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CERTAIN FORMULAS INVOLVING EULERIAN NUMBERS

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Abstract. In contrast with numerous identities involving the binomial coefficients and the Stirling numbers of the first and second kinds, a few identities involving the Eulerian numbers have been known. The objective of this note is to present certain interesting and (presumably) new identities involving the Eulerian numbers by mainly making use of Worpitzky's identity.

1. Introduction and preliminaries

Some sequences of numbers arise so often in mathematics that we recognize them immediately and give them special names, for example, square numbers, triangular numbers, prime numbers, Bernoulli numbers, Euler numbers, Eulerian numbers, Stirling numbers, Fibonacci numbers, and so on (see, *e.g.*, the references given here). We begin with a close relative of the binomial coefficients, the Stirling numbers of the first kind, named after James Stirling (1692-1770). There are also Stirling numbers of the second kind. Although they have a revered and many applications, their notations have not been used in one way, like the binomial coefficient $\binom{n}{k}$. Following Jovan Karamata [5], Graham *et al.* [4] have used $\left\{ {n \atop k} \right\}$ for Stirling numbers of the second kind and $\left[{n \atop k} \right]$ for Stirling numbers of the first kind by noting that these symbols turn out to be more user-friendly than the many other notations that people have tried. Here we agree to use the notation $\left[{n \atop k} \right]$ for Stirling numbers of the first kind, which counts the number of ways to arrange n objects into

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k *cycles*. Note that $\left[{n \atop k}\right]$ is also called the unsigned or absolute Stirling number of the first kind (see, *e.g.*, [2, p. 213]).

This $\left[{n \atop k}\right]$ satisfies the following recurrence:

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right] \quad (n \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ denotes the set of positive integers. We recall some values of $\left[{n \atop k}\right]$:

$$\begin{aligned} &\left[{n \atop 0}\right] = [n=0], \quad \left[{n \atop 1}\right] = (n-1)! \quad (n \in \mathbb{N}),\\ &\left[{n \atop n}\right] = 1, \quad \left[{n \atop n-1}\right] = \binom{n}{2}, \quad \left[{n \atop 2}\right] = (n-1)!\,H_{n-1} \quad (n \in \mathbb{N}), \end{aligned} \tag{1.2}$$

where the notation $[n=m]$ denotes 1 if $n=m$ and 0 otherwise, H_n are the harmonic numbers defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N}) \quad \text{and} \quad H_0 := 0. \tag{1.3}$$

For the last identity in (1.2), see also and compare its corresponding identity in [7, p. 77].

Another triangle of values which has often been appeared is due to Euler and called Eulerian numbers denoted by $\left\langle{n \atop k}\right\rangle$. The angle brackets in this case suggests *less than* and *greater than*; $\left\langle{n \atop k}\right\rangle$ is the number of permutations $\pi_1\,\pi_2 \cdots \pi_n$ of $\{1,\,2,\,\ldots,\,n\}$ that have k ascents, namely, k places where $\pi_j < \pi_{j+1}$. This $\left\langle{n \atop k}\right\rangle$ satisfies the following recurrence:

$$\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle \quad (n \in \mathbb{N}). \tag{1.4}$$

We recall some values of $\left\langle{n \atop k}\right\rangle$:

$$\begin{aligned} &\left\langle{n \atop n}\right\rangle = [n=0], \quad \left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle \quad (n \in \mathbb{N}),\\ &\left\langle{0 \atop k}\right\rangle = [k=0] \quad (k \in \mathbb{Z}), \end{aligned} \tag{1.5}$$

where $\mathbb{Z}$ denotes the set of integers. Graham *et al.* [4, p. 269] commented that Eulerian numbers are useful primarily because they provide an unusual connection between ordinary powers and consecutive binomial coefficients which is called Worpitzky's identity [8]:

$$x^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.6}$$

Graham *et al.* [4, p. 269] noted that the Eulerian recurrence (1.4) is a bit more complicated than the Stirling recurrences (1.1) and [4, p. 259, Equation (6.3)], so they don't expect the numbers $\left\langle {n \atop k} \right\rangle$ to satisfy as many simple identities, except for a few, for example,

$$\left\langle {n \atop m} \right\rangle = \sum_{k=0}^{m} \binom{n+1}{k} (m+1-k)^n \, (-1)^k. \tag{1.7}$$

Here we aim at presenting certain interesting identities involving the Eulerian numbers by making use of Worpitzky's identity (1.6). For our purpose, we recall the following functions. The *Psi* (or *Digamma*) *function* $\psi(z)$ is defined by

$$\psi(z) := \frac{d}{dz}\{\log\,\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\,\Gamma(z) = \int_1^z \psi(t)\,dt, \tag{1.8}$$

where $\Gamma(z)$ is the familiar Gamma function (see, *e.g.*, [6, Section 1.1] and [7, Section 1.1]). The *Polygamma functions* $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n}{dz^n}\psi(z) \quad (n \in \mathbb{N}_0;\ z \notin \mathbb{Z}_0^-), \tag{1.9}$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. In terms of the generalized (or Hurwitz) Zeta function $\zeta(s,a)$ (see, *e.g.*, [7, Section 2.2]), we can write

$$\begin{aligned} \psi^{(n)}(z) &= (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} \\ &= (-1)^{n+1}\, n!\, \zeta(n+1, z) \quad (n \in \mathbb{N};\ z \notin \mathbb{Z}_0^-), \end{aligned} \tag{1.10}$$

which may be used to deduce the properties of $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ from those of $\zeta(s,z)$ $(s = n+1;\ n \in \mathbb{N})$ and vice versa.

2. Identities involving the Eulerian numbers

We begin by using the expression (see, *e.g.*, [7, p. 5]):

$$\binom{x+k}{n} = \frac{\Gamma(x+k+1)}{n!\,\Gamma(x+k-n+1)}. \tag{2.1}$$

Differentiating each side of (2.1) with respect to the variable x, we get

$$\frac{d}{dx}\binom{x+k}{n} = \binom{x+k}{n}\{\psi(x+k+1)-\psi(x+k-n+1)\}. \tag{2.2}$$

Applying (2.2) to differentiate each side of (1.6) with respect to the variable x, we obtain the following identity:

$$n\,x^{n-1} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}\{\psi(x+k+1)-\psi(x+k-n+1)\} \ \ (n \in \mathbb{N}). \tag{2.3}$$

By using the following known formula for Psi function (see, *e.g.*, [7, p. 25]):

$$\psi(z+n) = \psi(z) + \sum_{j=1}^{n} \frac{1}{z+j-1} \quad (n \in \mathbb{N}), \tag{2.4}$$

a special case of (2.3) when $x = n$ yields the following interesting identity:

$$n^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{n+k}{k} (H_{n+k} - H_k) \quad (n \in \mathbb{N}), \tag{2.5}$$

where H_n are the harmonic numbers given in (1.3). Applying the formula (1.7) for $\left\langle {n \atop k} \right\rangle$ to (2.5), we obtain the following double series formulas: For $n \in \mathbb{N}$,

$$\begin{aligned} n^n &= \sum_{k=0}^{n}\sum_{j=0}^{k} (-1)^j \binom{n+k}{k}\binom{n+1}{j} (H_{n+k}-H_k)\,(k+1-j)^n \\ &= \sum_{j=0}^{n}\sum_{k=j}^{n} (-1)^j \binom{n+k}{k}\binom{n+1}{j} (H_{n+k}-H_k)\,(k+1-j)^n, \end{aligned} \tag{2.6}$$

where, for the second equality, we have used the following rearrangement formula for a double finite series (see, *e.g.*, [1, Eq. (1.24)]):

$$\sum_{k=0}^{n}\sum_{\ell=0}^{k} A_{k,\ell} = \sum_{\ell=0}^{n}\sum_{k=\ell}^{n} A_{k,\ell}. \tag{2.7}$$

Differentiating each side of (2.3) with respect to the variable x and using (1.10), we obtain the following identity:
(2.8)
$$n\,(n-1)\,x^{n-2} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \Big[\left(\psi(x+k+1)-\psi(x+k-n+1)\right)^2 + \zeta(2,x+k+1)-\zeta(2,x+k+1-n)\Big] \quad (n\in\mathbb{N}\setminus\{1\}).$$

Setting $x=n$ in (2.8) and using $\left\langle {n \atop 0} \right\rangle = [n=0]$, we have, for $n\in\mathbb{N}\setminus\{1\}$,

$$(2.9)\quad (n-1)\,n^{n-1} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{n+k}{k} \Big[(H_{n+k}-H_k)^2 + H_k^{(2)} - H_{n+k}^{(2)} \Big],$$

where $H_n^{(r)}$ denote the harmonic numbers of order r defined by

$$(2.10)\qquad H_n^{(r)} := \sum_{k=1}^{n} \frac{1}{k^r} \quad (n\in\mathbb{N}).$$

The Pochhammer symbol $(\lambda)_n$ is defined (for $\lambda\in\mathbb{C}$) by (see [6, p. 2 and p. 6] and [7, p. 2 and pp. 4–6]):

$$(2.11)\qquad (\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N}) \end{cases} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-),$$

where $\mathbb{C}$ denotes the set of complex numbers. Using (2.11), we have

$$(2.12)\qquad \binom{\lambda}{n} = \frac{\Gamma(\lambda+1)}{n!\,\Gamma(\lambda-n+1)} = \frac{(-1)^n\,(-\lambda)_n}{n!}.$$

Applying (2.12) to $\binom{x+k}{n}$ in (1.6), we have another expression of the Worpitzky's identity (1.6):

$$(2.13)\qquad x^n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle (-x-k)_n \quad (n\in\mathbb{N}_0)\,.$$

By recalling the following known expansion for $(x)_n$ (see [4, p. 264]):

$$(2.14)\qquad (x)_n = \sum_{\ell=0}^{n} \left[{n \atop \ell} \right] x^\ell \quad (n\in\mathbb{N}_0)\,,$$

we get

$$(2.15)\qquad \begin{aligned}(-x-k)_n &= \sum_{\ell=0}^{n} (-1)^{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} (x+k)^{\ell} \\ &= \sum_{\ell=0}^{n} (-1)^{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} \sum_{j=0}^{\ell} \binom{\ell}{j} k^{\ell-j}\, x^{j}.\end{aligned}$$

Applying (2.15) to (2.13), we obtain a triple series version of the (1.6) involving binomial coefficients, Striling numbers of the first kind and Eulerian numbers: For $n \in \mathbb{N}_0$,

$$(2.16)\qquad \begin{aligned}x^n &= \frac{(-1)^n}{n!} \sum_{k=0}^{n} \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} (-1)^{\ell} \left\langle {n \atop k} \right\rangle \begin{bmatrix} n \\ \ell \end{bmatrix} \binom{\ell}{j} k^{\ell-j}\, x^{j} \\ &= \frac{(-1)^n}{n!} \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{\ell=j}^{n} (-1)^{\ell} \left\langle {n \atop k} \right\rangle \begin{bmatrix} n \\ \ell \end{bmatrix} \binom{\ell}{j} k^{\ell-j}\, x^{j},\end{aligned}$$

where the second equality follows from (2.7). Setting $x = \mp 1$ in (2.16), we obtain the following four interesting representations for $n!$:

$$(2.17)\qquad \begin{aligned}n! &= \sum_{k=0}^{n} \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} (-1)^{\ell+j} \left\langle {n \atop k} \right\rangle \begin{bmatrix} n \\ \ell \end{bmatrix} \binom{\ell}{j} k^{\ell-j} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{\ell=j}^{n} (-1)^{\ell+j} \left\langle {n \atop k} \right\rangle \begin{bmatrix} n \\ \ell \end{bmatrix} \binom{\ell}{j} k^{\ell-j}\end{aligned}$$

and

$$(2.18)\qquad \begin{aligned}n! &= (-1)^n \sum_{k=0}^{n} \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} (-1)^{\ell} \left\langle {n \atop k} \right\rangle \begin{bmatrix} n \\ \ell \end{bmatrix} \binom{\ell}{j} k^{\ell-j} \\ &= (-1)^n \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{\ell=j}^{n} (-1)^{\ell} \left\langle {n \atop k} \right\rangle \begin{bmatrix} n \\ \ell \end{bmatrix} \binom{\ell}{j} k^{\ell-j}.\end{aligned}$$

Applying some known integral representations for the Psi function (see, e.g., [7, pp. 25-26]) to (2.3), for example, we get the following representations for $n\, x^{n-1}$ involving integrals: For $n \in \mathbb{N}$,

(2.19)

$$n\, x^{n-1} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \int_0^1 \left(t^{-n} - 1\right) \frac{t^{x+k}}{1-t}\, dt \quad (x > n-1).$$

(2.20) $$n\, x^{n-1} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \int_0^\infty \frac{(1+t)^n - 1}{t\,(1+t)^{x+k+1}}\, dt \quad (x > n-1).$$

(2.21)
$$n\, x^{n-1} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \int_0^\infty \frac{e^{nt} - 1}{(e^t - 1)\; e^{t(x+k)}}\, dt \quad (x > n-1).$$

We conclude this note by remarking that the special case of (2.19) when $x = n$ immediately yields the result (2.5).

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