterion are satisfied the endemic equilibrium E^* is locally asymptotically stable and if $R_0 < 1$ and $X < \Gamma$, the disease-free equilibrium (DFE) E_0 is globally asymptotically stable. Next all our important mathematical findings are numerically verified using MATLAB. We have numerically verified that, the disease-free equilibrium E_0 is stable when $R_0 < 1$ and when $R_0 > 1$, endemic equilibrium E^* becomes stable and disease-free equilibrium E_0 becomes unstable and forward bifurcation occurs for our set of values of parameters. We have also numerically verified the case when $R_0 < 1$ and disease free equilibrium E_0 and an endemic equilibrium E^* are stable and other endemic equilibrium E_1 is unstable (when backward bifurcation occurs).

As with many models, the mathematical model presented in this paper should be treated with circumspection due to the assumptions made and the difficulty in the estimation of the model parameters. As part of future work to improve the model in this paper, the model considered here can be refined to incorporate drinkers who start regular drinking on their own without having contact with heavy drinkers, age structure and recruitment by drinkers. The model can be refined to model a specific substance of abuse and be fitted to data. The model shows backward bifurcation which occurs because of the insufficient capacity for treatment policies. As a result the drinkers in treatment come to the direct contact of heavy drinkers and they re-enter into the heavy drinkers class. Despite its shortcomings, the model provides useful insights into the possible impact of

rehabilitation and reversion in communities struggling with alcohol abuse. Another important effect we want to include in our future work is the male/female distribution of alcohol abusers. For instance, in the 2005-2006 assessment, the prevalence rate of alcohol abusers in U.S. shows a big variation between male and female rates. It is estimated that of the 15.1 million alcohol-abusing individuals in the U.S., approximately 10.5 million are men and 4.6 million are women [25]. This involves complicated systems and association with the quantitative drinking research expert group is essential.

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Swarnali Sharma got her MSc in Applied Mathematics from Jadavpur University in 2010. Her areas of research are Mathematical Ecology and Epidemiology. She is now working as Junior Research Fellow under the supervision of Dr. G.P. Samanta in Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah, 711103, India.
Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah-711103, India.

e-mail: swarnali.sharma87@gmail.com

G. P. Samanta got his MSc and Ph.D. in Applied Mathematics from Calcutta University in 1985 and 1991 respectively. He was a Premchand Raychand Scholar of Calcutta University and received Mouat Medal at the convocation of Calcutta University in 1996. His areas of research are Mathematical Ecology and Operations Research. He is now working as Professor of the Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah, 711103, India.

Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah-711103, India.

e-mail: g_p_samanta@yahoo.co.uk