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REMARKS ON DIGITAL PRODUCTS WITH NORMAL ADJACENCY RELATIONS

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Abstract. To study product properties of digital spaces, we strongly need to formulate meaningful adjacency relations on digital products. Thus the paper [7] firstly developed a normal adjacency relation on a digital product which can play an important role in studying the multiplicative property of a digital fundamental group, and product properties of digital coverings and digitally continuous maps. The present paper mainly surveys the normal adjacency relation on a digital product, improves the assertion of Boxer and Karaca in the paper [4] and restates Theorem 6.4 of the paper [19].

1. Introduction

Let $\mathbf{Z}^n$ denote the set of points in the Euclidean nD space with integer coordinates and $\mathbf{N}$ denote the set of natural numbers. A basic operation of topology is a formation of the product space from two topological spaces. In [5], the impossibility of topologizing the digital plane $\mathbf{Z}^2$ in terms of graph theoretical approach was shown. Also, a digital space (X, k) can be considered as a simplicial complex via a geometric realization of a discrete object [8].

In graph theory it is well known that a *normal adjacency* for a Cartesian product of graphs [1] has substantially contributed to the study of a Cartesian product of graphs. Thus the paper [7] firstly developed a digital version of a normal adjacency of a Cartesian product of two graphs which can support digital continuities of the Cartesian product of digitally continuous maps and a projection map. This adjacency can be used for studying corresponding product properties of many kinds of digital

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topological properties [13, 14, 16]. For instance, given (digitally) (k_i, k_i')-continuous maps $f_i : (X_i, k_i) \to (Y_i, k_i')$, consider a Cartesian product map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ given by $f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$ and natural projection maps $p_i : X_1 \times X_2 \to X_i$ given by $p_i(x_1, x_2) = x_i$ for $i \in \{1, 2\}$. Then we need to propose respectively reasonable k- and k'-adjacency relations on the digital products $X_1 \times X_2$ and $Y_1 \times Y_2$ which make the map $f_1 \times f_2$ (resp. p_i) digitally (k, k')- (resp. digitally (k, k_i)-)continuous. Indeed, the existence of these adjacency relations depends on the situation. Unfortunately, the paper [7] misused an 8-adjacency of $SC_4^{2,8} \times SC_4^{2,8}$ as a normal adjacency (see [3, 9] and Example 3.2 of the present paper), where $SC_4^{2,8}$ means a simple closed 4-curve with eight elements on $\mathbf{Z}^2$ (see Figure 1). Besides, the paper improves the assertion of [4] related to the product property of digital coverings. This topic can be so important because it is helpful to study the corresponding product properties of many kinds of digital topological properties.

2. Preliminaries

In the paper we often use the integer interval: for $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ is called a *digital interval*. In order to study a digital space in $\mathbf{Z}^n, n \in \mathbf{N}$, we have often used the following adjacency relations of $\mathbf{Z}^n$: for two distinct points $p, q \in \mathbf{Z}^n$ let m be an integer such that $1 \leq m \leq n$. Consider all points $q := (q_i)_{i \in [1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ satisfying the property of (2.1) [7]

$$\left\{ \begin{array}{l} \bullet \text{ there are at most } m \text{ indices } i \text{ such that } |\, p_i - q_i \,| = 1 \text{ and} \\ \bullet \text{ for all other indices } i,\ p_i = q_i. \end{array} \right\} \tag{2.1}$$

We will say that two points $p, q \in \mathbf{Z}^n$ are k-adjacent if they satisfy the property (2.1). Hereafter, we may use the notation $k := k_m$ or $k(m, n)$, where $k := k_m := k(m, n)$ is the number of points q which are k-adjacent to a given point p. Consequently, as a generalization of k-adjacency relations of a low dimensional digital space in $\mathbf{Z}^2$ and $\mathbf{Z}^3$ [20, 21], the adjacency relations of $\mathbf{Z}^n$ can be represented [11] (for more details, see [15]), as follows.

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \tag{2.2}$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$.

For instance, $(n, m, k) \in \{(2,1,4), (2,2,8); (3,1,6), (3,2,18), (3,3,26); (4,1,8), (4,2,32), (4,3,64), (4,4,80); (5,1,10), (5,2,50), (5,3,130), (5,4,210), (5,5,242); (6,1,12), (6,2,72), (6,3,232), (6,4,472), (6,5,664), (6,6,728)\}$ [6, 7].

As usual, a (binary) digital space (X, k) is considered in a digital picture $(\mathbf{Z}^n, k, \bar{k}, X)$ with $k \neq \bar{k}$ in [20, 21]. But in the paper we will concern with only the k-adjacency of (X, k). For an adjacency relation k of $\mathbf{Z}^n$, a *simple k-path* with l elements on $\mathbf{Z}^n$ is assumed to be a *sequence* $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [20]. Then the number l is called the length of a simple k-path. Furthermore, a *simple closed k-curve* with l elements in $\mathbf{Z}^n$ is a simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$ with $x_0 = x_l$, where x_i and x_j are k-adjacent if and only if $j = i+1 (mod\, l)$ or $i = j+1 (mod\, l)$ [20]. By $SC_k^{n,l}$ we denote a simple closed k-curve with l elements on $\mathbf{Z}^n$ [7]. In addition, for $x \in \mathbf{Z}^n$ we follow the notations $N_k(x) := \{y \in \mathbf{Z}^n \mid x$ is k -adjacent to $y\}$ and $N_k^*(x) := N_k(x) \cup \{x\}$ [20].

The following notion has been often used for the study of digital k-curve and digital k-surface theory.

Definition 1. [6] (see also [7]) *For a digital space (X, k) in $\mathbf{Z}^n$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X $N_k(x_0, \varepsilon) = \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}$, where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.*

In view of Definition 1, for a digital space (X, k) on $\mathbf{Z}^n$ we can consider $N_k(x, 1)$ to be the set $N_k^*(x) \cap X$. Since every point $x \in X$ has $N_k(x, 1) \subset (X, k)$, motivated from both the digital continuity of [21] and the (k_0, k_1)-continuity in [2], we can establish the following version.

Proposition 2.1. [7] (see also [10]) *Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

Since a digital space can be recognized to be a digital k-graph, as discussed above, we can represent a (k_0, k_1)-*homeomorphism* in [2], as follows: for two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* [18] (see also [8]) if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then, we use the notation $X \approx_{(k_0, k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism* and use the notation $X \approx_{k_0} Y$.

The following simple closed 4- and 8-curves on $\mathbf{Z}^2$ and a simple closed 18-curves on $\mathbf{Z}^3$ in [9, 10] will be often used later in the present paper (see Figure 1).

$$\left\{\begin{array}{l} SC_4^{2,8} :\approx_4 ((0,0),(0,1),(0,2),(1,2),(2,2),(2,1),(2,0),(1,0)), \\ SC_8^{2,6} :\approx_8 ((0,0),(1,1),(1,2),(0,3),(-1,2),(-1,1)), \\ SC_{18}^{3,6} :\approx_{18} ((0,0,0),(1,0,1),(1,1,2),(0,2,2),(-1,1,2),(-1,0,1)), \\ MSC_{18} :\approx_{18} ((0,0,0),(0,0,1),(1,1,1),(0,2,1),(0,2,0),(-1,1,0)). \end{array}\right\}$$

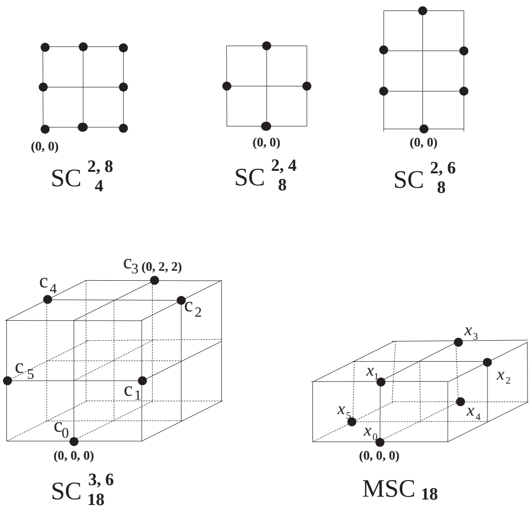

FIGURE 1. Various types of simple closed k-curves in [6, 7, 10]

3. Digital products with normal product adjacency relations

For digital spaces (X, k_1) on $\mathbf{Z}^{n_1}$ and (Y, k_2) on $\mathbf{Z}^{n_2}$, motivated from the *normal product adjacency* in [1], the paper [7] develops a k-adjacency of the Cartesian product (or digital product) $X \times Y = \{(x, y) \,|\, x \in X, y \in Y\} \subset \mathbf{Z}^{n_1+n_2}$, as follows.

Definition 2. [7] *For two digital space (X, k_1) on $\mathbf{Z}^{n_1}$, (Y, k_2) on $\mathbf{Z}^{n_2}$, we say that $(x, y) \in X \times Y \subset \mathbf{Z}^{n_1+n_2}$ is normally k-adjacent to $(x', y') \in X \times Y$ if and only if*

(1) y is equal to y' and x is k_1-adjacent to x', or
(2) x is equal to x' and y is k_2-adjacent to y', or
(3) x is k_1-adjacent to x' and y is k_2-adjacent to y'.

This k-adjacency of Definition 2 has strong merits of studying digital continuities of the corresponding products of both continuous maps and projection maps.

Example 3.1. $(SC_8^{2,4} \times [a,b]_{\mathbf{Z}}, 26)$, $(SC_8^{2,6} \times SC_8^{2,4}, 80)$ *and* $(SC_8^{2,6} \times SC_{18}^{3,6}, k), k \in \{210, 242\}$.

Even though in [3] there are some remarks on a normal adjacency of a digital product, we need to refer it in detail, as follows:

Remark 3.2. *No normal k-adjacency of $SC_4^{2,8} \times SC_4^{2,8}$, $MSC_{18} \times SC_{18}^{3,6}$ and $SC_4^{2,8} \times SC_8^{2,6}$ exists.*

Proof: (1) We prove that there is no normal k-adjacency of $SC_4^{2,8} \times SC_4^{2,8}$. Suppose a normal k-adjacency of $SC_4^{2,8} \times SC_4^{2,8} \subset \mathbf{Z}^4$, $k \in \{8, 32, 64, 80\}$. Then we observe that each k-adjacency of $\mathbf{Z}^4$ has a contradiction to Definition 2. To be specific, suppose $(SC_4^{2,8} \times SC_4^{2,8}, k)$ with the normal k-adjacency, $k \in \{8, 32\}$. Then we have a contradiction to (3) of Definition 2. Besides, if we take $(SC_4^{2,8} \times SC_4^{2,8}, k), k \in \{64, 80\}$, then this leads to a contradiction to (1) and (2) of Definition 2 either.

(2) Using the same method as the above case (1), we can prove that there is no normal k-adjacency of $MSC_{18} \times SC_{18}^{3,6} := (c_{i\,j})_{(i,j) \in [1,6]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}} \subset \mathbf{Z}^6$.

$$\left(\begin{array}{ccc} (0,0,0,0,0,0) & (0,0,0,1,0,1) & (0,0,0,1,1,2) \\ (0,0,1,0,0,0) & (0,0,1,1,0,1) & (0,0,1,1,1,2) \\ (1,1,1,0,0,0) & (1,1,1,1,0,1) & (1,1,1,1,1,2) \\ (0,2,1,0,0,0) & (0,2,1,1,0,1) & (0,2,1,1,1,2) \\ (0,2,0,0,0,0) & (0,2,0,1,0,1) & (0,2,0,1,1,2) \\ (-1,1,0,0,0,0) & (-1,1,0,1,0,1) & (-1,1,0,1,1,2) \end{array}\right.$$

$$\left.\begin{array}{ccc} (0,0,0,0,2,2) & (0,0,0,-1,1,2) & (0,0,0,-1,0,1) \\ (0,0,1,0,2,2) & (0,0,1,-1,1,2) & (0,0,1,-1,0,1) \\ (1,1,1,0,2,2) & (1,1,1,-1,1,2) & (1,1,1,-1,0,1) \\ (0,2,1,0,2,2) & (0,2,1,-1,1,2) & (0,2,1,-1,0,1) \\ (0,2,0,0,2,2) & (0,2,0,-1,1,2) & (0,2,0,-1,0,1) \\ (-1,1,0,0,2,2) & (-1,1,0,-1,1,2) & (-1,1,0,-1,0,1) \end{array}\right) \qquad (3.1)$$

Suppose $MSC_{18} \times SC_{18}^{3,6}$ with a normal k-adjacency, $k \in \{12, 72, 232, 472, 664, 728\}$. In view of Definition 2, the choice of $k(m,6)$, $m \in [1,4]_{\mathbf{Z}}$ cannot be meaningful. Thus let us suppose the normal $k(m,n)$-adjacency of $MSC_{18} \times SC_{18}^{3,6}$ for only $k \in \{664, 728\}$. Then we can prove that this approach cannot satisfy Definition 2. To be specific, consider the point $c_{4\,2} := (0, 2, 1, 1, 0, 1) \in MSC_{18} \times SC_{18}^{3,6}$. Then we can find that even though the point $c_{6\,2} := (-1, 1, 0, 1, 0, 1) \in MSC_{18} \times SC_{18}^{3,6}$ is k-adjacent to the point $c_{4\,2}$, $k \in \{664, 728\}$, the points $(0, 2, 1)$ and $(-1, 1, 0)$ in MSC_{18} which are the first parts of $c_{4,2}$ and $c_{6\,2}$ cannot be 18-adjacent to each other, which contradicts Definition 2.

(3) Let us examine $SC_4^{2,8} \times SC_8^{2,6}$. Suppose a normal k-adjacency of $SC_4^{2,8} \times SC_8^{2,6}$, $k \in \{8, 32, 64, 80\}$.
Then the 8-adjacency of $SC_4^{2,8} \times SC_8^{2,6}$ has a contradiction to (2) of Definition 2. Besides, the 32-adjacency of $SC_4^{2,8} \times SC_8^{2,6}$ cannot support (1) of Definition 2.
Finally, if we suppose a normal k-adjacency of $SC_4^{2,8} \times SC_8^{2,6}$, $k \in \{64, 80\}$. Then this approach contradicts both (1) and (2) of Definition 2. □

Remark 3.3. *Let* $SC_{k_1}^{n_1,l_1} := (c_i)_{i \in [0,l_1-1]_{\mathbf{Z}}}$ *and* $SC_{k_2}^{n_2,l_2} := (d_j)_{j \in [0,l_2-1]_{\mathbf{Z}}}$. *Then* m_i *is determined by the* $k_i := k(m_i, n_i)$*-adjacency of* $SC_{k_i}^{n_i,l_i}$ *via (2.1),* $i \in \{1, 2\}$.
In view of Remark 3.2, even though $m_1 = m_2$*, there is not always a normal* k*-adjacency of* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$.

Reminding the first bullet of (2.1), in particular, the phrase "at most", we obtain the following:

Theorem 3.4. *Given* $SC_{k_i}^{n_i,l_i}, i \in \{1, 2\}$*, if* $k_i = k(m_i, n_i)$*, where* $m_i = n_i$ *and further,* $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$*, then there is a normal* k*-adjacency on the digital product* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$*, where the* k*-adjacency is formulated by the number* $k(m, n_1 + n_2)$ *from (2.1),* $m = n_1 + n_2$ *and further,* $m \notin [1, n_1 + n_2 - 1]_{\mathbf{Z}}$*. The converse also holds.*

Proof: Before proving this theorem, we need to explain the hypothesis that $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$ and $m \notin [1, n_1 + n_2 - 1]_{\mathbf{Z}}$. Without the hypothesis, let us consider the following digital product $SC_8^{2,6} \times SC_{18}^{3,6} \subset \mathbf{Z}^5$ using the simple closed k-curves in Figure 1, i.e.

$$(c_{i\,j})_{(i,j) \in [1,6]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}} := SC_8^{2,6} \times SC_{18}^{3,6} \subset \mathbf{Z}^5$$

via the following matrix (see (3.2)), where for each $(i, j) \in [1, 6]_{\mathbf{Z}} \times [1, 6]_{\mathbf{Z}}$ $c_{i\,j} := (x_{i-1}, x_{j-1}), SC_8^{2,6} := (x_i)_{i \in [0,5]_{\mathbf{Z}}}$ and $SC_{18}^{3,6} := (x_j)_{j \in [0,5]_{\mathbf{Z}}}$ in Figure 1.

$$\left(\begin{array}{ccc}(0,0,0,0,0) & (0,0,1,0,1) & (0,0,1,1,2)\\ (1,1,0,0,0) & (1,1,1,0,1) & (1,1,1,1,2)\\ (1,2,0,0,0) & (1,2,1,0,1) & (1,2,1,1,2)\\ (0,3,0,0,0) & (0,3,1,0,1) & (0,3,1,1,2)\\ (-1,2,0,0,0) & (-1,2,1,0,1) & (-1,2,1,1,2)\\ (-1,1,0,0,0) & (-1,1,1,0,1) & (-1,1,1,1,2)\end{array}\right.$$

$$\left.\begin{array}{ccc}(0,0,0,2,2) & (0,0,-1,1,2) & (0,0,-1,0,1)\\ (1,1,0,2,2) & (1,1,-1,1,2) & (1,1,-1,0,1)\\ (1,2,0,2,2) & (1,2,-1,1,2) & (1,2,-1,0,1)\\ (0,3,0,2,2) & (0,3,-1,1,2) & (0,3,-1,0,1)\\ (-1,2,0,2,2) & (-1,2,-1,1,2) & (-1,2,-1,0,1)\\ (-1,1,0,2,2) & (-1,1,-1,1,2) & (-1,1,-1,0,1)\end{array}\right) \tag{3.2}$$

Owing to the phrase "at most" from the first bullet of (2.1), we can consider $SC_8^{2,6} \times SC_{18}^{3,6}$ with the normal k-adjacency, $k \in \{210, 242\}$, i.e. $k := 242 := k(5,5)$ or $k := 210 := k(4,5)$. But we see that $SC_{18}^{3,6}$ has an 18-adjacency instead of a 26-adjacency.
Let us now prove this theorem. Under the hypothesis, if $k_i = k(m_i, n_i)$, $m_i = n_i$, $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$ and $i \in \{1, 2\}$, then the k-adjacency of $X_1 \times X_2$ is normal if and only if $m = n_1 + n_2$, where $k := k(m, n_1 + n_2)$. $\square$

Remark 3.5. *Theorem 3.4 can play an important role in studying the multiplicative property of digital fundamental group and product properties of digital continuity, digital homeomorphism, a digital covering, etc.*

4. Improvement of the Boxer and Karaca's assertion

In the paper [4], the authors study the product property of digital coverings. However, we can find that many assertions in the paper [4] have been already studied in Han's paper in [12]. Hereafter, each digital space (X, k) is assumed to be k-connected. Based on the original version of the notion of a digital covering in [7], the recent paper make the original version simplified as follows: Let us follow the notion of digital covering space in [17] which is an advanced version of the original digital covering in [7].

Proposition 4.1. [17] *Let (E, k_0) and (B, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection. Suppose that for every $b \in B$ there exists $\varepsilon \in \mathbf{N}$ such that*

(DC 1) for some index set M,
$p^{-1}(N_{k_1}(b,\varepsilon)) = \cup_{i\in M} N_{k_0}(e_i,\varepsilon)$ *with* $e_i \in p^{-1}(b)$*;*
(DC 2) if $i,j \in M$ *and* $i \neq j$*, then* $N_{k_0}(e_i,\varepsilon) \cap N_{k_0}(e_j,\varepsilon) = \emptyset$*; and*
(DC 3) the restriction map $p\,|_{N_{k_0}(e_i,\varepsilon)} : N_{k_0}(e_i,\varepsilon) \to N_{k_1}(b,\varepsilon)$ *is a* (k_0,k_1)*-isomorphism for all* $i \in M$*.*

Then the map p is a (k_0,k_1)-covering map, (E,p,B) is said to be a (k_0,k_1)-covering, and (E,k_0) is called a (k_0,k_1)-covering space over (B,k_1).

In Proposition 4.1, $N_{k_1}(b,\varepsilon)$ is called an *elementary k_1-neighborhood* of b with radius ε. The collection $\{N_{k_0}(e_i,\varepsilon)\,|\,i \in M\}$ is a partition of $p^{-1}(N_{k_1}(b,\varepsilon))$ into slices. Furthermore, we may take $\varepsilon = 1$ which is a special case of Proposition 4.1 [10] (see also [11, 15]).

Remark 4.2. [17] *We observe that the digital covering map of Proposition 4.1 is more simplified than the versions of* [4, 7, 10, 15]*. Namely, the "(k_0,k_1)-continuous surjection" of the earlier versions in* [4, 7, 10, 15] *can be replaced by a "surjection".*

The paper [4] asserted the following: assume two digital coverings $p_i : \mathbf{Z} \to SC_{k_i}^{n_i,l_i} := (c_i)_{i\in[0,l_i-1]_{\mathbf{Z}}}$, $i \in \{1,2\}$ given by $p_i(t_i) = c_{t_i(mod\,l_i)}$. Consider the natural product map $p_1 \times p_2 : \mathbf{Z}\times\mathbf{Z} \to SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ given by $p_1\times p_2(t_1,t_2) = (p_1(t_1),p_2(t_2))$. Then the product map $p_1\times p_2$ is an $(8,k(m,n_1+n_2))$-covering, where $m = n_1+n_2$ if $k_i := k(m_i,n_i)$, where $m_i = n_i$.

In relation to the above Boxer and Karaca's assertion in [4], motivated from the digital product $SC_8^{2,6} \times SC_{18}^{3,6}$ in Example 3.1, consider the following product map $p_1\times p_2 : \mathbf{Z}\times\mathbf{Z} \to SC_8^{2,6}\times SC_{18}^{3,6}$ given by $p_1\times p_2(t_1,t_2) = (p_1(t_1),p_2(t_2))$, where $p_1 : \mathbf{Z} \to SC_8^{2,6} := (c_i)_{i\in[0,5]_{\mathbf{Z}}}$ given by $p_1(t_1) = c_{t_1(mod\,6)}$ and $p_2 : \mathbf{Z} \to SC_{18}^{3,6} := (c_j)_{j\in[0,5]_{\mathbf{Z}}}$ $p_2(t_2) = c_{t_2(mod\,6)}$. Then we can see that the product map $p_1\times p_2$ is an $(8,k)$-covering map, where $k \in \{210,242\}$. Even though the paper [4] investigate product property of two digital coverings, we can find that a well-presented version is already studied in the paper [12].

In view of Theorem 3.4, we can improve the above assertion as follows:

Theorem 4.3. *For two digital coverings* $p_i : \mathbf{Z} \to SC_{k_i}^{n_i,l_i} := (c_i)_{i\in[0,l_i-1]_{\mathbf{Z}}}$, $i \in \{1,2\}$ *given by* $p_i(t_i) = c_{t_i(mod\,l_i)}$*, consider the natural product map* $p_1\times p_2 : \mathbf{Z}\times\mathbf{Z} \to SC_{k_1}^{n_1,l_1}\times SC_{k_2}^{n_2,l_2}$ *given by* $p_1\times p_2(t_1,t_2)$*. Then the product map* $p_1\times p_2$ *is an* $(8,k(m,n_1+n_2))$*-covering, where* $m = n_1+n_2$ *if* $k_i := k(m_i,n_i)$*, where* $m_i = n_i$ *and further,* $m_i \notin [1,n_i-1]_{\mathbf{Z}}$*.*

In view of Example 3.1, we obtain the following:

Example 4.4. *Consider the following product map* $p_1 \times p_2 : \mathbf{Z} \times \mathbf{Z} \to SC_8^{2,6} \times SC_8^{2,4}$ *given by* $p_1 \times p_2(t_1, t_2) = (p_1(t_1), p_2(t_2))$*, where* $p_1 : \mathbf{Z} \to SC_8^{2,6} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ *given by* $p_1(t_1) = c_{t_1 (mod\, 6)}$ *and* $p_2 : \mathbf{Z} \to SC_8^{2,4} := (c_j)_{j \in [0,3]_{\mathbf{Z}}}$ $p_2(t_2) = c_{t_2 (mod\, 4)}$. *Then we can see that the product map* $p_1 \times p_2$ *is an* $(8, 80)$*-covering map.*

The recent paper [19] studies a classification of digital images up to an A-isomorphism. In particular, in Theorem 6.4 of the paper the part " if $l_0 = l_1$" was missing just before the part " Then f is an $\cdots$". Thus we can restate the theorem as follows:

Theorem 4.5 (Correcting of Theorem 6.4 of [19])**.** *Consider two spaces* $SC_A^{n,l_0} := (c_i)_{i \in [0, l_0 - 1]_{\mathbf{Z}}}$ *and* $SC_A^{n,l_1} := (x_i)_{i \in [0, l_1 - 1]_{\mathbf{Z}}}$. *Let* $f : SC_A^{n,l_0} \to SC_A^{n,l_1}$ *be the map given by* $f(c_i) = x_{i+m (mod\, l_1)}, m \in \mathbf{N}$. *If* $l_0 = l_1$*, then* f *is an* A*-isomorphism.*

5. Summary and further works

We have studied normal k-adjacencies on digital products which can contribute to the study of the corresponding product properties of all kind of digital topological properties. To further study this topic, we can suggest the following question.
Question: Even though the product property of digital coverings was studied in [12] by using the L_S-property, we can also study the property by using a normal adjacency as follows: Given two digital coverings, under what condition using a normal adjacency relation of a digital product is their product also a digital covering?

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