Honam Mathematical J. **35** (2013), No. 3, pp. 351–372
http://dx.doi.org/10.5831/HMJ.2013.35.3.351

REMARKS OF CONGRUENT ARITHMETIC SUMS OF THETA FUNCTIONS DERIVED FROM DIVISOR FUNCTIONS

Aeran Kim, Daeyeoul Kim*, and Nazli Yildiz İkikardes

Abstract. In this paper, we study a distinction the two generating functions : $\varphi^k(q) = \sum_{n=0}^{\infty} r_k(n)q^n$ and $\varphi^{*,k}(q) = \varphi^k(q) - \varphi^k(q^2)$ $(k = 2, 4, 6, 8, 10, 12, 16)$, where $r_k(n)$ is the number of representations of n as the sum of k squares. We also obtain some congruences of representation numbers and divisor function.

1. Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the sets of positive integers and $\mathbb{N} \cup \{0\}$, respectively. For $q \in \mathbb{C}$ with $|q| < 1$ the φ function is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Then we can see that

$$\varphi^k(q) = \sum_{n=0}^{\infty} r_k(n)q^n. \tag{1}$$

For $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we denote the number of representations of n as the sum of k squares by $r_k(n)$, that is

$$r_k(n) := card\left\{(x_1, \cdots, x_k) \in \mathbb{Z}^k | n = x_1^2 + \cdots + x_k^2\right\}.$$

The φ function is related to the theta function :

$$\varphi(q) = \theta_3(0, q),$$

Received March 20, 2013. Accepted June 11, 2013.
2010 Mathematics Subject Classification. 11A67.
Key words and phrases. Infinite product, Convolution sums, Congruent sums.
*Corresponding author

where $\theta_3(z,q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz$.

Proposition 1.1. ([3, p. 98]) *Let $n \in \mathbb{N}$. From Eq. (1) we can obtain*

(a)

$$\varphi^2(q) = 1 + 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{q^n}{1-q^n}.$$

(b)

$$\varphi^4(q) = 1 + 8\sum_{\substack{n=1\\4\nmid n}}^{\infty}\frac{nq^n}{1-q^n}.$$

(c)

$$\varphi^6(q) = 1 + 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}} - 4\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{(-1)^{(n-1)/2}n^2q^n}{1-q^n}.$$

The Legendre-Jacobi-Kronecker symbol for discriminant -4 in Proposition 1.1 is defined for $d \in \mathbb{N}$ by

$$\left(\frac{-4}{d}\right) := \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4, \\ -1, & \text{if } d \equiv 3 \pmod 4, \\ 0, & \text{if } d \equiv 0 \pmod 2. \end{cases} \tag{2}$$

We define

$$\begin{aligned} \varphi^{k,*}(q) &:= \varphi^k(q) - \varphi^k(q^2) := \sum_{n=1}^{\infty} r_k^*(n)q^n \\ &= \sum_{n=0}^{\infty} r_k(n)q^n - \sum_{n=0}^{\infty} r_k(n)q^{2n} = \sum_{n=1}^{\infty}\left\{r_k(n) - r_k(\frac{n}{2})\right\}q^n. \end{aligned} \tag{3}$$

Similarly, we define

$$\sigma_s^*(n) = \sum_{\substack{d|n\\ n/d \text{ odd}}} d^s.$$

Then $\sigma_s^*(n) = \sigma_s(n) - \sigma_s(\frac{n}{2})$.

In Section 2, noting the Proposition 1.1, we solve $\varphi^{k,*}(q)$ for $k = 2, 4, 6$. And we obtain the property $r_4^*(n) \equiv 8\sigma_1^*(n) \pmod{32}$. In Section 3, we calculate $\varphi^{k,*}(q)$ for $k = 8, 10, 12$. In Section 4, using the convolution sum for $r_k(n)$, we obtain $r_{16}(n)$. Through $r_{16}(n)$ we deduce some congruence relation by modulo 32 and 64 (see Theorem

4.7). And we also get the relation $b(n) = -8b(\frac{n}{2})$ with even n for $\sum_{n=1}^{\infty} b(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$. Using the formula of $\varphi^{16,*}(q)$, we obtain $r_{16}(n) \equiv 32\sigma_7(n) \pmod{512}$ for odd n (Theorem 4.9). Finally in Appendix we list some values for $b(n)$.

2. The property of $\varphi^{k,*}(q)$

Lemma 2.1. *Let $n \in \mathbb{N}$. Then we have*

$$\varphi^{2,*}(q) = 4\sum_{\substack{d|n \\ d \text{ odd}}} \left(\frac{-4}{d}\right) q^n = 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{q^n}{1-q^{2n}}.$$

Proof. As $r_2(n) = 4\sum_{d|n}\left(\frac{-4}{d}\right)$, we obtain

$$r_2(2n) - r_2(n) = 4\left\{\sum_{d|2n}\left(\frac{-4}{d}\right) - \sum_{d|n}\left(\frac{-4}{d}\right)\right\}.$$

Here, by (2), we observe that

$$\sum_{d|2n}\left(\frac{-4}{d}\right) = \sum_{\substack{d|2n \\ d \text{ odd}}}\left(\frac{-4}{d}\right) = \sum_{\substack{d|n \\ d \text{ odd}}}\left(\frac{-4}{d}\right) = \sum_{d|n}\left(\frac{-4}{d}\right).$$

Thus $r_2(2n) - r_2(n) = 0$, which leads $r_2^*(2n) = 0$. From this fact and (3) we can see that

$$\begin{aligned}\varphi^{2,*}(q) &= \sum_{n=1}^{\infty} r_2^*(n)q^n = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} r_2^*(n)q^n = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}\left\{r_2(n) - r_2(\frac{n}{2})\right\}q^n \\ &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} r_2(n)q^n = 4\sum_{\substack{d|n \\ n \text{ odd}}}\left(\frac{-4}{d}\right)q^n.\end{aligned}$$

By Proposition 1.1 (a) and (3), we get

$$\begin{aligned}\varphi^{2,*}(q) &= \varphi^2(q) - \varphi^2(q^2) \\ &= 1 + 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{q^n}{1-q^n} - \left\{1 + 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{q^{2n}}{1-q^{2n}}\right\} \\ &= 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\left\{\frac{q^n}{1-q^n} - \frac{q^{2n}}{1-q^{2n}}\right\}\end{aligned}$$

$$= 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\cdot\frac{q^n}{1-q^{2n}}\left(1+q^n-q^n\right)$$
$$= 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{q^n}{1-q^{2n}}.$$

Thus the proof is complete. □

Lemma 2.2. *Let $n \in \mathbb{N}$. Then we have*

$$\varphi^{4,*}(q) = \sum_{n=1}^{\infty}\left\{8\sigma_1^*(n) - 32\sigma_1^*(\frac{n}{4})\right\}q^n = 8\sum_{\substack{n=1\\4\nmid n}}^{\infty} n\frac{q^n}{1-q^{2n}}.$$

Proof. It is clear by Proposition 1.1 (b). □

Lemma 2.3. *Let $n \in \mathbb{N}$. Then we have*

$$\varphi^{6,*}(q) = \sum_{n=1}^{\infty}\left\{16\sum\nolimits_{n \text{ odd}}^{d|n}\left(\frac{-4}{n/d}\right)d^2q^n - 4\sum\nolimits_{n \text{ odd}}^{d|n}\left(\frac{-4}{d}\right)d^2q^n\right.$$
$$\left. +48\sum\nolimits_{n \text{ even}}^{d|\frac{n}{2}}\left(\frac{-4}{\frac{n}{2}/d}\right)d^2q^n\right\}$$
$$= 16\sum_{n=1}^{\infty}n^2\frac{q^n(1-q^n)(1-q^{3n})}{(1+q^{2n})(1+q^{4n})} - 4\sum_{\substack{n=1\\2\nmid n}}^{\infty}(-1)^{(n-1)/2}n^2\frac{q^n}{1-q^{2n}}.$$

Proof. There are given $r_6(n) = 16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2 - 4\sum_{d|n}\left(\frac{-4}{d}\right)d^2$ in [3, Theorem 9.6]. So

(4)
$r_6(2n) - r_6(n)$

$$= 16\sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^2 - 4\sum_{d|2n}\left(\frac{-4}{d}\right)d^2 - \left\{16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2 - 4\sum_{d|n}\left(\frac{-4}{d}\right)d^2\right\}$$
$$= 16\left\{\sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^2 - \sum_{d|n}\left(\frac{-4}{n/d}\right)d^2\right\} - 4\left\{\sum_{d|2n}\left(\frac{-4}{d}\right)d^2 - \sum_{d|n}\left(\frac{-4}{d}\right)d^2\right\}.$$

To simplify the first term in (4), we use the property of (2) and so we have

$$(5)\qquad \begin{aligned}\sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^2 &= \sum_{\frac{2n}{d}\text{ odd}}\left(\frac{-4}{2n/d}\right)d^2 = \sum_{\substack{e|2n\\ e\text{ odd}}}\left(\frac{-4}{e}\right)(\frac{2n}{e})^2\\ &= 4\sum_{\substack{e|n\\ e\text{ odd}}}\left(\frac{-4}{e}\right)(\frac{n}{e})^2,\end{aligned}$$

where we put $e := \frac{2n}{d}$. By letting $l = \frac{n}{e}$, (5) becomes

$$(6)\qquad \begin{aligned}4\sum_{\substack{e|n\\ e\text{ odd}}}\left(\frac{-4}{e}\right)(\frac{n}{e})^2 &= 4\sum_{\substack{l|n\\ \frac{n}{l}\text{ odd}}}\left(\frac{-4}{n/l}\right)l^2 = 4\sum_{l|n}\left(\frac{-4}{n/l}\right)l^2\\ &= 4\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2,\end{aligned}$$

in the last part we replace the index l with d. From (5) and (6), we derive that

$$(7)\qquad \sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^2 - \sum_{d|n}\left(\frac{-4}{n/d}\right)d^2 = 3\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2$$

and

$$(8)\qquad \sum_{\substack{d|n\\ n\text{ even}}}\left(\frac{-4}{n/d}\right)d^2 - \sum_{\substack{d|\frac{n}{2}\\ n\text{ even}}}\left(\frac{-4}{\frac{n}{2}/d}\right)d^2 = 3\sum_{\substack{d|\frac{n}{2}\\ n\text{ even}}}\left(\frac{-4}{\frac{n}{2}/d}\right)d^2.$$

And for the second term in (4) we have

$$\sum_{d|2n}\left(\frac{-4}{d}\right)d^2 = \sum_{\substack{d|2n\\ d\text{ odd}}}\left(\frac{-4}{d}\right)d^2 = \sum_{\substack{d|n\\ d\text{ odd}}}\left(\frac{-4}{d}\right)d^2 = \sum_{d|n}\left(\frac{-4}{d}\right)d^2$$

and

$$\sum_{d|2n}\left(\frac{-4}{d}\right)d^2 - \sum_{d|n}\left(\frac{-4}{d}\right)d^2 = 0.$$

It implies that

$$(9)\qquad \sum_{\substack{d|n\\ n\text{ even}}}\left(\frac{-4}{d}\right)d^2 - \sum_{\substack{d|\frac{n}{2}\\ n\text{ even}}}\left(\frac{-4}{d}\right)d^2 = 0.$$

From (8) and (9) we deduce that

$$\begin{aligned}
\varphi^{6,*}(q) &= \sum_{n=1}^{\infty} \left\{ r_6(n) - r_6(\frac{n}{2}) \right\} q^n \\
&= \sum_{n=1}^{\infty} \Bigg[16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\
&\qquad - \left\{ 16 \sum_{d|\frac{n}{2}} \left(\frac{-4}{\frac{n}{2}/d} \right) d^2 - 4 \sum_{d|\frac{n}{2}} \left(\frac{-4}{d} \right) d^2 \right\} \Bigg] q^n \\
&= \sum_{n=1}^{\infty} \Bigg[16 \left\{ \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - \sum_{d|\frac{n}{2}} \left(\frac{-4}{\frac{n}{2}/d} \right) d^2 \right\} \\
&\qquad - 4 \left\{ \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \sum_{d|\frac{n}{2}} \left(\frac{-4}{d} \right) d^2 \right\} \Bigg] q^n \\
&= \sum_{n=1}^{\infty} \Bigg\{ 16 \sum_{\substack{d|n \\ n \text{ odd}}} \left(\frac{-4}{n/d} \right) d^2 q^n + 48 \sum_{\substack{d|\frac{n}{2} \\ n \text{ even}}} \left(\frac{-4}{\frac{n}{2}/d} \right) d^2 q^n \\
&\qquad - 4 \sum_{\substack{d|n \\ n \text{ odd}}} \left(\frac{-4}{d} \right) d^2 q^n \Bigg\}.
\end{aligned}$$

By Proposition 1.1 (c), we obtain

$$\begin{aligned}
\varphi^{6,*}(q) &= \varphi^6(q) - \varphi^6(q^2) \\
&= \left\{ 1 + 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}} - 4 \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} \frac{(-1)^{(n-1)/2} n^2 q^n}{1-q^n} \right\} \\
&\qquad - \left\{ 1 + 16 \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{1+q^{4n}} - 4 \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} \frac{(-1)^{(n-1)/2} n^2 q^{2n}}{1-q^{2n}} \right\} \\
&= 16 \sum_{n=1}^{\infty} n^2 \frac{q^n}{(1+q^{2n})(1+q^{4n})} \left\{ 1 + q^{4n} - q^n(1+q^{2n}) \right\}
\end{aligned}$$

$$- 4\sum_{\substack{n=1\\2\nmid n}}^{\infty}(-1)^{(n-1)/2}n^2\cdot\frac{q^n}{1-q^{2n}}(1+q^n-q^n)$$

$$= 16\sum_{n=1}^{\infty}n^2\frac{q^n(1-q^n)(1-q^{3n})}{(1+q^{2n})(1+q^{4n})} - 4\sum_{\substack{n=1\\2\nmid n}}^{\infty}(-1)^{(n-1)/2}n^2\frac{q^n}{1-q^{2n}}. \qquad \square$$

Corollary 2.4. *Let $n\in\mathbb{N}$. Then we have $r_4^*(n)\equiv 8\sigma_1^*(n) \pmod{32}$.*

Proof. It is obvious by Lemma 2.2. $\square$

3. Property of φ function

From (1) we know that $\varphi^8(q) = 1+\sum_{n=1}^{\infty} r_8(n)q^n$. By [3, Theorem 19.1] we deduce that

$$\begin{aligned}\varphi^8(q) &= 1+\sum_{n=1}^{\infty}\left\{16(-1)^n\sum_{\substack{d\in\mathbb{N}\\d\mid n}}(-1)^d d^3\right\}q^n\\ &= 1+16\sum_{\substack{n=1\\2\nmid n}}^{\infty}n^3\sum_{k=1}^{\infty}(-1)^k q^{nk} + 16\sum_{\substack{n=1\\2\mid n}}^{\infty}n^3\sum_{k=1}^{\infty}q^{nk}.\end{aligned} \tag{10}$$

Lemma 3.1. *Let $n\in\mathbb{N}$. Then we have*

$$\begin{aligned}\varphi^{8,*}(q) &= \sum_{n=1}^{\infty}\left\{16\sigma_3^*(n)-32\sigma_3^*(\frac{n}{2})+256\sigma_3^*(\frac{n}{4})\right\}q^n\\ &= 16\left[\sum_{\substack{n=1\\2\nmid n}}^{\infty}n^3\frac{q^n(1-q^n)}{(1+q^n)(1+q^{2n})}+\sum_{\substack{n=1\\2\mid n}}^{\infty}n^3\frac{q^n}{1-q^{2n}}\right].\end{aligned}$$

Proof. Since $r_8(n) = 16\sigma_3(n)-32\sigma_3(\frac{n}{2})+256\sigma_3(\frac{n}{4})$ in [3, (19.10)], we have

$$\begin{aligned}\varphi^{8,*}(q) &= \sum_{n=1}^{\infty}\left\{r_8(n)-r_8(\frac{n}{2})\right\}q^n\\ &= \sum_{n=1}^{\infty}\left[\left\{16\sigma_3(n)-32\sigma_3(\frac{n}{2})+256\sigma_3(\frac{n}{4})\right\}\right.\end{aligned}$$

$$
\begin{aligned}
&- \left\{16\sigma_3(\frac{n}{2}) - 32\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{8})\right\}\Bigg] q^n \\
&= \sum_{n=1}^{\infty} \left\{16\sigma_3^*(n) - 32\sigma_3^*(\frac{n}{2}) + 256\sigma_3^*(\frac{n}{4})\right\} q^n.
\end{aligned}
$$

And using (10) we obtain

$$
\begin{aligned}
\varphi^{8,*}(q) &= 1 + 16\left\{\sum_{\substack{n=1\\2\nmid n}}^{\infty} \frac{n^3 q^n}{1+q^n} + \sum_{\substack{n=1\\2\mid n}}^{\infty} \frac{n^3 q^n}{1-q^n}\right\} \\
&\quad - \left[1 + 16\left\{\sum_{\substack{n=1\\2\nmid n}}^{\infty} \frac{n^3 q^{2n}}{1+q^{2n}} + \sum_{\substack{n=1\\2\mid n}}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}}\right\}\right] \\
&= 16\left[\sum_{\substack{n=1\\2\nmid n}}^{\infty} n^3 \frac{q^n(1-q^n)}{(1+q^n)(1+q^{2n})} + \sum_{\substack{n=1\\2\mid n}}^{\infty} n^3 \frac{q^n}{1-q^{2n}}\right]. \qquad \square
\end{aligned}
$$

In [3, p. 260], we can know that

$$
r_{10}(n) = \frac{4}{5}\sum_{d\mid n}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^4 + \frac{32}{5}w(n),
$$

where $\sum_{n=1}^{\infty} w(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^2(1-q^{4n})^4$. So we obtain that

(11)
$$
\varphi^{10}(q) = 1 + \frac{4}{5}\sum_{\substack{n=1\\d\mid n}}^{\infty}\left(\frac{-4}{d}\right)d^4 q^n + \frac{64}{5}\sum_{\substack{n=1\\d\mid n}}^{\infty}\left(\frac{-4}{n/d}\right)d^4 q^n + \sum_{n=1}^{\infty}\frac{32}{5}w(n)q^n.
$$

Then, by (2), the second term in the right hand side of Eq. (11) can be written as

$$\begin{aligned}\sum_{\substack{n=1\\d|n}}^{\infty}\left(\frac{-4}{d}\right)d^4q^n &= \left(\frac{-4}{1}\right)1^4q + \left\{\left(\frac{-4}{1}\right)1^4 + \left(\frac{-4}{2}\right)2^4\right\}q^2\\ &\quad + \left\{\left(\frac{-4}{1}\right)1^4 + \left(\frac{-4}{3}\right)3^4\right\}q^3\\ &\quad + \left\{\left(\frac{-4}{1}\right)1^4 + \left(\frac{-4}{2}\right)2^4 + \left(\frac{-4}{4}\right)4^4\right\}q^4\\ &\quad + \cdots\cdots\cdots\\ &= \sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{(-1)^{\frac{n-1}{2}}n^4q^n}{1-q^n}.\end{aligned}$$

We note that $\frac{q^n}{1-q^n} = q^n + q^{2n} + q^{3n} + \cdots$. Similarly, the third term in (11) becomes

$$\begin{aligned}\sum_{\substack{n=1\\d|n}}^{\infty}\left(\frac{-4}{n/d}\right)d^4q^n &= 1^4(q - q^3 + q^5 + \cdots) + 2^4(q^2 - q^6 + q^{10} + \cdots)\\ &\quad + 3^4(q^3 - q^9 + q^{15} + \cdots) + 4^4(q^4 - q^{12} + q^{20} + \cdots)\\ &\quad + \cdots\cdots\cdots\\ &= \sum_{n=1}^{\infty}\frac{n^4q^n}{1+q^{2n}}.\end{aligned}$$

Therefore, we obtain

(12)
$$\varphi^{10}(q) = 1 + \frac{4}{5}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{(-1)^{\frac{n-1}{2}}n^4q^n}{1-q^n} + \frac{64}{5}\sum_{n=1}^{\infty}\frac{n^4q^n}{1+q^{2n}} + \frac{32}{5}\sum_{n=1}^{\infty}w(n)q^n.$$

Lemma 3.2. *Let $n \in \mathbb{N}$. Then we obtain*

$$\begin{aligned}\varphi^{10,*}(q) &= \sum_{n=1}^{\infty}\left\{\frac{4}{5}\sum\nolimits_{\substack{d|n\\ n \text{ odd}}}\left(\frac{-4}{d}\right)d^4q^n + \frac{64}{5}\sum\nolimits_{\substack{d|n\\ n \text{ odd}}}\left(\frac{-4}{n/d}\right)d^4q^n\right.\\ &\qquad \left.+192\sum\nolimits_{\substack{d|\frac{n}{2}\\ n \text{ even}}}\left(\frac{-4}{\frac{n}{2}/d}\right)d^4q^n + \frac{24}{5}w^*(n)q^n\right\}\\ &= \frac{4}{5}\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}(-1)^{(n-1)/2}n^4\frac{q^n}{1-q^{2n}} + \frac{64}{5}\sum_{n=1}^{\infty}n^4\frac{q^n(1-q^n)(1-q^{3n})}{(1+q^{2n})(1+q^{4n})}\\ &\qquad + \frac{32}{5}\sum_{n=1}^{\infty}w^*(n)q^n,\end{aligned}$$

where

$$\begin{aligned}\sum_{n=1}^{\infty}w^*(n)q^n &:= \sum_{n=1}^{\infty}\left\{w(n)-w(\frac{n}{2})\right\}q^n\\ &= q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^2(1-q^{4n})^4\\ &\qquad - q^2\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^2(1-q^{8n})^4.\end{aligned}$$

Proof. Since $r_{10}(n) = \frac{4}{5}\sum_{d|n}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{d|n}\left(\frac{-4}{n/d}\right)d^4 + \frac{32}{5}w(n)$, so

$$\begin{aligned}&r_{10}(2n) - r_{10}(n)\\ &= \frac{4}{5}\sum_{d|2n}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^4 + \frac{32}{5}w(2n)\\ &\qquad - \left\{\frac{4}{5}\sum_{d|n}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{d|n}\left(\frac{-4}{n/d}\right)d^4 + \frac{32}{5}w(n)\right\}\\ &= \frac{4}{5}\left\{\sum_{d|2n}\left(\frac{-4}{d}\right)d^4 - \sum_{d|n}\left(\frac{-4}{d}\right)d^4\right\} + \frac{64}{5}\left\{\sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^4\right.\\ &\qquad \left.-\sum_{d|n}\left(\frac{-4}{n/d}\right)d^4\right\} + \frac{32}{5}\left\{w(2n)-w(n)\right\}.\end{aligned} \tag{13}$$

To simplify the first term in (13), we use the property of (2) and so we have

$$\sum_{\substack{d\in\mathbb{N}\\ d|2n}}\left(\frac{-4}{2n/d}\right)d^4=\sum_{\substack{d|2n\\ \frac{2n}{d}\text{ odd}}}\left(\frac{-4}{2n/d}\right)d^4=\sum_{\substack{e|2n\\ e\text{ odd}}}\left(\frac{-4}{e}\right)(\frac{2n}{e})^4 \tag{14}$$
$$=16\sum_{\substack{e|n\\ e\text{ odd}}}\left(\frac{-4}{e}\right)(\frac{n}{e})^4,$$

where we put $e:=\frac{2n}{d}$. By letting $l=\frac{n}{e}$, (14) becomes

$$16\sum_{\substack{e|n\\ e\text{ odd}}}\left(\frac{-4}{e}\right)(\frac{n}{e})^4=16\sum_{\substack{l|n\\ \frac{n}{l}\text{ odd}}}\left(\frac{-4}{n/l}\right)l^4=16\sum_{l|n}\left(\frac{-4}{n/l}\right)l^4 \tag{15}$$
$$=16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^4$$

in the last part we replace the index l with d. (14) and (15) show that

$$\sum_{d|2n}\left(\frac{-4}{2n/d}\right)d^4-\sum_{d|n}\left(\frac{-4}{n/d}\right)d^4=15\sum_{d|n}\left(\frac{-4}{n/d}\right)d^4. \tag{16}$$

(16) implies that

$$\sum_{\substack{d|n\\ n\text{ even}}}\left(\frac{-4}{n/d}\right)d^4-\sum_{\substack{d|\frac{n}{2}\\ n\text{ even}}}\left(\frac{-4}{\frac{n}{2}/d}\right)d^4=15\sum_{\substack{d|\frac{n}{2}\\ n\text{ even}}}\left(\frac{-4}{\frac{n}{2}/d}\right)d^4. \tag{17}$$

And for the second term in (13) we have

$$\sum_{d|2n}\left(\frac{-4}{d}\right)d^4=\sum_{\substack{d|2n\\ d\text{ odd}}}\left(\frac{-4}{d}\right)d^4=\sum_{\substack{d|n\\ d\text{ odd}}}\left(\frac{-4}{d}\right)d^4=\sum_{d|n}\left(\frac{-4}{d}\right)d^4.$$

So

$$\sum_{d|2n}\left(\frac{-4}{d}\right)d^4-\sum_{d|n}\left(\frac{-4}{d}\right)d^4=0.$$

It implies that

$$\sum_{\substack{d|n\\ n\text{ even}}}\left(\frac{-4}{d}\right)d^4-\sum_{\substack{d|\frac{n}{2}\\ n\text{ even}}}\left(\frac{-4}{d}\right)d^4=0. \tag{18}$$

From (17) and (18) we deduce that

$$\begin{aligned}\varphi^{10,*}(q) &= \sum_{n=1}^{\infty}\left\{r_{10}(n) - r_{10}(\frac{n}{2})\right\} q^n \\ &= \sum_{n=1}^{\infty}\left[\frac{4}{5}\sum_{d|n}\left(\frac{-4}{d}\right) d^4 + \frac{64}{5}\sum_{d|n}\left(\frac{-4}{n/d}\right) d^4 + \frac{32}{5} w(n)\right. \\ &\quad \left. -\left\{\frac{4}{5}\sum_{d|\frac{n}{2}}\left(\frac{-4}{d}\right) d^4 + \frac{64}{5}\sum_{d|\frac{n}{2}}\left(\frac{-4}{\frac{n}{2}/d}\right) d^4 + \frac{32}{5} w(\frac{n}{2})\right\}\right] q^n \\ &= \sum_{n=1}^{\infty}\left\{\frac{4}{5}\sum_{\substack{d|n \\ n \text{ odd}}}\left(\frac{-4}{d}\right) d^4 q^n + \frac{64}{5}\sum_{\substack{d|n \\ n \text{ odd}}}\left(\frac{-4}{n/d}\right) d^4 q^n \right. \\ &\quad \left. +192\sum_{\substack{d|\frac{n}{2} \\ n \text{ even}}}\left(\frac{-4}{\frac{n}{2}/d}\right) d^4 q^n + \frac{24}{5} w^*(n) q^n\right\}.\end{aligned}$$

Here, considering the definition of $\sum_{n=1}^{\infty} w(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^2(1-q^{4n})^4$ we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\left\{w(n) - w(\frac{n}{2})\right\} q^n &= \sum_{n=1}^{\infty}\left\{w(n)q^n - w(n)q^{2n}\right\} \\ &= q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^2(1-q^{4n})^4 \\ &\quad - q^2\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^2(1-q^{8n})^4.\end{aligned} \tag{19}$$

By (12), we get

$$\begin{aligned}&\varphi^{10,*}(q) = \varphi^{10}(q) - \varphi^{10}(q^2) \\ &= \left\{1 + \frac{4}{5}\sum_{\substack{n=1 \\ 2\nmid n}}^{\infty}\frac{(-1)^{\frac{n-1}{2}} n^4 q^n}{1-q^n} + \frac{64}{5}\sum_{n=1}^{\infty}\frac{n^4 q^n}{1+q^{2n}} + \frac{32}{5}\sum_{n=1}^{\infty} w(n)q^n\right\} \\ &\quad - \left\{1 + \frac{4}{5}\sum_{\substack{n=1 \\ 2\nmid n}}^{\infty}\frac{(-1)^{\frac{n-1}{2}} n^4 q^{2n}}{1-q^{2n}} + \frac{64}{5}\sum_{n=1}^{\infty}\frac{n^4 q^{2n}}{1+q^{4n}} + \frac{32}{5}\sum_{n=1}^{\infty} w(n)q^{2n}\right\}\end{aligned}$$

$$= \frac{4}{5}\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}(-1)^{(n-1)/2}n^4\frac{q^n}{1-q^{2n}} + \frac{64}{5}\sum_{n=1}^{\infty}n^4\frac{q^n(1-q^n)(1-q^{3n})}{(1+q^{2n})(1+q^{4n})}$$

$$+ \frac{32}{5}\sum_{n=1}^{\infty}w^*(n)q^n. \qquad \square$$

From [3, p. 256], we put $r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16\beta(n)$. Hence

$$\varphi^{12}(q) = 1 + 8\sum_{n=1}^{\infty}\left\{\sigma_5(n) - 4^5\sigma_5(\frac{n}{4})\right\}q^n + \frac{15}{2}\sum_{n=1}^{\infty}4^5\sigma_5(\frac{n}{4})q^n + 16\sum_{n=1}^{\infty}\beta(n)q^n$$

$$(20) \qquad = 1 + 8\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ 4\nmid d}}d^5q^n + \frac{15}{2}\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ 4|d}}d^5q^n + 16\sum_{n=1}^{\infty}\beta(n)q^n$$

$$= 1 + 8\sum_{\substack{n=1\\ 4\nmid n}}^{\infty}\frac{n^5q^n}{1-q^n} + \frac{15}{2}\sum_{\substack{n=1\\ 4|n}}^{\infty}\frac{n^5q^n}{1-q^n} + 16\sum_{n=1}^{\infty}\beta(n)q^n,$$

where $\sum_{n=1}^{\infty}\beta(n)q^n := q\prod_{n=1}^{\infty}(1-q^{2n})^{12}$, $q \in \mathbb{C}$, $|q| < 1$.

Lemma 3.3. *Let $n \in \mathbb{N}$. Then we obtain*

$$\varphi^{12,*}(q) = \sum_{n=1}^{\infty}\left\{8\sigma_5^*(n) - 512\sigma_5^*(\frac{n}{4}) + 16\beta^*(n)\right\}q^n$$

$$= 8\sum_{\substack{n=1\\ 4\nmid n}}^{\infty}n^5\frac{q^n}{1-q^{2n}} + \frac{15}{2}\sum_{\substack{n=1\\ 4|n}}^{\infty}n^5\frac{q^n}{1-q^{2n}} + 16\sum_{n=1}^{\infty}\beta^*(n)q^n,$$

where

$$\sum_{n=1}^{\infty}\beta^*(n)q^n := \sum_{n=1}^{\infty}\left\{\beta(n) - \beta(\frac{n}{2})\right\}q^n$$

$$= q\prod_{n=1}^{\infty}(1-q^{2n})^{12} - q^2\prod_{n=1}^{\infty}(1-q^{4n})^{12}.$$

Proof. Since $r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16\beta(n)$, we have

$$\begin{aligned}\varphi^{12,*}(q) &= \sum_{n=1}^{\infty}\left\{r_{12}(n) - r_{12}(\frac{n}{2})\right\}q^n \\ &= \sum_{n=1}^{\infty}\left[\left\{8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) + 16\beta(n)\right\}\right. \\ &\qquad \left. - \left\{8\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{8}) + 16\beta(\frac{n}{2})\right\}\right]q^n \\ &= \sum_{n=1}^{\infty}\left\{8\sigma_5^*(n) - 512\sigma_5^*(\frac{n}{4}) + 16\beta^*(n)\right\}q^n.\end{aligned}$$

Also, by (20), we obtain

$$\begin{aligned}&\varphi^{12,*}(q) = \varphi^{12}(q) - \varphi^{12}(q^2) \\ &= \left\{1 + 8\sum_{\substack{n=1\\4\nmid n}}^{\infty}\frac{n^5q^n}{1-q^n} + \frac{15}{2}\sum_{\substack{n=1\\4|n}}^{\infty}\frac{n^5q^n}{1-q^n} + 16\sum_{n=1}^{\infty}\beta(n)q^n\right\} \\ &\quad - \left\{1 + 8\sum_{\substack{n=1\\4\nmid n}}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}} + \frac{15}{2}\sum_{\substack{n=1\\4|n}}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}} + 16\sum_{n=1}^{\infty}\beta(n)q^{2n}\right\} \\ &= 8\sum_{\substack{n=1\\4\nmid n}}^{\infty}n^5\frac{q^n}{1-q^{2n}} + \frac{15}{2}\sum_{\substack{n=1\\4|n}}^{\infty}n^5\frac{q^n}{1-q^{2n}} + 16\sum_{n=1}^{\infty}\beta^*(n)q^n.\end{aligned}$$

□

Remark 3.4. From (10) we can know that

$$\varphi^8(-q) = 16\left\{\sum_{\substack{n=1\\2\nmid n}}^{\infty}n^3(\frac{-q^n}{1-q^n}) + \sum_{\substack{n=1\\2|n}}^{\infty}\frac{n^3q^n}{1-q^n}\right\}. \tag{21}$$

Then subtracting (10) by (21) we obtain

$$\varphi^8(q) - \varphi^8(-q) = 32\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{n^3q^n}{1-q^{2n}} = 32\sum_{n=1}^{\infty}\sigma_3(2n-1)q^{2n-1}. \tag{22}$$

In a similar manner, we can see that

$$\varphi^{12}(q) = 1 + 8\sum_{\substack{n=1\\4\nmid n}}^{\infty} \frac{n^5q^n}{1-q^n} + \frac{15}{2}\sum_{\substack{n=1\\4\mid n}}^{\infty} \frac{n^5q^n}{1-q^n} + 16\sum_{n=1}^{\infty}\beta(n)q^n$$

from (20). Thus

$$\varphi^{12}(-q) = 1 + 8\sum_{\substack{n=1\\4\nmid n}}^{\infty} \frac{n^5(-q)^n}{1-(-q)^n} + \frac{15}{2}\sum_{\substack{n=1\\4\mid n}}^{\infty} \frac{n^5q^n}{1-q^n} - 16\sum_{n=1}^{\infty}\beta(n)q^n.$$

Then we have

$$\varphi^{12}(q) - \varphi^{12}(-q) = 8\left\{\sum_{\substack{n=1\\4\nmid n}}^{\infty} \frac{n^5q^n}{1-q^n} - \sum_{\substack{n=1\\4\nmid n}}^{\infty} \frac{n^5(-q)^n}{1-(-q)^n}\right\} + 32\sum_{n=1}^{\infty}\beta(n)q^n$$

$$= 8\left[\left\{\sum_{\substack{n=1\\4\nmid n\\2\mid n}}^{\infty} \frac{n^5q^n}{1-q^n} + \sum_{\substack{n=1\\2\nmid n}}^{\infty} \frac{n^5q^n}{1-q^n}\right\} - \left\{\sum_{\substack{n=1\\4\nmid n\\2\mid n}}^{\infty} \frac{n^5q^n}{1-q^n} - \sum_{\substack{n=1\\2\nmid n}}^{\infty} \frac{n^5q^n}{1+q^n}\right\}\right]$$

$$+ 32\sum_{n=1}^{\infty}\beta(n)q^n$$

$$= 16\sum_{\substack{n=1\\2\nmid n}}^{\infty} \frac{n^5q^n}{1-q^{2n}} + 32\sum_{n=1}^{\infty}\beta(n)q^n$$

$$= 16\sum_{n=1}^{\infty}\sigma_5(2n-1)q^{2n-1} + 32q\prod_{n=1}^{\infty}(1-q^{2n})^{12}.$$

4. Property of $r_{16}(n)$

There is a simple convolution formula relating the number $r_k(n)$ of representations of a positive integer n as the sum of k squares to the number $r_e(n)$ of representations of n as the sum of e squares and $r_{k-e}(n)$. We have :

Proposition 4.1. (*See* [3, p. 120]) *Let* $n, e, k \in \mathbb{N}$ *satisfy* $2 \le e \le k-2$. *Then we have*

$$r_k(n) = \sum_{l=0}^{n} r_e(l) r_{k-e}(n-l).$$

Lemma 4.2. *Let* $n \in \mathbb{N}$. *Then we have*

$$r_{16}(n) = \frac{32}{17}\left\{\sigma_7(n) - 2\sigma_7(\frac{n}{2}) + 256\sigma_7(\frac{n}{4}) + 16b(n) + 256b(\frac{n}{2})\right\},$$

where $\sum_{n=1}^{\infty} b(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$.

Proof. Using Proposition 4.1 we can write as

$$r_{16}(n) = \sum_{k=0}^{n} r_8(k) r_8(n-k). \tag{23}$$

Since $r_8(0) = 1$ and [3, (19.10)], (23) becomes

$$\begin{aligned}
&r_{16}(n) - 2r_8(n) = \sum_{k=1}^{n-1} r_8(k) r_8(n-k)\\
&= \sum_{k=1}^{n-1}\left\{16\sigma_3(k) - 32\sigma_3(\frac{k}{2}) + 256\sigma_3(\frac{k}{4})\right\}\\
&\qquad \times \left\{16\sigma_3(N-k) - 32\sigma_3(\frac{N-k}{2}) + 256\sigma_3(\frac{N-k}{4})\right\}\\
&= 256\sum_{k=1}^{n-1}\sigma_3(k)\sigma_3(n-k) - 512\sum_{k<n/2}\sigma_3(n-2k)\sigma_3(k)\\
&\qquad + 4096\sum_{k<n/4}\sigma_3(n-4k)\sigma_3(k) - 512\sum_{k<n/2}\sigma_3(k)\sigma_3(n-2k)\\
&\qquad + 1024\sum_{k<n/2}\sigma_3(k)\sigma_3(\frac{n}{2}-k) - 8192\sum_{k<n/4}\sigma_3(\frac{n}{2}-2k)\sigma_3(k)\\
&\qquad + 4096\sum_{k<n/4}\sigma_3(k)\sigma_3(n-4k) - 8192\sum_{k<n/4}\sigma_3(k)\sigma_3(\frac{n}{2}-2k)\\
&\qquad + 65536\sum_{k<n/4}\sigma_3(k)\sigma_3(\frac{n}{4}-k).
\end{aligned}$$

Then we refer to

$$\sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k) = \frac{1}{120}\{\sigma_7(n) - \sigma_3(n)\}$$

in [2, (3.17)],

$$\sum_{k<n/2} \sigma_3(k)\sigma_3(n-2k) = \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{272}b(n)$$

in [1, Theorem 5.2] and

$$\sum_{k<n/2} \sigma_3(k)\sigma_3(n-4k) = \frac{1}{32640}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{255}\sigma_7(\frac{n}{4}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{4}) + \frac{9}{2176}b(n) + \frac{9}{136}b(\frac{n}{2})$$

in [1, Theorem 5.2]. □

Remark 4.3. In [1, p. 49] we can see that

$$r_{16}(n) = \frac{32}{17}(-1)^{n-1}\left\{\sigma_7(n) - 256\sigma_7(\frac{n}{2}) + 16b(n)\right\}, \tag{24}$$

$n \in \mathbb{N}$. When n is an even integer, equating (24) with Theorem 4.2, we have

$$128b(\frac{b}{2}) + 16b(n) = -\sigma_7(n) + 129\sigma_7(\frac{n}{2}) - 128\sigma_7(\frac{n}{4}) = -\sigma_{7,oo}(n),$$

where $\sigma_{7,oo}(n) := \sum_{\substack{d|n \\ d \text{ odd} \\ \frac{n}{d} \text{ odd}}} d^7$. Note that $\sigma_{s,oo}(n) = \sigma_s(n) - (2^s+1)\sigma_s(\frac{n}{2}) + 2^s\sigma_s(\frac{n}{4})$ in [3, p. 35]. And we can deduce that if n is even then $\sigma_{7,oo}(n) = 0$. Therefore,

$$b(n) = -8b(\frac{n}{2}).$$

Lemma 4.4. *Let $n \in \mathbb{N}$. Then we obtain*

$$\varphi^{16,*}(q) = \frac{32}{17}\sum_{n=1}^{\infty}\left\{\sigma_7^*(n) - 2\sigma_7^*(\frac{n}{2}) + 256\sigma_7^*(\frac{n}{4}) + 16b^*(n) + 256b^*(\frac{n}{2})\right\}q^n,$$

where

$$\sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=1}^{\infty} \left\{ b(n) - b(\frac{n}{2}) \right\} q^n$$
$$= q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8 - q^2 \prod_{n=1}^{\infty} (1-q^{2n})^8 (1-q^{4n})^8.$$

Proof. By Lemma 4.2 we have

$$\varphi^{16,*}(q) = \sum_{n=1}^{\infty} \left\{ r_{16}(n) - r_{16}(\frac{n}{2}) \right\} q^n$$
$$= \sum_{n=1}^{\infty} \left[\frac{32}{17} \left\{ \sigma_7(n) - 2\sigma_7(\frac{n}{2}) + 256\sigma_7(\frac{n}{4}) + 16b(n) + 256b(\frac{n}{2}) \right\} \right.$$
$$\left. - \frac{32}{17} \left\{ \sigma_7(\frac{n}{2}) - 2\sigma_7(\frac{n}{4}) + 256\sigma_7(\frac{n}{8}) + 16b(\frac{n}{2}) + 256b(\frac{n}{4}) \right\} \right] q^n$$
$$= \frac{32}{17} \sum_{n=1}^{\infty} \left\{ \sigma_7^*(n) - 2\sigma_7^*(\frac{n}{2}) + 256\sigma_7^*(\frac{n}{4}) + 16b^*(n) + 256b^*(\frac{n}{2}) \right\} q^n,$$

where

$$\sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=1}^{\infty} \left\{ b(n) - b(\frac{n}{2}) \right\} q^n = \sum_{n=1}^{\infty} \left\{ b(n)q^n - b(n)q^{2n} \right\}$$
$$= q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8 - q^2 \prod_{n=1}^{\infty} (1-q^{2n})^8 (1-q^{4n})^8.$$

Thus the proof is complete. □

Example 4.5. *We list first tenth values of* $r_{16}^*(n)$.

n	1	2	3	4	5	6	7	8	9	10
$r_{16}^*(n)$	32	448	4480	28672	140736	521472	1580800	3964928	8945824	18485376

TABLE 1. Some values of $r_{16}^*(n)$

Lemma 4.6. *Let n be an odd integer. Then we have* $\sigma_7(n) \equiv b(n)$ (mod 17).

Proof. Using Lemma 4.2 for odd n, we obtain

$$r_{16}(n) = \frac{32}{17} \left\{ \sigma_7(n) + 16b(n) \right\}.$$

So $\sigma_7(n) + 16b(n) \equiv 0 \pmod{17}$. It follows that

$$\sigma_7(n) \equiv -16b(n) \equiv b(n) \pmod{17}.$$ □

Theorem 4.7. *Let $n \in \mathbb{N}$. Then we have*

(a) $r_{16}(n) \equiv 32\sigma_7(n) \pmod{64}$.
(b) $r_{16}(n) \equiv 0 \pmod{32}$.
(c) *Let $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. If p_i and e_i are odd, then $r_{16}(n) \equiv 0 \pmod{64}$.*
(d)

$$r_{16}(n) \equiv \begin{cases} 32 \pmod{64}, & \text{if } n = 2^m \ (m \geq 0) \text{ or } t^2 \text{ or } 2^m t^2, \\ 0 \pmod{64}, & \text{otherwise.} \end{cases}$$

Proof. (a) By Lemma 4.2, we obtain

$$17r_{16}(n) = 32\sigma_7(n) - 64\sigma_7(\frac{n}{2}) + 32 \cdot 256\sigma_7(\frac{n}{4}) + 32 \cdot 16b(n) + 32 \cdot 256b(\frac{n}{2}).$$

So we have

$$17r_{16}(n) \equiv 32\sigma_7(n) \pmod{64}. \tag{25}$$

Since $(49, 64) = 1$, (25) becomes

$$49 \cdot 17r_{16}(n) \equiv 49 \cdot 32\sigma_7(n) \equiv (2 \cdot 24 + 1)32\sigma_7(n) \equiv 32\sigma_7(n) \pmod{64}.$$

Here $49 \cdot 17 \equiv 1 \pmod{64}$, so we deduce that $r_{16}(n) \equiv 32\sigma_7(n) \pmod{64}$.

(b) By (25), we have $17r_{16}(n) = 32\sigma_7(n) + 64k$ for some integer k. Thus $17r_{16}(n) \equiv 0 \pmod{32}$. Since $(17, 32) = 1$, we have $r_{16}(n) \equiv 0 \pmod{32}$.

(c) Let $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ and $p_i = p_1$, then we can see that

$$\begin{aligned} \sigma_7(n) &= \sigma_7(p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}) = \sigma_7(p_1^{e_1})\sigma_7(\frac{n}{p_1^{e_1}}) \\ &= \left\{1 + p_1^7 + p_1^{14} + \cdots + p_1^{7e_1}\right\} \sigma_7(\frac{n}{p_1^{e_1}}) \\ &\equiv 0 \pmod{2}. \end{aligned}$$

Therefore, by Theorem 4.7 (a), we obtain

$$17r_{16}(n) \equiv 32\sigma_7(n) \equiv 0 \pmod{64}.$$

Since $(17, 64) = 1$, so $r_{16}(n) \equiv 0 \pmod{64}$.

(d) The proof is similar to Theorem 4.7 (c). If $n = 2^m$, then

$$\sigma_7(n) = \sigma_7(2^m) = 1 + 2^7 + 2^{14} + \cdots + 2^{7m} \equiv 1 \pmod{2}.$$

It means that $\sigma_7(n) = 2k+1$ for some integer k. Thus, by Theorem 4.7 (a), we obtain

$$17r_{16}(n) \equiv 32\sigma_7(n) \equiv 32(2k+1) \equiv 32 \pmod{64}.$$

Since $(17, 64) = 1$, so $r_{16}(n) \equiv 32 \pmod{64}$. The other cases are similar. □

Corollary 4.8. *We have* $r_{16}(2^s t^{2l} m) \equiv r_{16}(m) \pmod{64}$.

Proof. We deduce from Theorem 4.7 (d). □

Theorem 4.9. *Let n be an odd integer. Then we have* $r_{16}(n) \equiv 32\sigma_7(n) \pmod{512}$.

Proof. In Lemma 4.4, if n is odd then

$$\varphi^{16,*}(q) = \sum_{r=1}^{\infty} r_{16}(n)q^n = \frac{32}{17}\sum_{n=1}^{\infty} \{\sigma_7(n) + 16b(n)\}\, q^n.$$

So

$$\sum_{r=1}^{\infty} 17r_{16}(n)q^n = \sum_{n=1}^{\infty} \{32\sigma_7(n) + 32\cdot 16b(n)\}\, q^n.$$

Therefore

$$17r_{16}(n) \equiv 32\sigma_7(n) \pmod{512}. \tag{26}$$

Since $(17, 512) = 1$, (26) is

$$\begin{aligned}241\cdot 17r_{16}(n) &\equiv 241\cdot 32\sigma_7(n) \equiv (512\cdot 15 + 32)\sigma_7(n)\\ &\equiv 32\sigma_7(n) \pmod{512}.\end{aligned}$$

Because $241\cdot 17 \equiv 1 \pmod{512}$, so we deduce that $r_{16}(n) \equiv 32\sigma_7(n) \pmod{512}$. □

5. Appendix

The first forty five values of $b(n)$ are given in Table 2.

n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	16	4096	31	227552
2	−8	17	14706	32	−32768
3	12	18	16344	33	13104
4	64	19	−39940	34	−117648
5	−210	20	−13440	35	−213360
6	−96	21	12192	36	−130752
7	1016	22	−8736	37	160526
8	−512	23	68712	38	319520
9	−2043	24	−6144	39	16584
10	1680	25	−34025	40	107520
11	1092	26	−11056	41	10842
12	768	27	−50760	42	−97536
13	1382	28	65024	43	−630748
14	−8128	29	−102570	44	69888
15	−2520	30	20160	45	429030

TABLE 2. $b(n)$ for n $(1 \le n \le 45)$

References

[1] N. Cheng and K. S. Williams, *Evaluation of some convolution sums involving the sum of divisors functions*, Yokohama Mathematical J., **52** (2005), 39-57.

[2] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams, *Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions*, Number theory for the millennium, II, (2002), 229-274.

[3] K. S. Williams, *Number Theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, (2011).

Aeran Kim
Department of Mathematics and Institute of Pure and Applied Mathematics,
Chonbuk National University,
Chonju, Chonbuk 561-756, Korea.
E-mail: ae_ran_kim@hotmail.com

Daeyeoul Kim
National Institute for Mathematical Sciences,
Yuseong-daero 1689-gil, Yuseong-Gu, Daejeon 305-811, South Korea.
E-mail: daeyeoul@nims.re.kr

Nazli Yildiz İkikardes
Department of Elementary Mathematics Education,
Necatibey Faculty of Education, Balikesir University,
10100 Balikesir, Turkey.
E-mail: nyildizikikardes@gmail.com