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A NEW PROOF OF THE EXTENDED SAALSCHÜTZ'S SUMMATION THEOREM FOR THE SERIES $_4F_3$ AND ITS APPLICATIONS

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Abstract. Very recently, Rakha and Rathie obtained an extension of the classical Saalschütz's summation theorem. Here, in this paper, we *first* give an elementary proof of the extended Saalschütz's summation theorem. By employing it, we *next* present certain extenstions of Ramanujan's result and another result involving hypergeometric series. The results presented in this paper are simple, interesting and (potentially) useful.

1. Introduction and preliminaries

In the theory of hypergeometric series $_2F_1$ and generalized hypergeometric series $_pF_q$ (see, *e.g.*, [9]; see also [12]), the following classical Saalschütz's summation theorem (see, *e.g.*, [9, p. 87]):

$$
{}_3F_2\left[\begin{array}{r} -n,\, \alpha,\, \beta\,; \\ \gamma,\, 1+\alpha+\beta-\gamma-n\,; \end{array} 1\right] = \frac{(\gamma-\alpha)_n\,(\gamma-\beta)_n}{(\gamma)_n\,(\gamma-\alpha-\beta)_n}, \tag{1.1}
$$

where $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}$ denotes the set of positive integers, and α, β, γ are independent of n, plays an important role.

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In a very interesting, popular and useful research paper, Baily [2] presented a large number of known as well as new results involving products of generalized hypergeometric series by employing the classical Saalschütz's summation theorem (1.1).

In [3, Chapters 10 and 11], there are a large number of Ramanujan's results on the hypergeometric series ${}_2F_1$ and generalized hypergeometric series ${}_pF_q$. Among those results, here we choose to recall the following three formulas (see [3, p. 99, **Entry 35(i)**, **Entry 35(ii)**, and **Entry 35(iii)**], respectively):

If n is arbitrary, then we have the following results:

$$\cos\left(2n\,\sin^{-1}x\right) = {}_2F_1\left[\begin{matrix} n,\ -n\,; \\ \frac{1}{2}\,; \end{matrix}\ x^2\right], \tag{1.2}$$

$$\sin\left(2n\,\sin^{-1}x\right) = 2nx\,{}_2F_1\left[\begin{matrix} \frac{1}{2}+n,\ \frac{1}{2}-n\,; \\ \frac{3}{2}\,; \end{matrix}\ x^2\right], \tag{1.3}$$

and

$$\left(1-x^2\right)^{-\frac{1}{2}}\cos\left(2n\,\sin^{-1}x\right) = {}_2F_1\left[\begin{matrix} \frac{1}{2}+n,\ \frac{1}{2}-n\,; \\ \frac{1}{2}\,; \end{matrix}\ x^2\right]. \tag{1.4}$$

It is interesting to observe that, if we use (1.2) in the left-hand side of (1.4), then we obtain the following form:

$$\left(1-x^2\right)^{-\frac{1}{2}}\ {}_2F_1\left[\begin{matrix} n,\ -n\,; \\ \frac{1}{2}\,; \end{matrix}\ x^2\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}+n,\ \frac{1}{2}-n\,; \\ \frac{1}{2}\,; \end{matrix}\ x^2\right]. \tag{1.5}$$

As pointed out by Berndt [3] that the results (1.4) or (1.5) can be obtained with the help of the classical Saalschütz's summation theorem (1.1). Also, very recently, Chopra and Rathie [5] presented the following result closely related to the Ramanujan's formula (1.4)

$$\left(1-x^2\right)^{-\frac{1}{2}}\sin\left(2n\,\sin^{-1}x\right) = 2nx\,{}_2F_1\left[\begin{matrix} 1+n,\ 1-n\,; \\ \frac{3}{2}\,; \end{matrix}\ x^2\right] \tag{1.6}$$

by using the Saalschütz's summation theorem (1.1). If we use (1.3) in the left-hand side of (1.6), then we also obtain the following form:

$$(1.7)\quad \left(1-x^2\right)^{-\frac{1}{2}}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}+n,\ \frac{1}{2}-n\,; \\ \frac{3}{2}\,;\end{matrix}\ x^2\right] = {}_2F_1\left[\begin{matrix}1+n,\ 1-n\,; \\ \frac{3}{2}\,;\end{matrix}\ x^2\right].$$

It is noted that the results (1.2) and (1.3) are also recorded in [1, p. 556, Entries **15.1.18** and **15.1.16**], respectively, in the following alternative forms:

$$(1.8)\quad {}_2F_1\left[\begin{matrix}a,\ 1-a\,; \\ \frac{1}{2}\,;\end{matrix}\ \sin^2 z\right] = \frac{\cos\left[(2a-1)z\right]}{\cos z}$$

and

$$(1.9)\quad {}_2F_1\left[\begin{matrix}a,\ 2-a\,; \\ \frac{3}{2}\,;\end{matrix}\ \sin^2 z\right] = \frac{\sin\left[(2a-2)z\right]}{(2a-1)\,\sin z}.$$

Very recently, Rakha and Rathie [11] established the following interesting extension of the classical Saalschütz's summation theorem (1.1):

(1.10)

$${}_4F_3\left[\begin{matrix}-n,\ d,\ a,\ c+1\,; \\ b+1,\ c,\ 1+a-b+d-n\,;\end{matrix}\ 1\right] = \frac{(b-a)_n\,(b-d)_n\,(1+g)_n}{(b+1)_n\,(b+a-d)_n\,(g)_n},$$

where

$$(1.11)\quad g = \frac{f(d-b)}{d-f}$$

and

$$(1.12)\quad f = \frac{c(a-b)}{a-c}.$$

It is remarked in passing that, for interesting applications of the result (1.10), the reader may be referred to the above-cited paper [11].

The remainder of this paper will be organized as follows. In Section 2, we provide an elementary and simple method to establish the extension of the Saalschütz's summation theorem (1.10). In Section 3, certain extensions of (1.5) and (1.7) will be presented. In Section 4, we will consider some special cases of those results presented here and give a concluding remark.

2. A new proof of (1.10)

In order to establish (1.10), we will make use of the following integral representation which is a special case of a more general formula recorded in Rainville [9, p. 85, Theorem 28]:

$$\begin{aligned}\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\,{}_3F_2\left[\begin{matrix}\alpha,\,\beta,\,d+1;\\ \gamma,\,d;\end{matrix}\;x\right]dx \\ = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_4F_3\left[\begin{matrix}\alpha,\,\beta,\,\rho,\,d+1;\\ \gamma,\,\rho+\sigma,\,d;\end{matrix}\;1\right],\end{aligned} \tag{2.1}$$

provided $\Re(\rho)>0$, $\Re(\sigma)>0$, $\Re(\sigma+\gamma-\alpha-\beta)>0$ and $d>0$.

In (2.1), if we take $\alpha=-n$, $\beta=a$, $\rho=d$, $d=c$, $\gamma=b+1$ and $\sigma=1+a-b-n$, after some simplifications, we have

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}-n,\,a,\,d,\,c+1\,;\\ b+1,\,1+a+d-b-n,\,c\,;\end{matrix}\;1\right]\\ &= \frac{\Gamma(1+a+d-b-n)}{\Gamma(d)\,\Gamma(1+a-b-n)}\int_0^1 x^{d-1}(1-x)^{a-b-n}\,{}_3F_2\left[\begin{matrix}-n,\,a,\,c+1;\\ b+1,\,c\,;\end{matrix}\;x\right]dx.\end{aligned}$$

Using the following relation (see, *e.g.*, [12, p. 5])

$$\Gamma(\alpha-n)=(-1)^n\,\frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\Gamma(1-\alpha+n)}=(-1)^n\,\frac{\Gamma(\alpha)}{(1-\alpha)_n},$$

the last identity can be expressed as follows:

$$\begin{aligned}{}_4F_3\left[\begin{matrix}-n,\,a,\,d,\,c+1\,;\\ b+1,\,1+a+d-b-n,\,c\,;\end{matrix}\;1\right] &= \frac{\Gamma(1+a-b+d)\,(b-a)_n}{\Gamma(d)\,\Gamma(1+a-b)\,(b-a-d)_n}\\ &\cdot\int_0^1 x^{d-1}(1-x)^{a-b-n}\,{}_3F_2\left[\begin{matrix}-n,\,a,\,c+1;\\ b+1,\,c\,;\end{matrix}\;x\right]dx,\end{aligned}$$

which, upon using the following transformation (see [10]):

$${}_3F_2\left[\begin{matrix}d,\,b-a,\,f+1\,;\\ b+1,\,f\,;\end{matrix}\;x\right]=(1-x)^{-d}\,{}_3F_2\left[\begin{matrix}d,\,a,\,c+1\,;\\ b+1,\,c\,;\end{matrix}\;-\frac{x}{1-x}\right],$$

f being the same as in (1.12), becomes

$$
{}_4F_3\left[\begin{matrix} -n,\, a,\, d,\, c+1\,; \\ b+1,\, 1+a+d-b-n,\, c\,; \end{matrix}\; 1\right] = \frac{\Gamma(1+a-b+d)\,(b-a)_n}{\Gamma(d)\,\Gamma(1+a-b)\,(b-a-d)_n}
$$
$$
\cdot \int_0^1 x^{d-1}\,(1-x)^{a-b}\,{}_3F_2\left[\begin{matrix} -n,\, b-a,\, f+1; \\ b+1,\, f\,; \end{matrix}\; -\frac{x}{1-x}\right] dx.
$$

Now, expressing the ${}_3F_2$ in the last identity as a series, changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, and evaluating the integral with the help of the Beta integral (see, *e.g.*, [12, p. 8]), after some algebra, we have

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} -n,\, a,\, d,\, c+1\,; \\ b+1,\, 1+a+d-b-n,\, c\,; \end{matrix}\; 1\right] \\
&\qquad = \frac{(b-a)_n}{(b-a-d)_n} \sum_{r=0}^{n} \frac{(-n)_r\,(d)_r\,(f+1)_r}{r!\,(b+1)_r\,(f)_r} \\
&\qquad = \frac{(b-a)_n}{(b-a-d)_n}\,{}_3F_2\left[\begin{matrix} -n,\, d,\, f+1; \\ b+1,\, f\,; \end{matrix}\; 1\right].
\end{aligned}
$$

Finally, using the known formula given by Miller [7]:

$$
(2.2) \qquad {}_3F_2\left[\begin{matrix} -n,\, a,\, c+1; \\ b+1,\, f\,; \end{matrix}\; 1\right] = \frac{(b-a)_n}{(b)_n} \cdot \frac{(f+1)_n}{(f)_n},
$$

where $f = c(a-b)/(a-c)$ as given in (1.12), we easily arrive at the right-hand side of (1.10). This completes the proof of the extended Saalschütz's summation theorem (1.10).

3. Extensions of the results (1.5) and (1.7)

Here we will establish the following results: For n arbitrary, we have

$$(3.1)\qquad \begin{aligned}&\left(1-x^2\right)^{-\frac{1}{2}}\ {}_3F_2\left[\begin{matrix}-n,\ n,\ a+1;\\ \frac{3}{2},\ a\,;\end{matrix}\ x^2\right]\\ &\qquad = {}_3F_2\left[\begin{matrix}\frac{1}{2}+n,\ \frac{1}{2}-n,\ A+1;\\ \frac{3}{2},\ A\,;\end{matrix}\ x^2\right],\end{aligned}$$

where, for convenience,

$$(3.2)\qquad A = \frac{a\left(n^2-\frac{1}{4}\right)}{n^2-\frac{a}{2}}$$

and

$$(3.3)\qquad \begin{aligned}&\left(1-x^2\right)^{-\frac{1}{2}}\ {}_3F_2\left[\begin{matrix}\frac{1}{2}+n,\ \frac{1}{2}-n,\ a+1;\\ \frac{5}{2},\ a\,;\end{matrix}\ x^2\right]\\ &\qquad = {}_3F_2\left[\begin{matrix}1+n,\ 1-n,\ B+1;\\ \frac{5}{2},\ B\,;\end{matrix}\ x^2\right],\end{aligned}$$

where, for convenience,

$$(3.4)\qquad B = \frac{a\left(n^2-1\right)}{n^2-\frac{a}{2}-\frac{1}{4}}.$$

Proof. In order to prove (3.1), for simplicity, let us denote the left-hand side of (3.1) by $\mathcal{S}(x)$. Then, expressing both functions in $\mathcal{S}(x)$, in particular, using the generalized binomial formula

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n\, x^n}{n!} \qquad (|x|<1)$$

for the former one, as series, we have the following double series:

$$\mathcal{S}(x) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!}\, \frac{(-n)_s\,(n)_s\,(a+1)_s}{\left(\frac{3}{2}\right)_s\,(a)_s\,s!}\, x^{2r+2s}.$$

Now using a familiar formal manipulation of double series (see, *e.g.*, [4, Eq. (1.4)]; see also [9, p. 56, Lemma 10]):

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{s,r} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} A_{s,r-s}, \tag{3.5}$$

we get

$$\mathcal{S}(x) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{\left(\frac{1}{2}\right)_{r-s}}{(r-s)!} \frac{(-n)_s\,(n)_s\,(a+1)_s}{\left(\frac{3}{2}\right)_s\,(a)_s\,s!}\,x^{2r}.$$

Applying a known identity (see, *e.g.*, [12, p. 5, Eq. (25)]):

$$(\lambda)_{r-s} = \frac{(-1)^s\,(\lambda)_r}{(1-\lambda-r)_s} \quad (0 \le s \le r), \tag{3.6}$$

after a little simplification, we obtain

$$\begin{aligned}
\mathcal{S}(x) &= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!}\,x^{2r} \sum_{s=0}^{r} \frac{(-r)_s\,(n)_s\,(-n)_s\,(a+1)_s}{\left(\frac{3}{2}\right)_s\,\left(\frac{1}{2}-r\right)_s\,(a)_s\,s!} \\
&= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!}\,x^{2r}\,{}_4F_3\left[\begin{matrix} -r,\ n,\ -n,\ a+1\,; \\ \frac{3}{2},\ \frac{1}{2}-r,\ a\,; \end{matrix}\ 1\right].
\end{aligned}$$

Now, it is easy to see that the ${}_4F_3$ in the last expression can be evaluated with the help of the extension of Saalschütz's summation theorem (1.10) and, after some simplification, we have

$$\mathcal{S}(x) = \sum_{r=0}^{\infty} \frac{x^{2r}}{r!} \frac{\left(\frac{1}{2}-n\right)_r\,\left(\frac{1}{2}+n\right)_r\,(A+1)_r}{\left(\frac{3}{2}\right)_r\,(A)_r},$$

where A is the same as given in (3.2). Finally, translating the last series into ${}_3F_2$ is seen to be equal to the right-hand side of (3.1).

A similar argument as in proving (3.1) can establish the result (3.3). This completes the proof. □

4. Concluding remarks

It is noted that the formula (2.2) is found to be a special case of a general formula recorded in [8, p. 534, Entry 10] (see also [6]).

Setting $a = \frac{1}{2}$ in (3.1) yields the Ramanujan's result (1.5). On the other hand, setting $a = \frac{1}{2}$ in (3.3) gives another result (1.7). Thus the formulas (3.1) and (3.3) may be regarded as certain extensions of the results (1.5) and (1.7), respectively.

By using a more known general integral formula with some suitable parametric changes in Section 2 and the arguments given in this paper, there may be a possibility to find certain new formulas involving the generalized hypergeometric series ${}_pF_q$ in which p and q are contiguously increasing. The work is under investigation and will be ready for publication soon.

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