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CONDITIONAL FOURIER-FEYNMAN TRANSFORMS AND CONVOLUTIONS OF UNBOUNDED FUNCTIONS ON A GENERALIZED WIENER SPACE

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Abstract. Let $C[0,t]$ denote the function space of real-valued continuous paths on $[0,t]$. Define $X_n : C[0,t] \to \mathbb{R}^{n+1}$ and $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ by $X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1}))$, respectively, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$. In the present paper, using simple formulas for the conditional expectations with the conditioning functions X_n and X_{n+1}, we evaluate the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the functions, which have the form $f_r((v_1, x), \ldots, (v_r, x)) \int_{L_2[0,t]} \exp\{i(v, x)\} d\sigma(v)$ for $x \in C[0,t]$, where $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,t]$, $f_r \in L_p(\mathbb{R}^r)$, and σ is the complex Borel measure of bounded variation on $L_2[0,t]$. We then investigate the inverse conditional Fourier-Feynman transforms of the function and prove that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions can be expressed by the products of the analytic conditional Fourier-Feynman transform of each function.

1. Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, that is, the space of real-valued continuous functions x on the closed interval $[0,t]$ with $x(0) = 0$. On the Wiener space $C_0[0,t]$, the concept of an analytic Fourier-Feynman transform was introduced by Brue [1]. Huffman, Park and Skoug [15] developed this theory to the Fourier-Feynman transform of functional involving multiple integrals. Furthermore, Chang and Skoug [6] examined the effects that drift has on the various relationships that occur among the Fourier-Feynman transform, the convolution product and the first variation for various functionals on the space.

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On the space, Yeh [25] introduced the conditional Wiener integral and derived several Fourier inversion formulas for retrieving the conditional Wiener integrals when the conditioning function is real-valued. But Yeh's inversion formula is very complicated in its applications when the conditioning function is vector-valued. Park and Skoug [21] derived a simple formula for conditional Wiener integrals on $C_0[0,t]$ with a vector-valued conditioning function. Using the simple formula, Chang and Skoug [4, 5] introduced the concepts of conditional Wiener integral, conditional Fourier-Feynman transform and conditional convolution product on $C_0[0,t]$. In those papers, they examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Further works were produced by Chang, Kim, Skoug, Song, Yoo and the author of [3, 13, 19]. In fact, they [3] introduced the L_1-analytic conditional Fourier-Feynman transform and the conditional convolution product over Wiener paths in abstract Wiener space and established the relationships between the transform and convolutions of certain functions similar to cylinder functions. The author [13] extended the relationships between the conditional convolution product and the $L_p(1 \le p \le 2)$-analytic conditional Fourier-Feynman transform of the functions. Moreover, on $C[0,t]$, the space of real-valued continuous paths on $[0,t]$, Kim [18] extended the relationships between the conditional convolution product and the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra $\mathcal{S}_{w_\varphi}$, which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [2]. The author [9] also did the same on the relationships between the convolution and the transform for the products of the functions in $\mathcal{S}_{w_\varphi}$ and the bounded cylinder functions of the Fourier-Stieltjes transforms of measures on the Borel class of $\mathbb{R}^r$. Furthermore, he [7, 8, 10] established several relationships between the L_p-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions on $C[0,t]$. In particular, he [7, 8] derived evaluation formulas for the L_p-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the same cylinder functions with the conditioning functions $X_n : C[0,t] \to \mathbb{R}^{n+1}$ and $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ given by $X_n(x) = (x(t_0), x(t_1), \dots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$, respectively, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$, and established their relationships. Note that the transforms and the convolutions given by X_n are independent of the present positions of the paths in $C[0,t]$, while those given by X_{n+1} wholly depend on the present positions of the paths.

In this paper, we further develop the relationships in [7, 8, 9, 18] on a more general space $(C[0,t], w_\varphi)$, an analogue of the Wiener space associated with the probability measure φ on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ [16, 22, 23]. For the conditioning functions X_n and X_{n+1}, we proceed to study the relationships between the conditional convolution products and the analytic conditional Fourier-Feynman transforms of unbounded functions on $C[0,t]$. In fact, using simple formulas for the conditional w_φ-integrals given X_n and X_{n+1}, we evaluate the L_p-analytic conditional Fourier-Feynman transforms and the conditional convolution products for the functions of the form

$$(1)\qquad f_r((v_1,x),\dots,(v_r,x))\int_{L_2[0,t]}\exp\{i(v,x)\}d\sigma(v)$$

for w_φ-a.e. $x\in C[0,t]$, where $\{v_1,\dots,v_r\}$ is an orthonormal subset of $L_2[0,t]$, $f_r\in L_p(\mathbb{R}^r)$, and σ is a complex Borel measure of bounded variation on $L_2[0,t]$. We then investigate various relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by (1). Finally, we derive the inverse conditional Fourier-Feynman transforms of the function and show that the L_p-analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions Ψ_1 and Ψ_2 of the form given by (1) can be expressed by the formula

(2)

$$\begin{aligned}&T_q^{(p)}[[(\Psi_1*\Psi_2)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\&=\left[T_q^{(p)}[\Psi_1|X_n]\left(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\right)\right]\left[T_q^{(p)}[\Psi_2|X_n]\left(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\right)\right]\end{aligned}$$

for a nonzero real q, w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions can be interpreted as the product of the analytic conditional Fourier-Feynman transform of each function.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of complex numbers, complex numbers with positive real parts and nonzero complex numbers with nonnegative real parts, respectively.

Now, we introduce the concrete form of the probability measure w_φ on $(C[0,t],\ \mathcal{B}(C[0,t]))$. For a positive real t, let $C=C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t}=(t_0,t_1,\dots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le t$, let $J_{\vec{t}}:C[0,t]\to\mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}}(x)=(x(t_0),x(t_1),\dots,x(t_n))$. For $B_j\,(j=0,1,\dots,n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, let

$$\begin{aligned}m_\varphi\left[J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)\right]=\left[\prod_{j=1}^n\frac{1}{2\pi(t_j-t_{j-1})}\right]^{\frac12}\int_{B_0}\int_{\prod_{j=1}^n B_j}\\ \exp\left\{-\frac12\sum_{j=1}^n\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\right\}d(u_1,\dots,u_n)d\varphi(u_0).\end{aligned}$$

Then $\mathcal{B}(C[0,t])$, the Borel σ-algebra of $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,t],\ \mathcal{B}(C[0,t]))$ such that $w_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ

is called an analogue of the Wiener measure associated with the probability measure φ [16, 22, 23].

Let $\{d_j : j = 1, 2, \ldots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each d_j is of bounded variation on $[0,t]$. For v in $L_2[0,t]$ and x in $C[0,t]$, let $(v,x) = \lim_{n\to\infty}\sum_{j=1}^{n}\langle v, d_j\rangle\int_0^t d_j(s)dx(s)$ if the limit exists, where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. (v,x) is called the Paley-Wiener-Zygmund integral of v according to x. Note we also denote by $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$ the dot product on the r-dimensional Euclidean space $\mathbb{R}^r$.

Applying Theorem 3.5 in [16], we can easily prove the following theorem.

Theorem 1.1. *Let $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal subset of $L_2[0,t]$. For $j = 1, 2, \ldots, r$, let $Z_j(x) = (v_j, x)$ on $C[0,t]$. Then $Z_1, Z_2, \ldots, Z_r$ are independent and each Z_j has the standard normal distribution. Moreover, if $f : \mathbb{R}^r \to \mathbb{R}$ is Borel measurable, then*

$$\int_C f(Z_1(x), Z_2(x), \ldots, Z_r(x))dw_\varphi(x)$$
$$\stackrel{*}{=} \left(\frac{1}{2\pi}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r} f(u_1, u_2, \ldots, u_r)\exp\left\{-\frac{1}{2}\sum_{j=1}^{r}u_j^2\right\}d(u_1, u_2, \ldots, u_r),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and X be a random vector on $C[0,t]$ assuming that the value space of X is a normed space equipped with the Borel σ-algebra. Then we have the conditional expectation $E[F|X]$ of F given X from a well-known probability theory. Furthermore, there exists a P_X-integrable $\mathbb{C}$-valued function ψ on the value space of X such that $E[F|X](x) = (\psi\circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let n be a positive integer and let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a fixed partition of $[0,t]$. For any x in $C[0,t]$, define the polygonal function $[x]$ of x by

$$[x](s) = \sum_{j=1}^{n+1}\chi_{(t_{j-1},t_j]}(s)\left(\frac{t_j - s}{t_j - t_{j-1}}x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}x(t_j)\right) + \chi_{\{t_0\}}(s)x(t_0) \tag{3}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. Similarly, for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ by the right-hand side of (3), where $x(t_j)$ is replaced by ξ_j for $j = 0, 1, \ldots, n+1$. Let $X_n : C[0,t] \to \mathbb{R}^{n+1}$ and $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ be given by

$$X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n)) \tag{4}$$

and

$$X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1})), \tag{5}$$

respectively. For a function $F : C[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$, $X_n^\lambda(x) = X_n(\lambda^{-\frac{1}{2}}x)$ and $X_{n+1}^\lambda(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x)$. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ-integral and (6) in [12, Theorem 2.9],

$$E[F^\lambda | X_{n+1}^\lambda](\vec{\xi}_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$$

for $P_{X_{n+1}^\lambda}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where the expectation is taken over the variable x and $P_{X_{n+1}^\lambda}$ is the probability distribution of X_{n+1}^λ on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. Throughout this paper, for $y \in C[0,t]$, let

$$I_F^\lambda(y, \vec{\xi}_{n+1}) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])].$$

Moreover, we can obtain from (2.6) in [11, Theorem 2.5]

$$\begin{aligned} &E[F^\lambda | X_n^\lambda](\vec{\xi}_n) \\ &= \left[\frac{\lambda}{2\pi(t - t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} I_F^\lambda(0, \vec{\xi}_{n+1}) \exp\left\{-\frac{\lambda}{2}\frac{(\xi_{n+1} - \xi_n)^2}{t - t_n}\right\} d\xi_{n+1} \end{aligned} \tag{6}$$

for $P_{X_n^\lambda}$-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n,\, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$ and $P_{X_n^\lambda}$ is the probability distribution of X_n^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. For $y \in C[0,t]$, let $K_F^\lambda(y, \vec{\xi}_n)$ be given by the right-hand side of (6), where 0 is replaced by y. If $I_F^\lambda(0, \vec{\xi}_{n+1})$ has the analytic extension $J_\lambda^*(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ-integral of F given X_{n+1} with parameter λ and denoted by

$$E^{anw_\lambda}[F | X_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F)(\vec{\xi}_{n+1})$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F | X_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F given X_{n+1} with parameter q and denoted by

$$E^{anf_q}[F | X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F | X_{n+1}](\vec{\xi}_{n+1}).$$

Similarly, the definitions of $E^{anw_\lambda}[F | X_n](\vec{\xi}_n)$ and $E^{anf_q}[F | X_n](\vec{\xi}_n)$ are understood with $K_F^\lambda(0, \vec{\xi}_n)$ if X_{n+1} is replaced by X_n.

For a given extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that $\lim_{n\to\infty} \int_C |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\text{l.i.m.}_{n\to\infty}(w^{p'})(F_n) = F$ and call F the limit in the mean of order p'.

A similar definition is understood when n is replaced by a continuously varying parameter. For $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$, let

$$T_\lambda[F|X_{n+1}](y,\vec{\xi}_{n+1}) = E^{anw_\lambda}[F(y+\cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For a nonzero real q and w_φ-a.e. $y \in C[0,t]$, define the L_1-analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_{n+1}]$ of F given X_{n+1} by the formula

$$T_q^{(1)}[F|X_{n+1}](y,\vec{\xi}_{n+1}) = E^{anf_q}[F(y+\cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For $1 < p \le \infty$, define the L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_{n+1}]$ of F given X_{n+1} by the formula

$$T_q^{(p)}[F|X_{n+1}](\cdot,\vec{\xi}_{n+1}) = \operatorname*{l.i.m.}_{\lambda\to -iq}(w^{p'})(T_\lambda[F|X_{n+1}](\cdot,\vec{\xi}_{n+1}))$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where λ approaches to $-iq$ through $\mathbb{C}_+$. Moreover, let G be defined on $C[0,t]$. We define the conditional convolution product $[(F*G)_\lambda|X_{n+1}]$ of F and G given X_{n+1} by the formula, for w_φ-a.e. $y \in C[0,t]$,

$$\begin{aligned}&[(F*G)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})\\ &= \begin{cases} E^{anw_\lambda}\left[F\left(\frac{y+\cdot}{\sqrt{2}}\right)G\left(\frac{y-\cdot}{\sqrt{2}}\right)\middle|X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda\in\mathbb{C}_+;\\ E^{anf_q}\left[F\left(\frac{y+\cdot}{\sqrt{2}}\right)G\left(\frac{y-\cdot}{\sqrt{2}}\right)\middle|X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda=-iq;\quad q\in\mathbb{R}-\{0\}\end{cases}\end{aligned}$$

if they exist for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. If $\lambda = -iq$, we replace $[(F*G)_\lambda|X_{n+1}]$ by $[(F*G)_q|X_{n+1}]$. Similar definitions and notations are understood with $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if X_{n+1} is replaced by X_n.

Remark 1.2. Note that if we let $\varphi = \delta_{\{0\}}$, the Dirac measure concentrated at 0, and let $n = 2$ in equation (2) above, then we obtain equation (3.9) in [6, p. 29], equation (3.7) in [15, p. 251], and also equation (2.5) in [19, p. 30]. These are among the first results expressing the conditional integral transform of the conditional convolution product as the product of conditional integral transforms.

2. The present position-dependent Fourier-Feynman transform and convolution

For $j = 1,\dots,n+1$, let $\alpha_j = (t_j - t_{j-1})^{-\frac{1}{2}}\chi_{(t_{j-1},t_j]}$ on $[0,t]$. Let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1,\dots,\alpha_{n+1}\}$ and $V^\perp$ denote the orthogonal complement of V. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0,t]$ to V and $V^\perp$, respectively. Throughout this paper, let $\{v_1, v_2,\dots,v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^\perp v_1,\dots,\mathcal{P}^\perp v_r\}$ is an independent set. Let $\{e_1,\dots,e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1,\dots,\mathcal{P}^\perp v_r\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1,\dots,r$, let

$\mathcal{P}^\perp v_l = \sum_{j=1}^r \beta_{lj} e_j$ be the linear combinations of the e_j's and let $A = [\beta_{lj}]_{r\times r}$ be the coefficient matrix of the combinations. Define the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ by

$$T_A \vec{z} = \vec{z} A^T, \tag{7}$$

where A^T is the transpose of A and $\vec{z}$ is any row-vector in $\mathbb{R}^r$. Note that A is invertible so that T_A is an isomorphism. For $v \in L_2[0,t]$, let

$$c_j(v) = \langle v, e_j\rangle \tag{8}$$

for $j = 1, \ldots, r$ and let $(\vec{v}, x) = ((v_1, x), \ldots, (v_r, x))$ for $x \in C[0,t]$. Furthermore, for $\lambda \in \mathbb{C}_+^\sim$ and $\vec{u} = (u_1, \ldots, u_r) \in \mathbb{R}^r$, let

$$A_1(\lambda, v, \vec{u}) = \exp\left\{\frac{1}{2\lambda}\left[\sum_{j=1}^r [\lambda i u_j + c_j(\mathcal{P}^\perp v)\right]^2 - \|\mathcal{P}^\perp v\|_2^2\right]\right\}, \tag{9}$$

where the c_j's are given by (8). For $\vec{z} \in \mathbb{R}^r$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, let

$$\begin{aligned} A_2(f, g; \lambda, \vec{z}, \vec{\xi}_{n+1}, v) = \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} & f(\vec{z} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{u}) \\ & \times g(\vec{z} - (\vec{v}, [\vec{\xi}_{n+1}]) - T_A\vec{u}) A_1(\lambda, v, \vec{u}) d\vec{u} \end{aligned} \tag{10}$$

if exists, where f and g are Borel measurable on $\mathbb{R}^r$. Note that by the Bessel's inequality, we have for $\lambda \in \mathbb{C}_+^\sim$,

$$|A_1(\lambda, v, \vec{u})| \le \exp\left\{-\frac{\mathrm{Re}\lambda}{2}\|\vec{u}\|_{\mathbb{R}^r}^2\right\} \le 1. \tag{11}$$

Using the same method as in the proof of [14, Theorem 3.3], we can prove the following lemma.

Lemma 2.1. *Let f be Borel measurable on $\mathbb{R}^r$. For $x \in C[0,t]$, $\lambda > 0$ and $v \in L_2[0,t]$, let*

$$A_3(f; \lambda, v, x) = f(\vec{v}, \lambda^{-\frac{1}{2}}(x - [x])) \exp\{i\lambda^{-\frac{1}{2}}(v, x - [x])\}. \tag{12}$$

Then

$$\int_C A_3(f; \lambda, v, x) dw_\varphi(x) \overset{*}{=} A_2(f, 1; \lambda, 0, 0, v),$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

For $1 \le p \le \infty$, let $\mathcal{A}_r^{(p)}$ be the space of the cylinder functions F_r given by

$$F_r(x) = f_r(\vec{v}, x) \tag{13}$$

for w_φ-a.e. $x \in C[0,t]$, where $f_r \in L_p(\mathbb{R}^r)$. Note that, without loss of generality, we can take f_r to be Borel measurable. Let $\mathcal{M} = \mathcal{M}(L_2[0,t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,t]$, and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which have the form for $\sigma \in \mathcal{M}$,

$$F(x) = \int_{L_2[0,t]} \exp\{i(v,x)\} d\sigma(v) \tag{14}$$

for w_φ-a.e. $x \in C[0,t]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ [16].

Now we have the following theorem by Lemma 2.1.

Theorem 2.2. *Let $1 \le p \le \infty$ and X_{n+1} be given by* (5). *For w_φ-a.e. $x \in C[0,t]$, let $\Psi(x) = F(x)F_r(x)$, where $F_r \in \mathcal{A}_r^{(p)}$ and $F \in \mathcal{S}_{w_\varphi}$ are given by* (13) *and* (14)*, respectively. Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,*

$$\begin{aligned} &T_\lambda[\Psi|X_{n+1}](y,\vec{\xi}_{n+1}) \\ &= \int_{L_2[0,t]} H_1(y,\vec{\xi}_{n+1},v,v) A_2(f_r,1;\lambda,(\vec{v},y),\vec{\xi}_{n+1},v) d\sigma(v), \end{aligned} \tag{15}$$

where A_2 is given by (10) *and H_1 is given by*

$$H_1(y,\vec{\xi}_{n+1},v_1,v_2) = \exp\{i[(v_1,y) + (v_2,[\vec{\xi}_{n+1}])]\} \tag{16}$$

for $v_1, v_2 \in L_2[0,t]$. Moreover, as a function of y, $T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in L_p(C[0,t])$. If $p = 1$, then for nonzero real q, $T_q^{(1)}[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})$ is given by (15) *replacing λ by $-iq$ and $T_q^{(1)}[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in L_\infty(C[0,t])$.*

Theorem 2.3. *Let the assumptions and notations be as given in Theorem 2.2 with one exception $1 \le p \le 2$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $\lambda \in \mathbb{C}_+$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in L_{p'}(C[0,t])$. Furthermore, suppose that σ is concentrated on V. Then for a nonzero real q and w_φ-a.e. $y \in C[0,t]$,*

$$\begin{aligned} &T_q^{(p)}[\Psi|X_{n+1}](y,\vec{\xi}_{n+1}) \\ &= (f_r(T_A\cdot) * \Phi(-iq,\cdot))(((\vec{v},[\vec{\xi}_{n+1}]) + (\vec{v},y))(A^T)^{-1}) \int_{L_2[0,t]} H_1(y,\vec{\xi}_{n+1},v,v) d\sigma(v), \end{aligned} \tag{17}$$

where Φ is given by

$$\Phi(\lambda,\vec{u}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \exp\left\{-\frac{\lambda}{2}\|\vec{u}\|_{\mathbb{R}^r}^2\right\} \tag{18}$$

for $\lambda \in \tilde{\mathbb{C}}_+$ and for $\vec{u} \in \mathbb{R}^r$. In this case, $T_q^{(p)}[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in L_{p'}(C[0,t])$.

Proof. Let $1 < p \le 2$. By the same process as in the proof of Theorem 2.2, we have for $\lambda \in \mathbb{C}_+$,

$$\int_C |T_\lambda[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'} dw_\varphi(y)$$

$$\leq |\det(A)|\,\|\sigma\|^{p'}\left(\frac{|\lambda|}{\mathrm{Re}\lambda}\right)^{\frac{rp'}{2}}\|\,|f_r(T_A\cdot)|*\Phi(\mathrm{Re}\lambda,\cdot)\|_{p'}^{p'}<\infty,$$

where the last inequality follows from [17, Lemma 1.1]. Now, we complete the proof of the first part of the theorem. Suppose that σ is concentrated on V. Then for σ-a.e. $v\in L_2[0,t]$ and $\lambda\in\mathbb{C}_+^{\sim}$,

$$\left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}}A_1(\lambda,v,\vec{u})=\Phi(\lambda,\vec{u}) \tag{19}$$

so that the formal form of $T_q^{(p)}[\Psi|X_{n+1}]$ is given by (17). By Theorem 1.1,

$$\int_C|T_q^{(p)}[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'}dw_\varphi(y)\leq|\det(A)|\,\|\sigma\|^{p'}\|\,f_r(T_A\cdot)*\Phi(-iq,\cdot)\|_{p'}^{p'},$$

which is finite by the change of variable theorem and [17, Lemma 1.1]. By Theorem 1.1 and the change of variable theorem, we have for $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$,

$$\int_C|T_\lambda[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})-T_q^{(p)}[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'}dw_\varphi(y)$$
$$\leq|\det(A)|\,\|\sigma\|^{p'}\int_{\mathbb{R}^r}|(f_r(T_A\cdot)*\Phi(\lambda,\cdot))(\vec{u})-(f_r(T_A\cdot)*\Phi(-iq,\cdot))(\vec{u})|^{p'}d\vec{u},$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by [17, Lemma 1.2]. If $p=1$, then the conclusions follow from Theorem 2.2. Now the proof is complete. □

Theorem 2.4. *Let X_{n+1} be given by* (5). *Let $F_r\in\mathcal{A}_r^{(p_1)}$, $G_r\in\mathcal{A}_r^{(p_2)}$ and f_r, g_r be related by* (13), *respectively, where $1\leq p_1,p_2\leq\infty$. Let $F_1,F_2\in\mathcal{S}_{w_\varphi}$ and $\sigma_1,\sigma_2\in\mathcal{M}(L_2[0,t])$ be related by* (14), *respectively. Furthermore, let $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $\frac{1}{p_2}+\frac{1}{p_2'}=1$, and let $\Psi_1(x)=F_r(x)F_1(x)$, $\Psi_2(x)=G_r(x)F_2(x)$ for w_φ-a.e. $x\in C[0,t]$. Then for $\lambda\in\mathbb{C}_+$, w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$,*

$$[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})$$
$$=\int_{L_2[0,t]}\int_{L_2[0,t]}H_1\bigg(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}\vec{\xi}_{n+1},v_1+v_2,v_1-v_2\bigg)$$
$$\times A_2\bigg(f_r,g_r;2\lambda,\bigg(\vec{v},\frac{1}{\sqrt{2}}y\bigg),\frac{1}{\sqrt{2}}\vec{\xi}_{n+1},v_1-v_2\bigg)d\sigma_1(v_1)d\sigma_2(v_2),$$

where A_2 and H_1 are given by (10) *and* (16), *respectively. Furthermore, as functions of y, $[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})\in L_1(C[0,t])$ if either $p_2\leq p_1'$ or $p_1\leq p_2'$, $[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})\in L_{p_2}(C[0,t])$ if $p_2\geq p_1'$, and $[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})\in L_{p_1}(C[0,t])$ if $p_1\geq p_2'$.*

Proof. For $\lambda>0$, w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$,

$$[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})$$

$$= \int_{L_2[0,t]} \int_{L_2[0,t]} H_1\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}\vec{\xi}_{n+1}, v_1+v_2, v_1-v_2\bigg) \int_C A_3(h_r; 2\lambda, v_1-v_2, x)dw_\varphi(x)d\sigma_1(v_1)d\sigma_2(v_2),$$

where $h_r(\vec{u}) = f_r((\vec{v}, \frac{1}{\sqrt{2}}(y+[\vec{\xi}_{n+1}])) + \vec{u})g_r((\vec{v}, \frac{1}{\sqrt{2}}(y-[\vec{\xi}_{n+1}])) - \vec{u})$ and A_3, H_1 are given by (12), (16), respectively. By Lemma 2.1,

$$\begin{aligned}
&[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1}) \\
&\overset{*}{=} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}\vec{\xi}_{n+1}, v_1+v_2, v_1-v_2\bigg) \\
&\quad A_2\bigg(f_r, g_r; 2\lambda, \bigg(\vec{v}, \frac{1}{\sqrt{2}}y\bigg), \frac{1}{\sqrt{2}}\vec{\xi}_{n+1}, v_1-v_2\bigg)d\sigma_1(v_1)d\sigma_2(v_2).
\end{aligned}$$

Then for $\lambda \in \mathbb{C}_+$, we have by (11), Theorem 1.1 and the change of variable theorem,

$$\begin{aligned}
&\int_C |[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1})| dw_\varphi(y) \\
&\le \|\sigma_1\| \, \|\sigma_2\| \bigg(\frac{|\lambda|}{\pi}\bigg)^{\frac{r}{2}} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \bigg| f_r\bigg(\frac{1}{\sqrt{2}}\vec{z} + \bigg(\vec{v}, \frac{1}{\sqrt{2}}[\vec{\xi}_{n+1}]\bigg) + T_A\vec{u}\bigg)\bigg| \\
&\quad \times \bigg| g_r\bigg(\frac{1}{\sqrt{2}}\vec{z} - \bigg(\vec{v}, \frac{1}{\sqrt{2}}[\vec{\xi}_{n+1}]\bigg) - T_A\vec{u}\bigg)\bigg| \exp\{-\mathrm{Re}\lambda \|\vec{u}\|_{\mathbb{R}^r}^2\} d\vec{u}d\vec{z} \\
&= \|\sigma_1\| \, \|\sigma_2\| \bigg(\frac{|\lambda|}{\mathrm{Re}\lambda}\bigg)^{\frac{r}{2}} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \bigg| f_r\bigg(\frac{1}{\sqrt{2}}[\vec{z} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{u}]\bigg)\bigg| \\
&\quad \times \bigg| g_r\bigg(\frac{1}{\sqrt{2}}[\vec{z} - (\vec{v}, [\vec{\xi}_{n+1}]) - T_A\vec{u}]\bigg)\bigg| \Phi(\mathrm{Re}\lambda, \vec{u}) d\vec{u}d\vec{z},
\end{aligned}$$

where Φ is given by (18). By the same method as in the proof of [8, Theorem 3.1], we have $[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_1(C[0,t])$ if either $p_2 \le p_1'$ or $p_1 \le p_2'$. Using the same method as in the proof of [8, Theorem 3.1] again, we have the remainder of the theorem. □

3. The present position-independent Fourier-Feynman transform and convolution

In this section, we evaluate the present position-independent conditional Fourier-Feynman transform and the conditional convolution product of the functions as given in the previous section.

Let $(\mathcal{P}\vec{v})(t) = ((\mathcal{P}v_1)(t), \ldots, (\mathcal{P}v_r)(t))$ and for $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, let $(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) = (\sum_{j=1}^n (\xi_j - \xi_{j-1})(\mathcal{P}v_1)(t_j), \ldots, \sum_{j=1}^n (\xi_j - \xi_{j-1})(\mathcal{P}v_r)(t_j))$. Let $\Gamma_t = \frac{1}{1+(t-t_n)\|T_A^{-1}(\mathcal{P}\vec{v})(t)\|_{\mathbb{R}^r}^2}$, where T_A is the linear transformation given by

(7). For $\lambda\in\mathbb{C}_+^{\sim}$, $v\in L_2[0,t]$ and $\vec{u}\in\mathbb{R}^r$, let

$$B_1(\lambda,v,\vec{u})=\exp\bigg\{\frac{1}{2\lambda}(t-t_n)\Gamma_t[i[(\mathcal{P}v)(t)-\langle\vec{c}(\mathcal{P}^{\perp}v), T_A^{-1}(\mathcal{P}\vec{v})(t)\rangle_{\mathbb{R}^r}]+\lambda\langle T_A^{-1}(\mathcal{P}\vec{v})(t),\vec{u}\rangle_{\mathbb{R}^r}]^2\bigg\}, \tag{20}$$

where $\vec{c}=(c_1,\dots,c_r)$ and the c_j's are given by (8). Furthermore, for $\vec{z}\in\mathbb{R}^r$, let

$$\begin{aligned} &B_2(f,g;\lambda,\vec{z},\vec{\xi}_n,v)\\ &=\Gamma_t^{\frac12}\left(\frac{\lambda}{2\pi}\right)^{\frac r2}\int_{\mathbb{R}^r}f(\vec{z}+(\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t}))+T_A\vec{u})\\ &\quad\times g(\vec{z}-(\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t}))-T_A\vec{u})A_1(\lambda,v,\vec{u})B_1(\lambda,v,\vec{u})d\vec{u}\end{aligned} \tag{21}$$

if exists, where f and g are Borel measurable on $\mathbb{R}^r$ and A_1 is given by (9). By the definition of the Paley-Wiener-Zygmund integral, it is not difficult to show that for $v\in L_2[0,t]$ and $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\dots,\xi_{n+1})\in\mathbb{R}^{n+2}$,

$$(v,[\vec{\xi}_{n+1}])=\sum_{j=1}^n(\mathcal{P}v)(t_j)(\xi_j-\xi_{j-1})+(\mathcal{P}v)(t)(\xi_{n+1}-\xi_n).$$

Using the above equality and the following well-known integration formula

$$\int_{\mathbb{R}}\exp\{-au^2+ibu\}du=\left(\frac{\pi}{a}\right)^{\frac12}\exp\left\{-\frac{b^2}{4a}\right\} \tag{22}$$

for $a\in\mathbb{C}_+$ and $b\in\mathbb{R}$, we can prove the following lemma from [14, Theorem 3.4].

Lemma 3.1. *For $\vec{\xi}_n=(\xi_0,\xi_1,\dots,\xi_n)\in\mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\dots,\xi_n,\xi_{n+1})$, where $\xi_{n+1}\in\mathbb{R}$. For $v\in L_2[0,t]$, let*

$$H_2(\vec{\xi}_n,v)=\exp\bigg\{i\sum_{j=1}^n(\xi_j-\xi_{j-1})(\mathcal{P}v)(t_j)\bigg\}. \tag{23}$$

Furthermore, let f be Borel measurable on $\mathbb{R}^r$ and let A_2 and H_1 be given by (10) *and* (16)*, respectively. Then for $\lambda>0$ and $v\in L_2[0,t]$,*

$$\begin{aligned}&\left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac12}\int_{\mathbb{R}}\exp\{i(v,[\vec{\xi}_{n+1}])\}A_2(f,1;\lambda,0,\vec{\xi}_{n+1},v)\\ &\times\exp\bigg\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\bigg\}d\xi_{n+1}\overset{*}{=}B_2(f,1;\lambda,0,\vec{\xi}_n,v)H_2(\vec{\xi}_n,v),\end{aligned}$$

where B_2 is given by (21).

Theorem 3.2. *Let X_n be given by* (4)*. Then under the assumptions as given in Theorem 2.2, we have for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$,*

$$T_\lambda[\Psi|X_n](y,\vec{\xi_n}) = \int_{L_2[0,t]} B_2(f_r,1;\lambda,(\vec{v},y),\vec{\xi_n},v)H_2(\vec{\xi_n},v)\exp\{i(v,y)\}d\sigma(v), \tag{24}$$

where B_2 and H_2 are given by (21) *and* (23)*, respectively. Moreover, as a function of y, $T_\lambda[\Psi|X_n](\cdot,\vec{\xi_n}) \in L_p(C[0,t])$. If $p=1$, then for nonzero real q, $T_q^{(1)}[\Psi|X_n](y,\vec{\xi_n})$ is given by the right-hand side of* (24) *replacing λ by $-iq$ and $T_q^{(1)}[\Psi|X_n](\cdot,\vec{\xi_n}) \in L_\infty(C[0,t])$.*

Proof. For $\vec{\xi_n} = (\xi_0,\xi_1,\dots,\xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\dots,\xi_n,\xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$, $y \in C[0,t]$ and $\vec{\xi_n} \in \mathbb{R}^{n+1}$,

$$\begin{aligned} &K_\Psi^\lambda(y,\vec{\xi_n}) \\ &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{L_2[0,t]} \exp\{i(v,y)\} \int_{\mathbb{R}} \exp\{i(v,[\vec{\xi}_{n+1}])\} \\ &\quad \times A_2(f_r((\vec{v},y)+\cdot),1;\lambda,0,\vec{\xi}_{n+1},v)\exp\left\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\}d\xi_{n+1}d\sigma(v) \end{aligned}$$

by Theorem 2.2. By Lemma 3.1,

$$K_\Psi^\lambda(y,\vec{\xi_n}) \stackrel{*}{=} \int_{L_2[0,t]} B_2(f_r,1;\lambda,(\vec{v},y),\vec{\xi_n},v)H_2(\vec{\xi_n},v)\exp\{i(v,y)\}d\sigma(v).$$

By (11), Bessel's inequality, and Schwarz's inequality, we have for $\lambda \in \mathbb{C}_+^\sim$,

$$|A_1(\lambda,v,\vec{u})B_1(\lambda,v,\vec{u})| \le \exp\left\{-\frac{\Gamma_t \mathrm{Re}\lambda}{2}\|\vec{u}\|_{\mathbb{R}^r}^2\right\} \le 1 \tag{25}$$

following the same method as in the proof of [14, Theorem 3.4]. Note that by the change of variable theorem,

$$\|f(\vec{z}+(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t}))+T_A\cdot)\|_p^p = |\det(A^T)^{-1}|\,\|f_r\|_p^p < \infty. \tag{26}$$

By (25), Hölder's inequality, Morera's theorem and the dominated convergence theorem, we have (24) as the analytic extension of $K_\Psi^\lambda(y,\vec{\xi_n})$ on $\mathbb{C}_+$. Using the same method as in the proof of Theorem 2.2, we have by (25) and (26),

$$\begin{aligned} &\int_C |T_\lambda[\Psi|X_n](y,\vec{\xi_n})|^p dw_\varphi(y) \\ &\le |\det(A)|\,\|\sigma\|^p\left(\frac{|\lambda|}{\Gamma_t\mathrm{Re}\lambda}\right)^{\frac{pr}{2}} \|\,|f_r(T_A\cdot)| * \Phi(\Gamma_t\mathrm{Re}\lambda,\cdot)\|_p^p < \infty \end{aligned}$$

if $1 \le p < \infty$, where Φ is given by (18), and $\|T_\lambda[\Psi|X_n](\cdot,\vec{\xi_n})\|_\infty \le \|\sigma\|\,\|f_r\|_\infty \times (\frac{|\lambda|}{\Gamma_t\mathrm{Re}\lambda})^{\frac{r}{2}}$ if $p=\infty$. If $p=1$, then the final result follows from the second inequality of (25) and the dominated convergence theorem. □

Theorem 3.3. *Let the assumptions be as given in Theorem 3.2 with one exception* $1 \le p \le 2$ *and let* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then for* $\lambda \in \mathbb{C}_+$ *and* P_{X_n}*-a.e.* $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $T_\lambda[\Psi|X_n](\cdot, \vec{\xi_n}) \in L_{p'}(C[0,t])$*. Furthermore, suppose that* σ *is concentrated on* V *and* $\{v_1, \ldots, v_r\} \subseteq V^\perp$*. Then for a nonzero real* q*,* w_φ*-a.e.* $y \in C[0,t]$ *and* P_{X_n}*-a.e.* $\vec{\xi_n} \in \mathbb{R}^{n+1}$,

$$(27) \qquad T_q^{(p)}[\Psi|X_n](y, \vec{\xi_n}) = (f_r * \Phi(-iq, \cdot))(\vec{v}, y) \int_{L_2[0,t]} B_{-iq}(v) H_2(\vec{\xi_n}, v) \exp\{i(v,y)\} d\sigma(v),$$

where Φ *is given by* (18), $B_\lambda(v) = \exp\{-\frac{t-t_n}{2\lambda}(v(t))^2\}$ *for* $\lambda \in \mathbb{C}_+^\sim$ *and* $H_2(\vec{\xi_n}, v) = \exp\{i \sum_{j=1}^n (\xi_j - \xi_{j-1}) v(t_j)\}$ *for* $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n)$*. In this case,*

$$T_q^{(p)}[\Psi|X_n](\cdot, \vec{\xi_n}) \in L_{p'}(C[0,t]).$$

Proof. Let $1 < p \le 2$. By the same processes as in the proofs of Theorems 2.3 and 3.2, we have for $\lambda \in \mathbb{C}_+$,

$$\int_C |T_\lambda[\Psi|X_n](y, \vec{\xi_n})|^{p'} dw_\varphi(y) \le |\det(A)| \, \|\sigma\|^{p'} \left(\frac{|\lambda|}{\Gamma_t \mathrm{Re}\lambda} \right)^{\frac{p'r}{2}} \| \, |f_r(T_A \cdot)| * \Phi(\Gamma_t \mathrm{Re}\lambda, \cdot) \|_{p'}^{p'} < \infty.$$

Now, suppose that σ is concentrated on V and $\{v_1, \ldots, v_r\} \subseteq V^\perp$. Then $\mathcal{P}\vec{v} = \vec{0} \in \mathbb{R}^r$, $\Gamma_t = 1$ and A^T is the identity matrix. Moreover, for σ-a.e $v \in L_2[0,t]$, $\mathcal{P}^\perp v = 0$ and $v = \mathcal{P}v$ so that we have $B_1(\lambda, v, \vec{u}) = B_\lambda(v)$ for $\lambda \in \mathbb{C}_+^\sim$ and $\vec{u} \in \mathbb{R}^r$. By Theorem 3.2 and (19), we have for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$,

$$T_\lambda[\Psi|X_n](y, \vec{\xi_n}) = (f_r * \Phi(\lambda, \cdot))(\vec{v}, y) \int_{L_2[0,t]} B_\lambda(v) H_2(\vec{\xi_n}, v) \exp\{i(v,y)\} d\sigma(v).$$

By Theorem 1.1 and [17, Lemma 1.1],

$$\int_C |T_q^{(p)}[\Psi|X_n](y, \vec{\xi_n})|^{p'} dw_\varphi(y) \le \|\sigma\|^{p'} \|f_r * \Phi(-iq, \cdot)\|_{p'}^{p'} < \infty.$$

By Theorem 1.1, the Minkowski inequality and the triangle inequality, we also have for $\lambda \in \mathbb{C}_+$,

$$\|T_\lambda[\Psi|X_n](\cdot, \vec{\xi_n}) - T_q^{(p)}[\Psi|X_n](\cdot, \vec{\xi_n})\|_{p'}$$
$$\le \left[\int_{\mathbb{R}^r} \left[\int_{L_2[0,t]} |B_\lambda(v)(f_r * \Phi(\lambda, \cdot))(\vec{u}) - B_{-iq}(v)(f_r * \Phi(-iq, \cdot))(\vec{u})| d|\sigma|(v) \right]^{p'} d\vec{u} \right]^{\frac{1}{p'}}$$
$$\le \left[\int_{\mathbb{R}^r} \left[|(f_r * \Phi(\lambda, \cdot))(\vec{u}) - (f_r * \Phi(-iq, \cdot))(\vec{u})| \int_{L_2[0,t]} |B_\lambda(v)| d|\sigma|(v) \right. \right.$$

$$+ |(f_r * \Phi(-iq,\cdot))(\vec{u})| \int_{L_2[0,t]} |B_\lambda(v) - B_{-iq}(v)| d|\sigma|(v)\Big]^{p'} d\vec{u}\Big]^{\frac{1}{p'}}$$

$$\le \|\sigma\| \, \|f_r * \Phi(\lambda,\cdot) - f_r * \Phi(-iq,\cdot)\|_{p'}$$

$$+ \|f_r * \Phi(-iq,\cdot)\|_{p'} \int_{L_2[0,t]} |B_\lambda(v) - B_{-iq}(v)| d|\sigma|(v),$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem and [17, Lemma 1.2]. If $p = 1$, then the conclusions follow from Theorem 3.2. Now the proof is complete. □

Remark 3.4. If σ is concentrated on V, then for σ-a.e. $v \in L_2[0,t]$, $v = \mathcal{P}v = \sum_{j=1}^{n+1} \langle v, \alpha_j \rangle \alpha_j = \sum_{j=1}^{n+1} \frac{\chi_{(t_{j-1},t_j]}}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v(s)ds$ so that other expressions of H_2 and B_λ in (27) can be given by $H_2(\vec{\xi_n}, v) = \exp\{i \sum_{j=1}^{n} \frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v(s)ds\}$ and $B_\lambda(v) = \exp\{-\frac{1}{2\lambda(t-t_n)}[\int_{t_n}^{t} v(s)ds]^2\}$.

Theorem 3.5. *Let the assumptions be as given in Theorem 2.4 and let X_n be given by* (4)*. Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$,*

$$[(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi_n})$$
$$= \int_{L_2[0,t]} \int_{L_2[0,t]} \exp\Big\{ i\Big(v_1 + v_2, \frac{1}{\sqrt{2}} y\Big)\Big\} H_2\Big(\frac{1}{\sqrt{2}}\vec{\xi_n}, v_1 - v_2\Big)$$
$$\times B_2\Big(f_r, g_r; 2\lambda, \Big(\vec{v}, \frac{1}{\sqrt{2}}y\Big), \frac{1}{\sqrt{2}}\vec{\xi_n}, v_1 - v_2\Big) d\sigma_1(v_1) d\sigma_2(v_2),$$

where B_2 and H_2 are given by (21) *and* (23)*, respectively. Furthermore, as functions of y, $[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi_n}) \in L_1(C[0,t])$ if either $p_2 \le p_1'$ or $p_1 \le p_2'$, $[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi_n}) \in L_{p_2}(C[0,t])$ if $p_2 \ge p_1'$, and $[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi_n}) \in L_{p_1}(C[0,t])$ if $p_1 \ge p_2'$.*

Proof. For $\vec{\xi_n} = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and $y \in C[0,t]$, we have by Theorem 2.4,

$$[(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi_n})$$
$$= \Big[\frac{\lambda}{2\pi(t-t_n)}\Big]^{\frac{1}{2}} \int_{L_2[0,t]} \int_{L_2[0,t]} \int_{\mathbb{R}} H_1\Big(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}\vec{\xi}_{n+1}, v_1 + v_2, v_1 - v_2\Big)$$
$$\times A_2\Big(f_r, g_r; 2\lambda, \Big(\vec{v}, \frac{1}{\sqrt{2}}y\Big), \frac{1}{\sqrt{2}}\vec{\xi}_{n+1}, v_1 - v_2\Big) \exp\Big\{-\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t-t_n)}\Big\}$$
$$d\xi_{n+1} d\sigma_1(v_1) d\sigma_2(v_2).$$

Note that by the change of variable theorem,

$$A_2\Big(f_r, g_r; 2\lambda, \Big(\vec{v}, \frac{1}{\sqrt{2}}y\Big), \frac{1}{\sqrt{2}}\vec{\xi}_{n+1}, v_1 - v_2\Big)$$

$$= A_2\left(h_r, 1; \lambda, 0, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}(v_1 - v_2)\right),$$

where $h_r(\vec{z}) = f_r(\frac{1}{\sqrt{2}}[(\vec{v}, y) + \vec{z}])g_r(\frac{1}{\sqrt{2}}[(\vec{v}, y) - \vec{z}])$ for $\vec{z} \in \mathbb{R}^r$. Then

$$\begin{aligned}
&[(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi}_n) \\
&= \int_{L_2[0,t]}\int_{L_2[0,t]} \exp\left\{\frac{i}{\sqrt{2}}(v_1 + v_2, y)\right\} B_2\left(h_r, 1; \lambda, 0, \vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2)\right) \\
&\qquad \times H_2\left(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2)\right) d\sigma_1(v_1) d\sigma_2(v_2).
\end{aligned}$$

By (21) and the change of variable theorem,

$$\begin{aligned}
&[(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi}_n) \\
&= \int_{L_2[0,t]}\int_{L_2[0,t]} \exp\left\{i\left(v_1 + v_2, \frac{1}{\sqrt{2}}y\right)\right\} H_2\left(\frac{1}{\sqrt{2}}\vec{\xi}_n, v_1 - v_2\right) \\
&\quad \times B_2\left(f_r, g_r; 2\lambda, \left(\vec{v}, \frac{1}{\sqrt{2}}y\right), \frac{1}{\sqrt{2}}\vec{\xi}_n, v_1 - v_2\right) d\sigma_1(v_1) d\sigma_2(v_2).
\end{aligned}$$

Now, for $\lambda \in \mathbb{C}_+$, we have by (25), Theorem 1.1 and the change of variable theorem,

$$\begin{aligned}
&\int_C |[(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi}_n)| dw_\varphi(y) \\
&\le \|\sigma_1\|\,\|\sigma_2\|\left(\frac{|\lambda|}{\Gamma_t \mathrm{Re}\lambda}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \left| f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + T_A\vec{z}]\right)\right| \\
&\quad \times \left| g_r\left(\frac{1}{\sqrt{2}}[\vec{u} - (\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) - T_A\vec{z}]\right)\right| \Phi(\Gamma_t \mathrm{Re}\lambda, \vec{z}) d\vec{z} d\vec{u},
\end{aligned}$$

where Φ is given by (18). By the same method as in the proof of [8, Theorem 3.1], we have the theorem. □

4. Relationships between conditional Fourier-Feynman transforms and convolutions

In this section, we investigate the inverse conditional transforms of the conditional Fourier-Feynman transforms of the functions as given in the previous sections. We also show that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions can be expressed as the products of the analytic conditional Fourier-Feynman transform of each function.

Theorem 4.1. *Let q be a nonzero real number. Then, under the assumptions as given in Theorem 2.2 with one exception that σ is concentrated on V, we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,*

$$\|T_{\overline{\lambda}}[T_\lambda[\Psi | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](\cdot, \vec{\zeta}_{n+1}) - \Psi(\cdot + [\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}])\|_p \longrightarrow 0$$

for $1 \le p < \infty$*, and for* $1 \le p \le \infty$*,*

$$T_{\overline{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \longrightarrow \Psi(y+[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}])$$

for w_φ*-a.e.* $y \in C[0,t]$ *as* λ *approaches to* $-iq$ *through* $\mathbb{C}_+$*.*

Proof. For $\lambda_1 > 0$, $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$, we have by (19), Lemma 2.1 and Theorem 2.2,

$$\begin{aligned}
&I^{\lambda_1}_{T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})}(y,\vec{\zeta}_{n+1})\\
&= \int_{L_2[0,t]} \exp\{i[(v,y)+(v,[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}])]\}\int_C \exp\{i\lambda_1^{\frac{1}{2}}(v,x-[x])\}\int_{\mathbb{R}^r} f_r\\
&\quad ((\vec{v},\lambda_1^{-\frac{1}{2}}(x-[x])+y+[\vec{\zeta}_{n+1}])+(\vec{v},[\vec{\xi}_{n+1}])+T_A\vec{u})\Phi(\lambda,\vec{u})d\vec{u}dw_\varphi(x)d\sigma(v)\\
&= \int_{L_2[0,t]} H_1(y,\vec{\zeta}_{n+1}+\vec{\xi}_{n+1},v,v)\int_{\mathbb{R}^r}\int_{\mathbb{R}^r} f_r((\vec{v},y)+(\vec{v},[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}])\\
&\quad + T_A(\vec{z}+\vec{u}))\Phi(\lambda_1,\vec{z})\Phi(\lambda,\vec{u})d\vec{z}d\vec{u}d\sigma(v),
\end{aligned}$$

where H_1 and Φ are given by (16) and (18), respectively. By the analytic continuation, we have $T_{\lambda_1}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})$ for $\lambda_1 \in \mathbb{C}_+$. For $\vec{u},\vec{l} \in \mathbb{R}^r$ and $\lambda \in \mathbb{C}_+$, we have

$$\Phi(\lambda,\vec{u})\Phi(\overline{\lambda},\vec{l}-\vec{u}) = \left(\frac{|\lambda|}{2\pi}\right)^r \exp\left\{-\mathrm{Re}\lambda\|\vec{u}\|^2_{\mathbb{R}^r}+\overline{\lambda}\langle\vec{u},\vec{l}\rangle_{\mathbb{R}^r}-\frac{\overline{\lambda}}{2}\|\vec{l}\|^2_{\mathbb{R}^r}\right\} \tag{28}$$

so that by the change of variable theorem and (22),

$$\begin{aligned}
&T_{\overline{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\
&= \left(\frac{|\lambda|^2}{4\pi\mathrm{Re}\lambda}\right)^{\frac{r}{2}}\int_{L_2[0,t]} H_1(y,\vec{\zeta}_{n+1}+\vec{\xi}_{n+1},v,v)\\
&\quad \int_{\mathbb{R}^r} f_r(T_A((\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])(A^T)^{-1}-\vec{l}))\exp\left\{-\frac{|\lambda|^2}{4\mathrm{Re}\lambda}\|\vec{l}\|^2_{\mathbb{R}^r}\right\}d\vec{l}d\sigma(v)\\
&= \epsilon^{-r}(f_r(T_A\cdot)*\Phi(1,\cdot/\epsilon))((\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])(A^T)^{-1})\\
&\quad \int_{L_2[0,t]} H_1(y,\vec{\zeta}_{n+1}+\vec{\xi}_{n+1},v,v)d\sigma(v),
\end{aligned}$$

where $\epsilon = (2\mathrm{Re}\lambda/|\lambda|^2)^{1/2} > 0$. Let $1 \le p < \infty$. Then we have by Theorem 1.1 and the change of variable theorem,

$$\begin{aligned}
&\int_C |T_{\overline{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})-\Psi(y+[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}])|^p dw_\varphi(y)\\
&= \int_C \bigg|[\epsilon^{-r}(f_r(T_A\cdot)*\Phi(1,\cdot/\epsilon))((\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])(A^T)^{-1})\\
&\quad - f_r(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])]\int_{L_2[0,t]} H_1(y,\vec{\xi}_{n+1}+\vec{\zeta}_{n+1},v,v)d\sigma(v)\bigg|^p dw_\varphi(y)
\end{aligned}$$

$$\leq |\det(A)|\,\|\sigma\|^p \int_{\mathbb{R}^r} |\epsilon^{-r}(f_r(T_A\cdot) * \Phi(1,\cdot/\epsilon))(\vec{u}) - f_r(T_A\vec{u})|^p d\vec{u}.$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, which satisfies $\epsilon \to 0$, we have the first part of the theorem by [24, Theorem 1.18]. If $1 \leq p \leq \infty$, then the remainder of the theorem follows from [24, Theorem 1.25]. □

Theorem 4.2. *Let q be a nonzero real number. Then, under the assumptions as given in Theorem 3.2 with exceptions that σ is concentrated on V and $\{v_1, \ldots, v_r\} \subseteq V^\perp$, we have for P_{X_n}-a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$,*

$$\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](\cdot,\vec{\zeta}_n) - \Psi_{\vec{\xi}_n,\vec{\zeta}_n}\|_p \longrightarrow 0$$

for $1 \leq p < \infty$, and for $1 \leq p \leq \infty$,

$$T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n) \longrightarrow \Psi_{\vec{\xi}_n,\vec{\zeta}_n}(y)$$

for w_φ-a.e. $y \in C[0,t]$ as λ approaches to $-iq$ through $\mathbb{C}_+$, where $\Psi_{\vec{\xi}_n,\vec{\zeta}_n}(y) = F_r(y)\int_{L_2[0,t]} H_2(\vec{\zeta}_n + \vec{\xi}_n, v)\exp\{i(v,y)\}d\sigma(v)$ and H_2 is as given in Theorem 3.3.

Proof. Note that $\Gamma_t = 1$, $\mathcal{P}\vec{v} = \vec{0}$ and T_A is the identity transformation on $\mathbb{R}^r$. For $\lambda \in \mathbb{C}_+$, $\lambda_1 > 0$, $y \in C[0,t]$, $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and $\vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$, we have by (19), Lemma 2.1 and Theorem 3.2,

$$\begin{aligned}
&I^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_{n+1})\\
&= \int_{L_2[0,t]} B_\lambda(v)H_1(y,\vec{\zeta}_{n+1},v,v)H_2(\vec{\xi}_n,v)\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\Phi(\lambda,\vec{u})\Phi(\lambda_1,\vec{z})f_r((\vec{v},y)\\
&\quad + \vec{u} + \vec{z})d\vec{z}d\vec{u}d\sigma(v),
\end{aligned}$$

where H_1 and Φ are given by (16) and (18), respectively, and B_λ is as given in Theorem 3.3. For $\vec{\zeta}_n = (\zeta_0, \zeta_1, \ldots, \zeta_n) \in \mathbb{R}^{n+1}$, let $\vec{\zeta}_{n+1} = (\zeta_0, \zeta_1, \ldots, \zeta_{n+1})$, where $\zeta_{n+1} \in \mathbb{R}$. Then we have by Lemma 3.1,

$$\begin{aligned}
&K^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_n)\\
&= \left[\frac{\lambda_1}{2\pi(t-t_n)}\right]^{\frac{1}{2}}\int_{\mathbb{R}} I^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_{n+1})\exp\left\{-\frac{\lambda_1(\zeta_{n+1}-\zeta_n)^2}{2(t-t_n)}\right\}d\zeta_{n+1}\\
&= \int_{L_2[0,t]} B_\lambda(v)B_{\lambda_1}(v)H_2(\vec{\zeta}_n + \vec{\xi}_n, v)\exp\{i(v,y)\}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\Phi(\lambda,\vec{u})\Phi(\lambda_1,\vec{z})\\
&\quad f_r((\vec{v},y) + \vec{u} + \vec{z})d\vec{z}d\vec{u}d\sigma(v),
\end{aligned}$$

which holds for $\lambda_1 \in \mathbb{C}_+$ by the analytic continuation. Let $\epsilon = (2\mathrm{Re}\lambda/|\lambda|^2)^{1/2} > 0$ for $\lambda \in \mathbb{C}_+$. Now, we have for σ-a.e. $v \in L_2[0,t]$,

$$B_\lambda(v)B_{\overline{\lambda}}(v) = \exp\{-(\epsilon^2/2)(t-t_n)(v(t))^2\} = B_{1/\epsilon^2}(v)$$

so that by (22), (28) and the change of variable theorem,

$$T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$$

$$= \left(\frac{1}{2\pi\epsilon^2}\right)^{\frac{r}{2}} \int_{L_2[0,t]} H_2(\vec{\zeta}_n + \vec{\xi}_n, v) \exp\{i(v,y)\} B_{1/\epsilon^2}(v) \int_{\mathbb{R}^r} f_r((\vec{v},y) - \vec{l}) \times \exp\left\{-\frac{\|\vec{l}\|^2_{\mathbb{R}^r}}{2\epsilon^2}\right\} d\vec{l} d\sigma(v)$$
$$= \epsilon^{-r}(f_r * \Phi(1,\cdot/\epsilon))(\vec{v},y) \int_{L_2[0,t]} H_2(\vec{\zeta}_n + \vec{\xi}_n, v) \exp\{i(v,y)\} B_{1/\epsilon^2}(v) d\sigma(v).$$

Let $1 \le p < \infty$. Then we have by Theorem 1.1 and the triangle inequality,

$$\int_C |T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n) - \Psi_{\vec{\xi}_n,\vec{\zeta}_n}(y)|^p dw_\varphi(y)$$
$$= \int_C \left| \int_{L_2[0,t]} H_2(\vec{\zeta}_n + \vec{\xi}_n, v) \exp\{i(v,y)\} [\epsilon^{-r}(f_r * \Phi(1,\cdot/\epsilon))(\vec{v},y) B_{1/\epsilon^2}(v) - f_r(\vec{v},y)] d\sigma(v) \right|^p dw_\varphi(y)$$
$$\le \int_{\mathbb{R}^r} \left[|\epsilon^{-r}(f_r * \Phi(1,\cdot/\epsilon))(\vec{u}) - f_r(\vec{u})| \int_{L_2[0,t]} B_{1/\epsilon^2}(v) d|\sigma|(v) + |f_r(\vec{u})| \times \int_{L_2[0,t]} |B_{1/\epsilon^2}(v) - 1| d|\sigma|(v) \right]^p d\vec{u}$$

so that by the Minkowski inequality,

$$\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](\cdot,\vec{\zeta}_n) - \Psi_{\vec{\xi}_n,\vec{\zeta}_n}\|_p$$
$$\le \|\sigma\| \, \|\epsilon^{-r}(f_r * \Phi(1,\cdot/\epsilon)) - f_r\|_p + \|f_r\|_p \int_{L_2[0,t]} |B_{1/\epsilon^2}(v) - 1| d|\sigma|(v).$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, which satisfies $\epsilon \to 0$, we have the first part of the theorem by the dominated convergence theorem and [24, Theorem 1.18]. By the dominated convergence theorem again,

$$\int_{L_2[0,t]} H_2(\vec{\zeta}_n + \vec{\xi}_n, v) \exp\{i(v,y)\} B_{1/\epsilon^2}(v) d\sigma(v)$$
$$\to \int_{L_2[0,t]} H_2(\vec{\zeta}_n + \vec{\xi}_n, v) \exp\{i(v,y)\} d\sigma(v)$$

as $\epsilon \to 0$ so that if $1 \le p \le \infty$, then the remainder of the theorem follows from [24, Theorem 1.25]. □

Theorem 4.3. *Let Ψ_1 and Ψ_2 be as given in Theorem 2.4. Let X_{n+1} be given by* (5) *and q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have the following*:

(1) *if $F_r, G_r \in \mathcal{A}_r^{(1)}$, then $[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_1(C[0,t])$,*
(2) *if $F_r, G_r \in \mathcal{A}_r^{(2)}$, then $[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_\infty(C[0,t])$,*
(3) *if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(2)}$, then $[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_2(C[0,t])$,*

(4) *if* $F_r \in \mathcal{A}_r^{(1)}$ *and* $G_r \in \mathcal{A}_r^{(1)} \cap \mathcal{A}_r^{(2)}$, *then* $[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_1(C[0,t]) \cap L_2([0,t])$, *and*

(5) *if* $F_r \in \mathcal{A}_r^{(1)}$ *and* $G_r \in \mathcal{A}_r^{(\infty)}$, *then* $[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_\infty(C[0,t])$.

Proof. Let $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$. For $y \in C[0,t]$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have by Theorem 2.4 and (11),

$$\begin{aligned} &|[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1})| \\ &\leq \|\sigma_1\| \, \|\sigma_2\| \left(\frac{|\lambda|}{\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \left| f_r\left(\left(\vec{v}, \frac{1}{\sqrt{2}} y\right) + \left(\vec{v}, \frac{1}{\sqrt{2}}[\vec{\xi}_{n+1}]\right) + T_A \vec{u}\right) \right. \\ &\left. g_r\left(\left(\vec{v}, \frac{1}{\sqrt{2}} y\right) - \left(\vec{v}, \frac{1}{\sqrt{2}}[\vec{\xi}_{n+1}]\right) - T_A \vec{u}\right) \right| d\vec{u}. \end{aligned}$$

Now, using the same method as in the proof of [8, Theorem 3.2] and the above inequality, we can prove the theorem. □

Using the same method as in the proof of [8, Theorem 3.2] with Theorem 3.5 and (25), we can prove the following theorem.

Theorem 4.4. *Let* X_n *be given by* (4). *If we replace* X_{n+1} *by* X_n *in Theorem 4.3, then the conclusions of the theorem still hold, where* $\vec{\xi}_{n+1}$ *is replaced by* $\vec{\xi}_n \in \mathbb{R}^{n+1}$.

Theorem 4.5. *Under the assumptions as given in Theorem 2.4, we have for* $\lambda \in \mathbb{C}_+$, w_φ*-a.e.* $y \in C[0,t]$ *and* $P_{X_{n+1}}$*-a.e.* $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned} &T_\lambda[[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1}) \\ &= \left[T_\lambda[\Psi_1 | X_{n+1}]\left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1})\right)\right] \\ &\quad \times \left[T_\lambda[\Psi_2 | X_{n+1}]\left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1})\right)\right]. \end{aligned}$$

Remark 4.6. Theorem 4.5 above follows quite easily using the same method as in the proof of [20, Theorem 2].

By Lemma 2.1, Theorems 2.2, 3.5 and the analytic continuation, we have the following theorem.

Theorem 4.7. *Under the assumptions as given in Theorem 2.4, we have for* $\lambda, \lambda_1 \in \mathbb{C}_+$, w_φ*-a.e.* $y \in C[0,t]$, P_{X_n}*-a.e.* $\xi_n \in \mathbb{R}^{n+1}$ *and* $P_{X_{n+1}}$*-a.e.* $\vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned} &T_{\lambda_1}[[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n) | X_{n+1}](y, \vec{\zeta}_{n+1}) \\ &= \Gamma_t^{\frac{1}{2}} \left(\frac{\lambda}{\pi}\right)^{\frac{r}{2}} \left(\frac{\lambda_1}{\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1\left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} \vec{\zeta}_{n+1}, v_1 + v_2, v_1 + v_2\right) \end{aligned}$$

$$\times H_2\bigg(\frac{1}{\sqrt{2}}\vec{\xi}_n, v_1 - v_2\bigg)\int_{\mathbb{R}^r}\int_{\mathbb{R}^r} A_1(2\lambda, v_1 - v_2, \vec{u})A_1(2\lambda_1, v_1 + v_2, \vec{z})B_1$$
$$(2\lambda, v_1 - v_2, \vec{u})f_r\bigg(\frac{1}{\sqrt{2}}(\vec{v}, y + [\vec{\zeta}_{n+1}]) + \frac{1}{\sqrt{2}}(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + T_A(\vec{u} + \vec{z})\bigg)$$
$$\times g_r\bigg(\frac{1}{\sqrt{2}}(\vec{v}, y + [\vec{\zeta}_{n+1}]) - \frac{1}{\sqrt{2}}(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) - T_A(\vec{u} - \vec{z})\bigg)d\vec{z}d\vec{u}d\sigma_1(v_1)$$
$$d\sigma_2(v_2),$$

where A_1, H_1, B_1 and H_2 are given by (9), (16), (20) *and* (23), *respectively.*

Theorem 4.8. *If we replace X_{n+1} by X_n in Theorem 4.5, then the conclusion of the theorem still holds, where $\vec{\xi}_{n+1}$, $\vec{\zeta}_{n+1}$ are replaced by $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$, respectively.*

Proof. Note that $B_1(2\lambda_1, v_1 + v_2, \vec{z}) = B_1(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \vec{\alpha})$ and $A_1(2\lambda_1, v_1 + v_2, \vec{z}) = A_1(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \vec{\alpha})$ if $\vec{\alpha} = \sqrt{2}\vec{z}$, where A_1 and B_1 are given by (9) and (20), respectively. For $\vec{\zeta}_n = (\zeta_0, \zeta_1, \dots, \zeta_n) \in \mathbb{R}^{n+1}$, let $\vec{\zeta}_{n+1} = (\zeta_0, \zeta_1, \dots, \zeta_n, \zeta_{n+1})$, where $\zeta_{n+1} \in \mathbb{R}$. For $\lambda \in \mathbb{C}_+$ and $\lambda_1 > 0$, we have by Lemma 3.1, Theorem 4.7 and the change of variable theorem,

$$K^{\lambda_1}_{[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n)}(y, \vec{\zeta}_n)$$
$$= \Gamma_t^{\frac{1}{2}}\bigg(\frac{\lambda}{\pi}\bigg)^{\frac{r}{2}}\bigg(\frac{\lambda_1}{2\pi}\bigg)^{\frac{r}{2}}\bigg[\frac{\lambda_1}{2\pi(t - t_n)}\bigg]^{\frac{1}{2}}\int_{L_2[0,t]}\int_{L_2[0,t]}\exp\bigg\{\frac{i}{\sqrt{2}}(v_1 + v_2, y)\bigg\}H_2$$
$$\bigg(\frac{1}{\sqrt{2}}\vec{\xi}_n, v_1 - v_2\bigg)\int_{\mathbb{R}^r} A_1(2\lambda, v_1 - v_2, \vec{u}))B_1(2\lambda, v_1 - v_2, \vec{u})\int_{\mathbb{R}}\exp\bigg\{\frac{i}{\sqrt{2}}$$
$$(v_1 + v_2, [\vec{\zeta}_{n+1}])\bigg\}\int_{\mathbb{R}^r} A_1\bigg(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \vec{\alpha}\bigg)f_r\bigg(\frac{1}{\sqrt{2}}((\vec{v}, y + [\vec{\zeta}_{n+1}]) +$$
$$T_A\vec{\alpha}) + \frac{1}{\sqrt{2}}(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + T_A\vec{u}\bigg)g_r\bigg(\frac{1}{\sqrt{2}}((\vec{v}, y + [\vec{\zeta}_{n+1}]) + T_A\vec{\alpha}) - \frac{1}{\sqrt{2}}$$
$$(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) - T_A\vec{u}\bigg)\exp\bigg\{-\frac{\lambda_1}{2}\frac{(\zeta_{n+1} - \zeta_n)^2}{t - t_n}\bigg\}d\vec{\alpha}d\zeta_{n+1}d\vec{u}d\sigma_1(v_1)d\sigma_2(v_2)$$
$$= \Gamma_t\bigg(\frac{\lambda}{\pi}\bigg)^{\frac{r}{2}}\bigg(\frac{\lambda_1}{\pi}\bigg)^{\frac{r}{2}}\int_{L_2[0,t]}\int_{L_2[0,t]}\exp\bigg\{\frac{i}{\sqrt{2}}(v_1 + v_2, y)\bigg\}H_2\bigg(\frac{1}{\sqrt{2}}\vec{\xi}_n, v_1 - v_2\bigg)$$
$$H_2\bigg(\frac{1}{\sqrt{2}}\vec{\zeta}_n, v_1 + v_2\bigg)\int_{\mathbb{R}^r}\int_{\mathbb{R}^r} A_1(2\lambda, v_1 - v_2, \vec{u}))B_1(2\lambda, v_1 - v_2, \vec{u})A_1$$
$$(2\lambda_1, v_1 + v_2, \vec{z})B_1(2\lambda_1, v_1 + v_2, \vec{z})f_r\bigg(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\vec{\zeta}_n + \vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t}))$$
$$+ T_A(\vec{u} + \vec{z})\bigg)g_r\bigg(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\vec{\zeta}_n - \vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + T_A(\vec{z} - \vec{u})\bigg)d\vec{z}d\vec{u}$$
$$d\sigma_1(v_1)d\sigma_2(v_2).$$

By the analytic continuation, we have the existence of $T_{\lambda_1}[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ for $\lambda_1\in\mathbb{C}_+$. Let $\vec{w}=\vec{z}+\vec{u}$ and $\vec{l}=\vec{z}-\vec{u}$. By a long calculation that is tedious but not difficult, we can prove that for $\lambda\in\mathbb{C}_+$, $B_1(2\lambda,v_1-v_2,\frac{1}{2}(\vec{w}-\vec{l}))B_1(2\lambda,v_1+v_2,\frac{1}{2}(\vec{w}+\vec{l}))=B_1(\lambda,v_1,\vec{w})B_1(\lambda,v_2,\vec{l})$ so that by Theorem 3.2 and the change of variable theorem,

$$
\begin{aligned}
&T_\lambda[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
&=\Gamma_t\Big(\frac{\lambda}{2\pi}\Big)^r\int_{L_2[0,t]}\int_{L_2[0,t]}\exp\Big\{\frac{i}{\sqrt{2}}(v_1+v_2,y)\Big\}H_2\Big(\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n),v_1\Big)H_2\\
&\quad\Big(\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n),v_2\Big)\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}A_1(\lambda,v_1,\vec{w})A_1(\lambda,v_2,\vec{l})B_1(\lambda,v_1,\vec{w})B_1(\lambda,v_2,\vec{l})\\
&\quad f_r\Big(\frac{1}{\sqrt{2}}(\vec{v},y)+\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t}))+T_A\vec{w}\Big)\\
&\quad g_r\Big(\frac{1}{\sqrt{2}}(\vec{v},y)+\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t}))+T_A\vec{l}\Big)d\vec{w}d\vec{l}d\sigma_1(v_1)d\sigma_2(v_2)\\
&=\Big[T_\lambda[\Psi_1|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\Big)\Big]\Big[T_\lambda[\Psi_2|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\Big)\Big],
\end{aligned}
$$

which completes the proof. □

Now, we have the following relationships between the conditional Fourier-Feynman transforms and the conditional convolution products from Theorems 2.2, 2.4, 3.2, 3.5, 4.3, 4.4, 4.5 and 4.8.

Theorem 4.9. *Let Ψ_1 and Ψ_2 be as given in Theorem 2.4. Let X_n be given by* (4) *and q be a nonzero real number. Then we have the following*:

(1) *if $F_r,G_r\in\mathcal{A}_r^{(1)}$, then we have for w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$,*

$$
\begin{aligned}
&T_q^{(1)}[[(\Psi_1*\Psi_2)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
&=\Big[T_q^{(1)}[\Psi_1|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\Big)\Big]\Big[T_q^{(1)}[\Psi_2|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\Big)\Big],
\end{aligned}
$$

(2) *if $F_r\in\mathcal{A}_r^{(1)}$ and $G_r\in\mathcal{A}_r^{(2)}$, then we have for w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$,*

$$
\begin{aligned}
&T_q^{(2)}[[(\Psi_1*\Psi_2)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
&=\Big[T_q^{(1)}[\Psi_1|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\Big)\Big]\Big[T_q^{(2)}[\Psi_2|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\Big)\Big].
\end{aligned}
$$

Theorem 4.10. *If we replace X_n by X_{n+1} in Theorem 4.9, then the conclusions of the theorem still hold, where $\vec{\xi}_n$, $\vec{\zeta}_n$ are replaced by $\vec{\xi}_{n+1},\vec{\zeta}_{n+1}\in\mathbb{R}^{n+2}$, respectively.*

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