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# FIXED POINTS AND KERNEL OF THE PROJECTIVE HOLONOMY OF AN AFFINE MANIFOLD

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**Abstract.** It is an interesting problem to study fixed points of an element in the holonomy group of an affine manifold. We compute the limit of a sequence of projective transformations and verify relations between fixed points and kernels.

## 1. Introduction

An affine manifold $M$ is a smooth manifold whose coordinate transitions are affine maps. Since affine maps are analytic, an affine manifold is an analytic manifold. See [1], [2] and [6].

A coordinate chart can be analytically continued along a curve and induces a map $D$ from the universal covering space $\tilde{M}$ to $\mathbb{R}^n$. Obviously, $D$ depends on the choice of the initial chart. The map $D$ is called a *developing map* and the image $D(\tilde{M})$ is called a *developing image*.

For a deck transformation $\alpha \in \pi_1(M)$ there is an affine transformation $\rho(\alpha)$ such that

$$D(\alpha x) = \rho(\alpha) D(x).$$

The map $\rho$ is a homomorphism from $\pi_1(M)$ into the affine group $\mathrm{Aff}(n)$ of $\mathbb{R}^n$. $\rho$ is called the *holonomy homomorphism* and $\Gamma = \rho(\pi_1(M))$ the *holonomy group*.

Now we especially consider the homomorphism $\psi$ from $\mathrm{Aff}(n)$ to $\mathrm{PGL}(n+1, \mathbb{R})$ given by the composition of the following two homomorphisms:

$$\begin{array}{ccccc} \mathrm{Aff}(n) & \longrightarrow & \mathrm{GL}(n+1,\mathbb{R}) & \longrightarrow & \mathrm{PGL}(n+1,\mathbb{R}) \\ (a, A) & \longmapsto & \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} & \longmapsto & \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}, \end{array}$$

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where $\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}$ is the equivalence class containing $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$. The group $\psi(\Gamma)$ is called the *projective holonomy group* of the affine manifold $M$.

In this paper, we show that there is a singular projective transformation $\sigma \in \overline{\psi(\Gamma)}$ such that $[\ker \sigma] \cap \mathbb{R}^n \neq \emptyset$ if $M$ is closed and an element of $\Gamma$ has a fixed point in $\mathbb{R}^n$.

T. Nagano and K. Yagi proved a result related to fixed points of holonomy group([4]): If the holonomy group $\Gamma$ of a closed affine manifold has a fixed point then the developing image avoids the fixed point.

## 2. Limit of a sequence of projective transformations

In this section we describe limits of sequences of projective transformations.

Let $\mathrm{M}(n)$ be the set of $n \times n$ matrices whose entries are real numbers. $\mathrm{M}(n)$ can be identified with $\mathbb{R}^{n^2}$ and therefore, projectivized. The projectivization of $\mathrm{M}(n)$ is denoted by $\mathrm{PM}(n)$. The equivalence class $[L] \in \mathrm{PM}(n)$ containing $L$ is called the *projectivization* of $L$. That is,

$$[L] = \{X \in \mathrm{M}(n) - \{O\} \mid X = kL \text{ for some } k \in \mathbb{R} - \{0\}\}.$$

The equivalence class $[L]$ can be considered as a map from $\mathbb{RP}^{n-1}$ onto itself whenever $L$ is not singular. It is called a *nonsingular projective transformation*. The projective general linear group $\mathrm{PGL}(n)$, or $\mathrm{PGL}(n, \mathbb{R})$, is the group of non-singular projective transformations.

For a singular matrix $L$ the projectivization of $\ker L$ is a projective subspace of $\mathbb{RP}^{n-1}$. We call it the *projective kernel* of $[L]$ and denote $\ker[L]$. The equivalence class $[L]$ can be considered as a map from $\mathbb{RP}^{n-1} - \ker[L]$ to $\mathbb{RP}^{n-1}$. It is called a *singular projective transformation*. See also [3].

Let $L$ be an $n \times n$ matrix with real entries and $\lambda_1, \ldots, \lambda_s$ complex eigenvalues of $L$. There exist square matrices $\Lambda_1, \ldots, \Lambda_s$ such that

$$\Lambda_r = \begin{pmatrix} \lambda_r & 1 & & \\ & & \ddots & 1 \\ & & & \lambda_r \end{pmatrix}, \tag{1}$$

and $L$ is similar to

$$\begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_s \end{pmatrix}.$$

When $\lambda_r = a_r + ib_r$ is not a real number, $\bar{\lambda}_r$ also is an eigenvalue and one of $\Lambda_q$ coincides with

$$\bar{\Lambda}_r = \begin{pmatrix} \bar{\lambda}_r & 1 & & \\ & & \ddots & 1 \\ & & & \bar{\lambda}_r \end{pmatrix}.$$

We note that $\begin{pmatrix} \Lambda_r & 0 \\ 0 & \bar{\Lambda}_r \end{pmatrix}$ is similar to

$$\begin{pmatrix} C_r & I & & \\ & & \ddots & I \\ & & & C_r \end{pmatrix} \tag{2}$$

where $C_r = \begin{pmatrix} a_r & -b_r \\ b_r & a_r \end{pmatrix}$. Hence $L$ is similar to

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_l \end{pmatrix} \tag{3}$$

where $A_k$ is of the form (1) with a real number $\lambda_r$ or (2) with a complex number $\lambda_r = a_r + ib_r$.

Suppose that $\Lambda \in \mathrm{M}(n)$ is a matrix of the form (1) with a real number $\lambda = \lambda_r$. For sufficiently large $m$ we obtain

$$\Lambda^m = \begin{pmatrix} \binom{m}{0}\lambda^m & \binom{m}{1}\lambda^{m-1} & \cdots & \binom{m}{n-1}\lambda^{m-n+1} \\ & \binom{m}{0}\lambda^m & \cdots & \binom{m}{n-2}\lambda^{m-n+2} \\ & & \ddots & \\ & & & \binom{m}{0}\lambda^m \end{pmatrix}.$$

The $(1,n)$-entry $\binom{m}{n-1}\lambda^{m-n+1}$ has the largest absolute value among entries of $\Lambda^m$. Hence $[\Lambda]^m$ converges to a projective transformation $[\Lambda_0]$ and $\Lambda_0$ has only a nonzero entry at $(1,n)$. The matrix $\Lambda_0$ is not singular if and only if $n = 1$ and $\lambda \neq 0$. The kernel $\Lambda_0$ is of dimension $n-1$ if $\lambda \neq 0$.

Suppose that $\Lambda \in \mathrm{M}(n)$ is a matrix of the form (2) with a complex number $\lambda_r$. The similar arguments implies that the sequence $[\Lambda]^m$ has a convergent subsequence. The limit $[\Lambda_0]$ of the subsequence is a singular projective transformation if and only if $n > 2$. The kernel $\ker \Lambda_0$ is of dimension $n-2$.

In general, suppose that $L$ is of the form (3). Let $[L_0]$ be the limit of a convergent subsequence of $[L]^m$ and

$$\mu = \max\{|\lambda_1|, \dots, |\lambda_s|\}.$$

If $|\lambda_r| < \mu$ for some $r$ and the submatrix $A_k$ is of the form (1) with a real number $\lambda_r$ or (2) with a complex number $\lambda_r$ then $L_0$ is singular.

If $|\lambda_r| = \mu$ for all $r$ and $L$ is diagonalizable then $L_0$ is nonsingular. As a result,

**Proposition 1.** *Suppose that $L \in \mathrm{M}(n)$ is a non-singular matrix and a subsequence of $[L^m]$ converges to $[L_0]$. $L_0$ is non-singular if and only if $L$ is similar to $cR$ for a nonzero real number $c$ and an orthogonal matrix $R$.*

*Proof.* We note that a matrix is orthogonal if and only if it is similar to (3) where $A_k$ is one of the following 4 forms:

$$(1), (-1), \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Applying the above arguments, our result follows. □

## 3. Fixed point of an affine holonomy

Suppose that an affine transformation $(a, A)$ has a fixed point, That is, there exists a point $x \in \mathbb{R}^n$ such that $a + Ax = x$. Let

$$L = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Then the nonsingular matrix $L$ has a point $\xi$ as a fixed point. Hence $\xi$ is an eigenvector of $L$ with the corresponding eigenvalue 1.

Suppose that $L$ has an eigenvalue $\lambda$ such that $|\lambda| > 1$. The arguments in section 2 implies that $[L]^m$ has a subsequence which converges to $[L_0]$ and $\xi \in \ker L_0$.

Suppose that $L$ has an eigenvalue $\lambda$ and $|\lambda| < 1$. Then $\xi$ is a fixed point of

$$L^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}a \\ 0 & 1 \end{pmatrix}$$

and $L^{-1}$ has an eigenvalue $1/\lambda$ and $|1/\lambda| > 1$.

If $L$ is not similar to an orthogonal matrix and any eigenvalue of $L$ is of absolute value 1, then $[L]^m$ has a subsequence which converges to

$[L_0]$ and $\xi \in \ker L_0$ as section 2. Hence we have

**Theorem 1.** *Let $M$ be an affine manifold with the holonomy group $\Gamma$. If $\gamma \in \Gamma$ has a fixed point and is not similar to an orthogonal matrix, then there is a singular projective transformation $\sigma \in \overline{\psi(\Gamma)}$ such that* $\ker \sigma \cap \mathbb{R}^n \neq \emptyset$.

Now we introduce a set, which is called a *limit set*, defined in [5]. Let $M$ be an affine manifold with the developing map $D : \tilde{M} \to \mathbb{R}^n$. Let $E_M$ be the set of those points $y$ which is the end point $c(1)$ of a continuous curve $c$ in $\mathbb{R}^n$ such that :

(4) there exists a curve $\tilde{c}(t) \in \tilde{M}$ for $0 \leq t < 1$, $D(\tilde{c}(t)) = c(t)$ and $\tilde{c}(1)$ can not be defined continuously in $\tilde{M}$.

The following Theorem is proved in [5].

**Theorem 2.** *Let $M$ be a closed affine manifold. For any $x \in E_M$ there exists $\sigma \in \overline{\psi(\Gamma)}$ such that $x \in \ker \sigma$.*

Now, suppose that $M$ is a closed affine manifold and $(a, A) \in \Gamma$ has a fixed point $x$. If $x \notin \Omega$ then obviously $E_M$ is not empty.

We assume that $x = D(u) \in \Omega$ for some $u \in \tilde{M}$. Let $\alpha : \tilde{M} \to \tilde{M}$ be a deck transformation satisfying $\rho(\alpha) = (a, A)$ and $h : [0,1] \to \tilde{M}$ a curve starting at $u$ ending at $\alpha(u)$. The curve $D \circ h$ is a loop such that $(D \circ h)(0) = (D \circ h)(1) = x$. Hence there is an homotopy

$$H : [0,1] \times [0,1] \to \mathbb{R}^n$$

satisfying

$$H(\cdot, 0) = D \circ h$$

and

$$H(\cdot, 1) = H(0, \cdot) = H(1, \cdot) = x.$$

Since $u \neq \alpha(u)$, there is no homotopy

$$\tilde{H} : [0,1] \times [0,1] \to \tilde{M}$$

such that $D \circ \tilde{H} = H$. Hence a curve $c = H(t_0, \cdot)$ starting at $(D \circ h)(t_0)$ for some $t_0$ satisfies the condition (4).

In fact, $\tilde{c}$ in (4) with $D \circ \tilde{c} = c$ is not defined on $[0, 1)$ but on $[0, t_1)$ for some $t_1 \in (0, 1)$ in this case.

**Theorem 3.** *Let $M$ be a closed affine manifold. If an element in the holonomy group $\Gamma$ has a fixed point, then there is a singular projective* *transformation* $\sigma \in \overline{\psi(\Gamma)}$ *such that* $\ker \sigma \cap \mathbb{R}^n \neq \emptyset$.

*Proof.* Since $E_M$ is not empty, Theorem 3 implies our result. □

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