J. Korean Math. Soc. **50** (2013), No. 5, pp. 1067–1082
http://dx.doi.org/10.4134/JKMS.2013.50.5.1067

STRUCTURE RELATIONS OF CLASSICAL MULTIPLE ORTHOGONAL POLYNOMIALS BY A GENERATING FUNCTION

Dong Won Lee

Abstract. In this paper, we will find some recurrence relations of classical multiple OPS between the same family with different parameters using the generating functions, which are useful to find structure relations and their connection coefficients. In particular, the differential-difference equations of Jacobi-Piñeiro polynomials and multiple Bessel polynomials are given.

1. Introduction

Let $r \geq 2$ be a fixed positive integer, $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}_0^r$ a multi-index, and $e_i = (0, \ldots, 1, \ldots, 0)$ the i-th standard unit vector in $\mathbb{R}^r$ with $e = \sum_{i=1}^r e_i$. For any vector $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_r)$, we let $|\mathbf{t}| = t_1 + t_2 + \cdots + t_r$ and the product $\alpha \cdot \mathbf{t} = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_r t_r$.

A sequence $\{P_{\vec{n}}(x)\}_{|\vec{n}|=0}^{\infty}$ of polynomials is called a multiple orthogonal polynomial system (multiple OPS) if

(i) $\deg(P_{\vec{n}}) = |\vec{n}|$;
(ii) there exist r positive weights w_i such that for $i = 1, 2, \ldots, r$,

$$\int_{-\infty}^{\infty} x^k P_{\vec{n}}(x) w_i(x) dx = 0 \quad \text{for } k = 0, 1, 2, \ldots, n_i - 1.$$

The multiple OPS was originated from the paper of Angelesco in dealing with simultaneous Padé approximants ([1]). These families of polynomials attracted big interest in the area of simultaneous Padé approximation, random matrix, asymptotics, number theory, and so on (see [6, 7, 9, 10, 11] for recent relevant references and therein).

Recently many results are obtained on so-called classical multiple OPS's whose orthogonalizing weights w_i are classical. More precisely, they are multiple Hermite polynomials, multiple Laguerre I polynomials, multiple Laguerre

Received November 30, 2012.
2010 *Mathematics Subject Classification.* 33C45, 42C05.
Key words and phrases. multiple orthogonal polynomial, classical multiple orthogonal polynomial, recurrence relation, generating function.

II polynomials, Jacobi-Piñeiro polynomials, and multiple Bessel polynomials (see [2, 3, 16, 21] and references therein).

Among the classical multiple OPS's the generating functions are developed for three families of them, which are the case of multiple Hermite polynomials, multiple Laguerre I polynomials, and multiple Laguerre II polynomials ([13]). For these multiple OPS's the orthogonalizing weights and the generating functions are as follow.

(a) multiple Hermite polynomials $\{H_{\vec{n}}^{(\vec{\alpha})}(x)\}_{|\vec{n}|=0}^{\infty}$: The orthogonalizing weights are $w_i(x) = e^{\frac{\delta}{2}x^2+\alpha_i x}$ on $(-\infty,\infty)$, where $\delta < 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. The generating function is

$$G^{(\alpha)}(x,\mathbf{t}) = e^{\delta x|\mathbf{t}|+\frac{\delta}{2}|\mathbf{t}|^2+|\vec{\alpha}\cdot\mathbf{t}|},$$

that means for $x \in \mathbb{R}$ and $t_i \in \mathbb{R}$ for $i = 1, 2, \ldots, r$,

$$G^{(\alpha)}(x,\mathbf{t}) = \sum_{\vec{n}=0}^{\infty} H_{\vec{n}}^{(\vec{\alpha})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}.$$

(b) multiple Laguerre I polynomials $\{L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\}_{|\vec{n}|=0}^{\infty}$: The orthogonalizing weights are $w_i(x) = x^{\alpha_i}e^{\beta x}$ on $(0,\infty)$, where $\alpha_i > -1$, $\beta < 0$, and $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$. The generating function is

$$G^{(\vec{\alpha};\beta)}(x,\mathbf{t}) = \prod_{i=1}^{r}\frac{1}{(1-t_i)^{\alpha_i+1}}e^{\beta x\left(\frac{1}{\prod_{i=1}^{r}(1-t_i)}-1\right)},$$

that means for $x \in (0,\infty)$ and $|t_i| < 1$ for $i = 1, 2, \ldots, r$,

$$G^{(\vec{\alpha};\beta)}(x,\mathbf{t}) = \sum_{\vec{n}=0}^{\infty} L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}.$$

(c) multiple Laguerre II polynomials $\{L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\}_{|\vec{n}|=0}^{\infty}$: The orthogonalizing weights are $w_i(x) = x^{\alpha}e^{\beta_i x}$ on $(0,\infty)$, where $\alpha > -1$, $\beta_i < 0$, and $\beta_i \neq \beta_j$ for $i \neq j$. The generating function is

$$G^{(\alpha;\vec{\beta})}(x,\mathbf{t}) = \frac{1}{(1-|\mathbf{t}|)^{\alpha+1}}e^{\frac{|\vec{\beta}\cdot\mathbf{t}|x}{1-|\mathbf{t}|}},$$

that means for $x \in (0,\infty)$ and $|\mathbf{t}| < 1$,

$$G^{(\alpha;\vec{\beta})}(x,\mathbf{t}) = \sum_{\vec{n}=0}^{\infty} L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}.$$

Here, we used multi-index notations $\vec{n}! = n_1!n_2!\cdots n_r!$, $\mathbf{t}^{\vec{n}} = t_1^{n_1}t_2^{n_2}\cdots t_r^{n_r}$, and $\sum_{\vec{n}=0}^{\infty} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}$.

By the generating function, many properties of multiple Hermite polynomials and the multiple Laguerre polynomials were developed such as differential-difference relation and differential equations (see [13]).

On the other hand, recurrence relations of Jacobi-Piñeiro polynomials and multiple Bessel polynomials are not rigorously studied until now. In order to get their recurrence relations, we need their generating functions. Recently we found the generating function for them in the case of $r = 2$ by Lagrange expansion method (see [14]). More precisely, the author proved:

(d) Jacobi-Piñeiro polynomials $\{P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)\}_{n_1+n_2=0}^{\infty}$: The orthogonalizing weights are $w_i(x) = x^{\alpha_i}(x-1)^{\alpha}$ $(i = 1, 2)$ on $(0, 1)$, where $\alpha_1, \alpha_2, \alpha > -1$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$. The generating function is

$$G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t}) = \frac{1}{[(1+t_1-2t_1z)(1+t_2-2t_2z)-t_1t_2z^2]}\frac{(1+t_1-t_1z)^{-\alpha_1}(1+t_2-t_2z)^{-\alpha_2}}{[(1-t_1z)(1-t_2z)-t_1t_2z]^{\alpha}},$$

that means for $|t_1| < 1$, $|t_2| < 1$, and $x \in (0, 1)$,

$$G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!},$$

where z is a solution of $z(1+t_1-t_1z)(1+t_2-t_2z) = x$ with $z \to x$ as $t_1, t_2 \to 0$.

(e) multiple Bessel polynomials $\{B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\}_{n_1+n_2=0}^{\infty}$: The orthogonalizing weights are $w_i(x) = x^{\alpha_i}e^{\frac{\gamma}{x}}$ $(i = 1, 2)$ on the unit circle in complex plane, where $\alpha_1, \alpha_2 > -1$, $\gamma \neq 0$, and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$. The generating function is

$$G^{(\alpha_1,\alpha_2;\gamma)}(x,\mathbf{t}) = \frac{(1-t_1z)^{-\alpha_1}(1-t_2z)^{-\alpha_2}}{(1-2t_1z)(1-2t_2z)-t_1t_2z^2}e^{\gamma(\frac{1}{z}-\frac{1}{x})},$$

that means for $|t_1| < 1$, $|t_2| < 1$, and $|x| = 1$ on complex plane,

$$G^{(\alpha_1,\alpha_2;\gamma)}(x,\mathbf{t}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!},$$

where z is a solution of $z(1-t_1z)(1-t_2z) = x$ with $z \to x$ as $t_1, t_2 \to 0$.

There are tremendous recurrence relations for orthogonal polynomials such as three term recurrence relation, differential-difference equation, and so on. We refer to [8, 17, 18]. The recurrence relation plays a key role in applications of orthogonal polynomials in area of rational approximation, quadrature formula, special functions, combinatorics, differential equations and so on.

For multiple OPS's, many recurrence relations are obtained as an extension of orthogonal polynomials and so they would be used in many areas of multiple OPS's (see [4, 5, 12, 20] and references therein). Most of these papers treated the recurrence relations with the same polynomials or differential-difference relation of the same polynomials. Comparing to ordinary classical OPS, we can deduce that the recurrence relations of the same family of classical multiple OPS with different parameter will also play an important role in investigating the properties of multiple OPSs.

In this paper, we find some recurrence relations of classical multiple OPS between the same family with different parameters using the generating function, which are useful to find the structure relations and their connection coefficients. In particular, the differential-difference equations of Jacobi-Piñeiro polynomials and multiple Bessel polynomials are given.

2. Multiple Hermite polynomials

Using the identities of $G^{(\vec{\alpha})}(x,\mathbf{t})$ for multiple Hermite polynomials

$$\frac{\partial}{\partial x}G^{(\vec{\alpha})}(x,\mathbf{t}) = \delta|\mathbf{t}|G^{(\vec{\alpha})}(x,\mathbf{t})$$

and

$$\frac{\partial}{\partial t_i}G^{(\vec{\alpha})}(x,\mathbf{t}) = (\delta x+\alpha_i)G^{(\vec{\alpha})}(x,\mathbf{t}) + \frac{\partial}{\partial x}G^{(\vec{\alpha})}(x,\mathbf{t}),$$

the author found ([13, Theorem 2.4]) differential-difference equations

$$\frac{d}{dx}H_{\vec{n}}^{(\vec{\alpha})}(x) = \delta\sum_{j=1}^{r}H_{\vec{n}-e_j}^{(\vec{\alpha})}(x) = H_{\vec{n}+e_i}^{(\vec{\alpha})}(x) - (\delta x+\alpha_i)H_{\vec{n}}^{(\alpha)}(x), \quad i=1,2,\dots,r.$$

These relations can be regarded as generalizations of the relation

$$H_n'(x) = nH_{n-1}(x) = 2xH_n(x) - 2H_{n+1}(x),$$

where $\{H_n(x)\}_{n=0}^{\infty}$ is the monic Hermite polynomials orthogonal with respect to $w(x)=e^{-x^2}$ on $(-\infty,\infty)$.

By a simple calculation we have an identity

$$G^{(\vec{\alpha}+e_i)}(x,\mathbf{t}) = e^{t_i}G^{(\vec{\alpha})}(x,\mathbf{t}), \tag{2.1}$$

from which a new recurrence relation for multiple Hermite polynomials immediately follows.

Theorem 2.1. *Let $\{H_{\vec{n}}^{(\vec{\alpha})}(x)\}_{|\vec{n}|=0}^{\infty}$ be the multiple Hermite polynomials. Then we have for $i=1,2,\dots,r$,*

$$H_{\vec{n}}^{(\vec{\alpha}+e_i)}(x) = \sum_{j=0}^{n_i}\binom{n_i}{j}H_{\vec{n}-je_i}^{(\alpha)}(x).$$

Proof. From the definition of generating function, we have for $i=1,2,\dots,r$,

$$G^{(\vec{\alpha}+e_i)}(x,\mathbf{t}) = \sum_{\vec{n}=0}^{\infty}H_{\vec{n}}^{(\vec{\alpha}+e_i)}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}. \tag{2.2}$$

On the other hand,

$$\begin{aligned} e^{t_i}G^{(\vec{\alpha})}(x,\mathbf{t}) &= \left(\sum_{j=0}^{\infty}\frac{t_i^j}{j!}\right)\left(\sum_{\vec{n}=0}^{\infty}H_{\vec{n}}^{(\vec{\alpha})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}\right) \\ &= \sum{}'\left(\sum_{n_i=0}^{\infty}\sum_{j=0}^{\infty}H_{\vec{n}}^{(\vec{\alpha})}(x)\frac{t_i^{n_i+j}}{n_i!j!}\right)\frac{t_1^{n_1}\cdots t_{i-1}^{n_{i-1}}t_{i+1}^{n_{i+1}}\cdots t_r^{n_r}}{n_1!\cdots n_{i-1}!n_{i+1}!\cdots n_r!}, \end{aligned}$$

where $\sum' := \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{i-1}=0}^{\infty}\sum_{n_{i+1}=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}$. By change of variables we have

$$\begin{aligned}\sum_{n_i=0}^{\infty}\sum_{j=0}^{\infty}H_{\vec{n}}^{(\vec{\alpha})}(x)\frac{t_i^{n_i+j}}{n_i!j!} &= \sum_{n_i=0}^{\infty}\sum_{j=0}^{n_i}H_{\vec{n}-je_i}^{(\vec{\alpha})}(x)\frac{t_i^{n_i}}{(n_i-j)!j!}\\ &= \sum_{n_i=0}^{\infty}\sum_{j=0}^{n_i}\binom{n_i}{j}H_{\vec{n}-je_i}^{(\vec{\alpha})}(x)\frac{t_i^{n_i}}{n_i!},\end{aligned}$$

so that

$$e^{t_i}G^{(\vec{\alpha})}(x,\mathbf{t}) = \sum_{\vec{n}=0}^{\infty}\sum_{j=0}^{n_i}\binom{n_i}{j}H_{\vec{n}-je_i}^{(\vec{\alpha})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}. \tag{2.3}$$

From the equation (2.1), the conclusion follows by comparing the coefficients of (2.2) and (2.3). □

In particular, if $r=2$, then

$$H_{n_1,n_2}^{(\alpha_1+1,\alpha_2)}(x) = \sum_{j=0}^{n_1}\binom{n_1}{j}H_{n_1-j,n_2}^{(\alpha_1,\alpha_2)}(x)$$

and

$$H_{n_1,n_2}^{(\alpha_1,\alpha_2+1)}(x) = \sum_{j=0}^{n_2}\binom{n_2}{j}H_{n_1,n_2-j}^{(\alpha_1,\alpha_2)}(x).$$

Applying Theorem 2.1 iteratively we obtain

$$H_{\vec{n}}^{(\vec{\alpha}+e)}(x) = \sum_{\vec{k}=0}^{\vec{n}}\binom{\vec{n}}{\vec{k}}H_{\vec{n}-\sum_{j=1}^{r}k_je_j}^{(\vec{\alpha})}(x),$$

where $\vec{k}=(k_1,k_2,\ldots,k_r)$. Here, we used the notations $\sum_{\vec{k}=0}^{\vec{n}} = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}$ and $\binom{\vec{n}}{\vec{k}} = \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}$.

The recurrence relation in Theorem 2.1 is quite interesting because we could not find a similar relation for Hermite polynomials. Hence, the relation is a new property that distinguishes multiple Hermite polynomials from Hermite polynomials.

3. Multiple Laguerre I polynomials

Using the identities of $G^{(\vec{\alpha};\beta)}(x,\mathbf{t})$ for multiple Laguerre I polynomials

$$\frac{\partial}{\partial x}G^{(\vec{\alpha};\beta)}(x,\mathbf{t}) = \beta\left(\frac{1}{\prod_{j=1}^{r}(1-t_j)}-1\right)G^{(\vec{\alpha};\beta)}(x,\mathbf{t})$$

and

$$\frac{\partial}{\partial t_i}G^{(\vec{\alpha};\beta)}(x,\mathbf{t}) = \frac{\beta x+\alpha_i+1}{1-t_i}G^{(\vec{\alpha};\beta)}(x,\mathbf{t}) + \frac{x}{1-t_i}\frac{\partial}{\partial x}G^{(\vec{\alpha};\beta)}(x,\mathbf{t}), \tag{3.1}$$

the author found ([13, Theorem 2.7]) differential-difference equations

$$\frac{d}{dx}L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) = \beta\left(L_{\vec{n}}^{(\vec{\alpha}+e;\beta)}(x) - L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\right)$$

and for $i = 1, 2, \ldots, r$,

$$L_{\vec{n}+e_i}^{(\vec{\alpha};\beta)}(x) = (\beta x + \alpha_i + 1)L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x) + x\frac{d}{dx}L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x). \tag{3.2}$$

These equations can be regarded as a generalization of the relations

$$\frac{d}{dx}L_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) - L_n^{(\alpha+1)}(x)$$

and

$$L_{n+1}^{(\alpha)}(x) = (x - \alpha - 1)L_n^{(\alpha+1)}(x) - x\frac{d}{dx}L_n^{(\alpha+1)}(x),$$

where $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic Laguerre polynomials orthogonal with respect to $w(x) = x^{\alpha}e^{-x}$ on $(0, \infty)$.

For a new recurrence relation for the multiple Laguerre I polynomials we use the identity

$$G^{(\vec{\alpha}+e_i;\beta)}(x, \mathbf{t}) = \frac{1}{1 - t_i}G^{(\vec{\alpha};\beta)}(x, \mathbf{t})$$

or equivalently

$$G^{(\vec{\alpha};\beta)}(x, \mathbf{t}) = (1 - t_i)G^{(\vec{\alpha}+e_i;\beta)}(x, \mathbf{t}). \tag{3.3}$$

Theorem 3.1. *Let $\{L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\}_{|\vec{n}|=0}^{\infty}$ be the multiple Laguerre I polynomials. Then we have for $i = 1, 2, \ldots, r$,*

$$L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) = L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x) - n_i L_{\vec{n}-e_i}^{(\vec{\alpha}+e_i;\beta)}(x) \tag{3.4}$$

and

$$L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x) = \sum_{j=0}^{n_i}\frac{n_i!}{(n_i - j)!}L_{\vec{n}-je_i}^{(\alpha;\beta)}(x). \tag{3.5}$$

Proof. We prove here only the equation (3.5) because the equation (3.4) is an easy consequence of the identity (3.3). Since

$$\sum_{n_i=0}^{\infty}\sum_{j=0}^{\infty}L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\frac{t_i^{n_i+j}}{n_i!} = \sum_{n_i=0}^{\infty}\sum_{j=0}^{n_i}L_{\vec{n}-je_i}^{(\vec{\alpha};\beta)}(x)\frac{t_i^{n_i}}{(n_i - j)!},$$

we have

(3.6)

$$
\begin{aligned}
\frac{1}{1-t_i}G^{(\vec{\alpha};\beta)}(x,\mathbf{t}) &= \left(\sum_{j=0}^{\infty} t_i^j\right)\left(\sum_{\vec{n}=0}^{\infty} L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\frac{\mathbf{t}^n}{\vec{n}!}\right) \\
&= \sum{}' \left(\sum_{n_i=0}^{\infty}\sum_{j=0}^{\infty} L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\frac{t_i^{n_i+j}}{n_i!}\right)\frac{t_1^{n_1}\cdots t_{i-1}^{n_{i-1}}t_{i+1}^{n_{i+1}}\cdots t_r^{n_r}}{n_1!\cdots n_{i-1}!n_{i+1}!\cdots n_r!} \\
&= \sum_{\vec{n}=0}^{\infty}\left(\sum_{j=0}^{n_i}\frac{n_i!}{(n_i-j)!}L_{\vec{n}-je_i}^{(\vec{\alpha};\beta)}(x)\right)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!},
\end{aligned}
$$

where $\sum{}' := \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{i-1}=0}^{\infty}\sum_{n_{i+1}=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}$. Since

$$G^{(\vec{\alpha}+e_i;\beta)}(x,\mathbf{t}) = \sum_{\vec{n}=0}^{\infty} L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}, \tag{3.7}$$

we obtain the result by comparing the coefficients of (3.6) and (3.7). □

Combining (3.2) and (3.4) we have

$$L_{\vec{n}+e_i}^{(\vec{\alpha};\beta)}(x) - n_i L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) = (\beta x+\alpha_i+1)L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) + x\frac{d}{dx}L_{\vec{n}}^{(\vec{\alpha};\beta)}(x),$$

which can be obtained from (3.1) directly. In case of $r=2$, Theorem 3.1 implies

$$
\begin{aligned}
L_{n_1,n_2}^{(\alpha_1,\alpha_2;\beta)}(x) &= L_{n_1,n_2}^{(\alpha_1+1,\alpha_2;\beta)}(x) - n_1 L_{n_1-1,n_2}^{(\alpha_1+1,\alpha_2;\beta)}(x) \\
&= L_{n_1,n_2}^{(\alpha_1,\alpha_2+1;\beta)}(x) - n_2 L_{n_1,n_2-1}^{(\alpha_1,\alpha_2+1;\beta)}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
L_{n_1,n_2}^{(\alpha_1+1,\alpha_2;\beta)}(x) &= \sum_{j=0}^{n_1}\frac{n_1!}{(n_1-j)!}L_{n_1-j,n_2}^{(\alpha_1,\alpha_2;\beta)}(x); \\
L_{n_1,n_2}^{(\alpha_1,\alpha_2+1;\beta)}(x) &= \sum_{j=0}^{n_2}\frac{n_2!}{(n_2-j)!}L_{n_1,n_2-j}^{(\alpha_1,\alpha_2;\beta)}(x).
\end{aligned}
$$

If we adopt the relation (3.5) in Theorem 3.1 iteratively, we have

$$L_{\vec{n}}^{(\vec{\alpha}+e;\beta)}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\frac{\vec{n}!}{(\vec{n}-\vec{k})!}L_{\vec{n}-\sum_{j=1}^{r}k_je_j}^{(\vec{\alpha};\beta)}(x),$$

where $\vec{k}=(k_1,k_2,\ldots,k_r)$. The recurrence relations in Theorem 3.1 can also be regraded as a generalization of

(3.8)

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x) \quad \text{and} \quad L_n^{(\alpha+1)}(x) = \sum_{j=0}^{n}\frac{(-1)^j n!}{(n-j)!}L_{n-j}^{(\alpha)}(x),$$

where $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic Laguerre polynomials.

4. Multiple Laguerre II polynomials

From the identities of $G^{(\alpha;\vec{\beta})}(x,\mathbf{t})$ for multiple Laguerre II polynomials

$$\frac{\partial}{\partial x}G^{(\alpha;\vec{\beta})}(x,\mathbf{t})=\frac{|\vec{\beta}\cdot\mathbf{t}|}{1-|\mathbf{t}|}G^{(\alpha;\vec{\beta})}(x,\mathbf{t})=|\vec{\beta}\cdot\mathbf{t}|G^{(\alpha+1;\vec{\beta})}(x,\mathbf{t})$$

and for $i=1,2,\dots,r$,

$$\frac{\partial}{\partial t_i}G^{(\alpha;\vec{\beta})}(x,\mathbf{t})=\frac{\beta_i x+\alpha+1}{1-|\mathbf{t}|}G^{(\alpha;\vec{\beta})}(x,\mathbf{t})+\frac{x}{1-|\mathbf{t}|}\frac{\partial}{\partial x}G^{(\alpha;\vec{\beta})}(x,\mathbf{t}), \tag{4.1}$$

the author found ([13, Theorem 2.8] differential-difference equations

$$\frac{d}{dx}L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)=\sum_{j=1}^{r}\beta_j n_j L_{\vec{n}-e_j}^{(\alpha+1;\vec{\beta})}(x),$$

$$\frac{d}{dx}L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)-\sum_{j=1}^{r}n_j\frac{d}{dx}L_{\vec{n}-e_j}^{(\alpha;\vec{\beta})}(x)=\sum_{j=1}^{r}n_j\beta_j L_{\vec{n}-e_j}^{(\alpha;\vec{\beta})}(x)$$

and for $i=1,2,\dots,r$,

$$L_{\vec{n}+e_i}^{(\alpha;\vec{\beta})}(x)-\sum_{j=1}^{r}n_j L_{\vec{n}-e_j+e_i}^{(\alpha;\vec{\beta})}(x)=(\beta_i x+\alpha+1)L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)+x\frac{d}{dx}L_{\vec{n}}^{(\alpha;\vec{\beta})}(x). \tag{4.2}$$

These equations can be regarded as generalizations of

$$\frac{d}{dx}L_n^{(\alpha)}(x)=nL_{n-1}^{(\alpha+1)}(x),$$

$$\frac{d}{dx}\left(L_n^{(\alpha)}(x)+nL_{n-1}^{(\alpha)}(x)\right)=nL_{n-1}^{(\alpha)}(x)$$

and

$$L_{n+1}^{(\alpha)}(x)+nL_n^{(\alpha)}(x)=(x-\alpha-1)L_n^{(\alpha)}(x)-x\frac{d}{dx}L_n^{(\alpha)}(x),$$

where $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic Laguerre polynomials as in Section 3.

For a new recurrence relation for multiple Laguerre II polynomials we use the identity

$$G^{(\alpha+1;\vec{\beta})}(x,\mathbf{t})=\frac{1}{1-|\mathbf{t}|}G^{(\alpha;\vec{\beta})}(x,\mathbf{t})$$

or equivalently

$$G^{(\alpha;\vec{\beta})}(x,\mathbf{t})=(1-|\mathbf{t}|)G^{(\alpha+1;\vec{\beta})}(x,\mathbf{t}).$$

Theorem 4.1. *Let $\{L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\}_{|\vec{n}|=0}^{\infty}$ be the multiple Laguerre II polynomials. Then we have*

$$L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)=L_{\vec{n}}^{(\alpha+1;\vec{\beta})}(x)-\sum_{j=1}^{r}n_j L_{\vec{n}-e_j}^{(\alpha+1;\vec{\beta})}(x) \tag{4.3}$$

and

$$L_{\vec{n}}^{(\alpha+\gamma+1;\vec{\beta}+\vec{\delta})}(x)=\sum_{\vec{k}=0}^{\vec{n}}\binom{\vec{n}}{\vec{k}}L_{\vec{k}}^{(\gamma;\vec{\delta})}(x)L_{\vec{n}-\sum_{j=1}^{r}k_je_j}^{(\alpha;\vec{\beta})}(x), \tag{4.4}$$

where $\gamma>-1$, $\vec{\delta}=\sum_{i=1}^{r}\delta_ie_i$ *with* $\delta_i<0$, *and* $\vec{k}=(k_1,k_2,\ldots,k_r)$.

Proof. From $G^{(\alpha;\vec{\beta})}(x,\mathbf{t})=(1-|\mathbf{t}|)G^{(\alpha+1;\vec{\beta})}(x,\mathbf{t})$, the equation (4.3) can be proved by the equation

$$\begin{aligned}\sum_{\vec{n}=0}^{\infty}L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}&=\sum_{\vec{n}=0}^{\infty}(1-|\mathbf{t}|)L_{\vec{n}}^{(\alpha+1;\vec{\beta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}\\&=\sum_{\vec{n}=0}^{\infty}L_{\vec{n}}^{(\alpha+1;\vec{\beta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}-\sum_{\vec{n}=0}^{\infty}\sum_{j=1}^{r}n_jL_{\vec{n}-e_j}^{(\alpha+1;\vec{\beta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}.\end{aligned}$$

For the proof of the equation (4.4), note that

$$G^{(\alpha+\gamma+1;\vec{\beta}+\vec{\delta})}(x,\mathbf{t})=\frac{1}{(1-|\mathbf{t}|)^{\alpha+\gamma+2}}e^{\frac{|(\vec{\beta}+\vec{\delta})\cdot\mathbf{t}|x}{1-|\mathbf{t}|}}=G^{(\gamma;\vec{\delta})}(x,\mathbf{t})G^{(\alpha;\vec{\beta})}(x,\mathbf{t})$$

so that

$$\begin{aligned}\sum_{\vec{n}=0}^{\infty}L_{\vec{n}}^{(\alpha+\gamma+1;\vec{\beta}+\vec{\delta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}&=\left(\sum_{\vec{k}=0}^{\infty}L_{\vec{k}}^{(\gamma;\vec{\delta})}(x)\frac{\mathbf{t}^{\vec{k}}}{\vec{k}!}\right)\left(\sum_{\vec{n}=0}^{\infty}L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}\right)\\&=\sum_{i=1}^{r}\left(\sum_{n_i=0}^{\infty}\sum_{k_i=0}^{\infty}L_{\vec{k}}^{(\gamma;\vec{\delta})}(x)L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\frac{t_i^{n_i+k_i}}{k_i!n_i!}\right)\\&=\sum_{i=1}^{r}\left(\sum_{n_i=0}^{\infty}\sum_{k_i=0}^{n_i}\binom{n_i}{k_i}L_{\vec{k}}^{(\gamma;\vec{\delta})}(x)L_{\vec{n}-k_ie_i}^{(\alpha;\vec{\beta})}(x)\frac{t_i^{n_i}}{n_i!}\right)\\&=\sum_{\vec{n}=0}^{\infty}\left(\sum_{\vec{k}=0}^{\vec{n}}\binom{\vec{n}}{\vec{k}}L_{\vec{k}}^{(\gamma;\vec{\delta})}(x)L_{\vec{n}-\sum_{j=1}^{r}k_je_j}^{(\alpha;\vec{\beta})}(x)\right)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}.\end{aligned}\tag{4.5}$$

By comparing the coefficients of (4.5), the relation (4.4) follows. □

Combining (4.2) and (4.3), we have

$$L_{\vec{n}+e_i}^{(\alpha-1;\vec{\beta})}(x)=(\beta_ix+\alpha+1)L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)+x\frac{d}{dx}L_{\vec{n}}^{(\alpha;\vec{\beta})}(x).$$

In case of $r=2$, Theorem 4.1 implies

$$L_{n_1,n_2}^{(\alpha;\beta_1,\beta_2)}(x)=L_{n_1,n_2}^{(\alpha+1;\beta_1,\beta_2)}(x)-n_1L_{n_1-1,n_2}^{(\alpha+1;\beta_1,\beta_2)}(x)-n_2L_{n_1,n_2-1}^{(\alpha+1;\beta_1,\beta_2)}(x)$$

and

$$L_{n_1,n_2}^{(\alpha+\gamma+1;\beta_1-1,\beta_2-1)}(x)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}L_{k_1,k_2}^{(\gamma;-1,-1)}(x)L_{n_1-k_1,n_2-k_2}^{(\alpha;\beta_1,\beta_2)}(x).$$

The equation (4.3) is a generalization of the first equation (3.8) and the equation (4.4) is quite similar to

$$L_n^{(\alpha+\beta+1)}(x)=\sum_{k=0}^{n}\binom{n}{k}L_k^{(\alpha)}(0)L_{n-k}^{(\beta)}(x).$$

5. Jacobi-Piñeiro polynomials

For the generating function $G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t})$ of Jacobi-Piñeiro polynomials, we can easily prove the identities

- $t_2G^{(\alpha_1-1,\alpha_2;\alpha)}(x,\mathbf{t})-t_1G^{(\alpha_1,\alpha_2-1;\alpha)}(x,\mathbf{t})=(t_2-t_1)G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t})$;
- $t_1xG^{(\alpha_1+1,\alpha_2;\alpha)}(x,\mathbf{t})=(1+t_1)G^{(\alpha_1,\alpha_2-1;\alpha)}(x,\mathbf{t})-G^{(\alpha_1-1,\alpha_2-1;\alpha)}(x,\mathbf{t})$;
  $t_2xG^{(\alpha_1,\alpha_2+1;\alpha)}(x,\mathbf{t})=(1+t_2)G^{(\alpha_1-1,\alpha_2;\alpha)}(x,\mathbf{t})-G^{(\alpha_1-1,\alpha_2-1;\alpha)}(x,\mathbf{t})$;
- $(x-1)G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t})=xG^{(\alpha_1+1,\alpha_2+1;\alpha-1)}(x,\mathbf{t})-G^{(\alpha_1,\alpha_2;\alpha-1)}(x,\mathbf{t})$,

from which the following recurrence relations follow.

Theorem 5.1. *Let $\{P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)\}_{n_1+n_2=0}^{\infty}$ be the Jacobi-Piñeiro polynomials. Then we have*

(i) $n_2P_{n_1,n_2-1}^{(\alpha_1-1,\alpha_2;\alpha)}(x)-n_1P_{n_1-1,n_2}^{(\alpha_1,\alpha_2-1;\alpha)}(x)$
$\quad=n_2P_{n_1,n_2-1}^{(\alpha_1,\alpha_2;\alpha)}(x)-n_1P_{n_1-1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)$;

(ii) $n_1xP_{n_1-1,n_2}^{(\alpha_1+1,\alpha_2;\alpha)}(x)=P_{n_1,n_2}^{(\alpha_1,\alpha_2-1;\alpha)}(x)-P_{n_1,n_2}^{(\alpha_1-1,\alpha_2-1;\alpha)}(x)$
$\quad+n_1P_{n_1-1,n_2}^{(\alpha_1,\alpha_2-1;\alpha)}(x)$;
$n_2xP_{n_1,n_2-1}^{(\alpha_1,\alpha_2+1;\alpha)}(x)=P_{n_1,n_2}^{(\alpha_1-1,\alpha_2;\alpha)}(x)-P_{n_1,n_2}^{(\alpha_1-1,\alpha_2-1;\alpha)}(x)$
$\quad+n_2P_{n_1,n_2-1}^{(\alpha_1-1,\alpha_2;\alpha)}(x)$;

(iii) $(x-1)P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)=xP_{n_1,n_2}^{(\alpha_1+1,\alpha_2+1;\alpha-1)}(x)-P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha-1)}(x)$.

Proof. From the first identity of generating function, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}^{(\alpha_1-1,\alpha_2;\alpha)}(x)\frac{t_1^{n_1}t_2^{n_2+1}}{n_1!n_2!}-\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}^{(\alpha_1,\alpha_2-1;\alpha)}(x)\frac{t_1^{n_1+1}t_2^{n_2}}{n_1!n_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(t_2-t_1)P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}$$

so that (i) is proved. (ii) and (iii) can be proved by the same way. □

Now, we will give a differential-difference relation for Jacobi-Piñeiro polynomials. In order to obtain the relation, we need a differential equation of generating function.

Lemma 5.2. *The generating function $G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t})$ of Jacobi-Piñeiro polynomials satisfies*

$$x(x-1)G_x+[\alpha x+(\alpha_1+1)(x-1)]G+t_1(x-1)G_{t_1}=G_{t_1}^{(\alpha_1,\alpha_2;\alpha-1)} \tag{5.1}$$

and

$$x(x-1)G_x+[\alpha x+(\alpha_2+1)(x-1)]G+t_2(x-1)G_{t_2}=G_{t_2}^{(\alpha_1,\alpha_2;\alpha-1)}, \tag{5.2}$$

where $G=G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t})$, $G_x=\frac{dG}{dx}$, $G_{t_1}=\frac{dG}{dt_1}$, *and* $G_{t_2}=\frac{dG}{dt_2}$.

Proof. For convenience we let $A=1+t_1-t_1z$, $B=1+t_2-t_2z$, $C=(1-t_1z)(1-t_2z)-t_1t_2z$, and $\Phi=AB-t_1zB-t_2zA$. Then we have $C=\frac{x-1}{z-1}$,

$$x=zAB \quad \text{and} \quad G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t})=A^{-\alpha_1}B^{-\alpha_2}C^{-\alpha}\Phi^{-1}.$$

A direct calculation shows

$$\begin{aligned}G_{t_1}&=-\left(\frac{\alpha_1}{A}\frac{dA}{dt_1}+\frac{\alpha_2}{B}\frac{dB}{dt_1}+\frac{\alpha}{C}\frac{dC}{dt_1}+\frac{1}{\Phi}\frac{\partial\Phi}{\partial t_1}\right)G\\&=(z-1)\left(\alpha_1(B-t_2z)+\alpha_2t_2z+\frac{\alpha zB}{z-1}+\frac{(2z-1)B}{z-1}-t_2z-\frac{zB}{\Phi}\frac{\partial\Phi}{\partial z}\right)\frac{G}{\Phi}\end{aligned}$$

and

$$\begin{aligned}G_x&=-\left(\frac{\alpha_1}{A}\frac{dA}{dx}+\frac{\alpha_2}{B}\frac{dB}{dx}+\frac{\alpha}{C}\frac{dC}{dx}+\frac{1}{\Phi}\frac{\partial\Phi}{\partial x}\right)G\\&=-\frac{\alpha}{x-1}G+\left(\frac{\alpha_1t_1}{A}+\frac{\alpha_2t_2}{B}+\frac{\alpha}{z-1}-\frac{1}{\Phi}\frac{\partial\Phi}{\partial z}\right)\frac{G}{\Phi}.\end{aligned}$$

Hence,

$$\begin{aligned}xG_x+\frac{\alpha x}{x-1}G-\frac{A}{z-1}G_{t_1}&=-\left(\alpha_1\Phi+\Phi+\frac{zAB}{z-1}+t_1zB\right)\frac{G}{\Phi}\\&=-(\alpha_1+1)G-\left(\frac{zAB}{z-1}+t_1zB\right)\frac{G}{\Phi}.\end{aligned} \tag{5.3}$$

Since $\frac{\partial z}{\partial t_1}=\frac{z(z-1)B}{\Phi}$, $zB-zAB-t_1zB(z-1)=0$, and

$$\begin{aligned}\frac{A}{z-1}G_{t_1}&=-t_1G_{t_1}+\frac{1}{z-1}G_{t_1}\\&=-t_1G_{t_1}+\frac{1}{x-1}\left(\frac{x-1}{z-1}G\right)_{t_1}+\frac{zB}{(z-1)\Phi}G,\end{aligned} \tag{5.4}$$

we obtain by the equations (5.3) and (5.4)

$$xG_x+\frac{\alpha x}{x-1}G+t_1G_{t_1}+(\alpha_1+1)G=\frac{1}{x-1}G_{t_1}^{(\alpha_1,\alpha_2,\alpha-1)}$$

which is (5.1). The equation (5.2) can be proved by the same method. □

Theorem 5.3. *The Jacobi-Piñeiro polynomial* $\{P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)\}_{n_1+n_2=0}^{\infty}$ *satisfies a differential-difference equation*

$$P_{n_1+1,n_2}^{(\alpha_1,\alpha_2;\alpha-1)}(x)=x(x-1)\frac{d}{dx}P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)+[\alpha x+(\alpha_1+1+n_1)(x-1)]P_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x) \tag{5.5}$$

and
(5.6)

$$P^{(\alpha_1,\alpha_2;\alpha-1)}_{n_1,n_2+1}(x)=x(x-1)\frac{d}{dx}P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2}(x)+[\alpha x+(\alpha_2+1+n_2)(x-1)]P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2}(x).$$

Proof. Since

$$\begin{aligned}
t_1G_{t_1} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!},\\
G_{t_1} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P^{(\alpha_1,\alpha_2;\alpha)}_{n_1+1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!},\\
G_x &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{d}{dx}P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!},
\end{aligned}$$

we obtain the equation (5.5) from the equation (5.1) in Lemma 5.2. The second equation (5.6) can also be proved by the same method using the equation (5.2) in Lemma 5.2. □

The equations (5.5) and (5.6)) are kinds of raising operators that are very useful to find a differential equation for orthogonal polynomials. It is well known that the Jacobi polynomial $\{P^{(\alpha,\beta)}_n(x)\}_{n=0}^{\infty}$ satisfies a differential-difference equation

$$\begin{aligned}
(1-x^2)\frac{d}{dx}P^{(\alpha,\beta)}_n(x) &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}[(2n+\alpha+\beta+2)x+\alpha-\beta]P^{(\alpha,\beta)}_n(x)\\
&\quad -\frac{2(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+2}P^{(\alpha,\beta)}_{n+1}(x),
\end{aligned}$$

where $\{P^{(\alpha,\beta)}_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$ on $[-1,1]$ and normalized by $P^{(\alpha,\beta)}_n(1)=\binom{n+\alpha}{n}$. Hence, the results of Theorem 5.3 is a generalization of the relation for classical Jacobi polynomials.

Subtracting (5.6) from (5.5) in Theorem 5.3 gives

$$(\alpha_1-\alpha_2+n_1-n_2)(x-1)P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2}(x)=P^{(\alpha_1,\alpha_2;\alpha-1)}_{n_1+1,n_2}(x)-P^{(\alpha_1,\alpha_2;\alpha-1)}_{n_1,n_2+1}(x)$$

and subtracting the equation (5.6) of the case (n_1,n_2-1) from the equation (5.5) of the case (n_1-1,n_2), we have

$$\begin{aligned}
x(x-1)\ &\frac{d}{dx}\left(P^{(\alpha_1,\alpha_2;\alpha)}_{n_1-1,n_2}(x)\ -\ P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2-1}(x)\right)\\
&= \alpha x\left(P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2-1}(x)-P^{(\alpha_1,\alpha_2;\alpha)}_{n_1-1,n_2}(x)\right)\\
&\quad +(x-1)\left((\alpha_2+n_2)P^{(\alpha_1,\alpha_2;\alpha)}_{n_1,n_2-1}(x)-(\alpha_1+n_1)P^{(\alpha_1,\alpha_2;\alpha)}_{n_1-1,n_2}(x)\right),
\end{aligned}$$

which seems to be a new differential-difference equation for Jacobi-Piñeiro polynomials.

6. Multiple Bessel polynomials

For the generating function $G^{(\alpha_1,\alpha_2;\gamma)}(x,\mathbf{t})$ of multiple Bessel polynomials, we can easily prove the identities

- $t_2G^{(\alpha_1-1,\alpha_2;\gamma)}(x,\mathbf{t}) - t_1G^{(\alpha_1,\alpha_2-1;\gamma)}(x,\mathbf{t}) = (t_2-t_1)G^{(\alpha_1,\alpha_2;\gamma)}(x,\mathbf{t})$.
- $t_1xG^{(\alpha_1+1,\alpha_2;\gamma)}(x,\mathbf{t}) = G^{(\alpha_1,\alpha_2-1;\gamma)}(x,\mathbf{t}) - G^{(\alpha_1-1,\alpha_2-1;\gamma)}(x,\mathbf{t})$;
  $t_2xG^{(\alpha_1,\alpha_2+1;\gamma)}(x,\mathbf{t}) = G^{(\alpha_1-1,\alpha_2;\gamma)}(x,\mathbf{t}) - G^{(\alpha_1-1,\alpha_2-1;\gamma)}(x,\mathbf{t})$

from which the following recurrence relations follow.

Theorem 6.1. *Let $\{B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\}_{n_1+n_2=0}^{\infty}$ be the multiple Bessel polynomials. Then we have*

(i) $n_2B_{n_1,n_2-1}^{(\alpha_1-1,\alpha_2;\gamma)}(x) - n_1B_{n_1-1,n_2}^{(\alpha_1,\alpha_2-1;\gamma)}(x)$
$= n_2B_{n_1,n_2-1}^{(\alpha_1,\alpha_2;\gamma)}(x) - n_1B_{n_1-1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)$.

(ii) $n_1xB_{n_1-1,n_2}^{(\alpha_1+1,\alpha_2;\gamma)}(x) = B_{n_1,n_2}^{(\alpha_1,\alpha_2-1;\gamma)}(x) - B_{n_1,n_2}^{(\alpha_1-1,\alpha_2-1;\gamma)}(x)$;
$n_2xB_{n_1,n_2-1}^{(\alpha_1,\alpha_2+1;\gamma)}(x) = B_{n_1,n_2}^{(\alpha_1-1,\alpha_2;\gamma)}(x) - B_{n_1,n_2}^{(\alpha_1-1,\alpha_2-1;\gamma)}(x)$.

Proof. From the first equation of generating function, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} B_{n_1,n_2}^{(\alpha_1-1,\alpha_2;\alpha)}(x)\frac{t_1^{n_1}t_2^{n_2+1}}{n_1!n_2!} - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} B_{n_1,n_2}^{(\alpha_1,\alpha_2-1;\alpha)}(x)\frac{t_1^{n_1+1}t_2^{n_2}}{n_1!n_2!}$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(t_2-t_1)B_{n_1,n_2}^{(\alpha_1,\alpha_2;\alpha)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}$$

so that (i) is proved. (ii) and (iii) can be proved by the same way. $\square$

Subtracting the second equation from the first equation in Theorem 6.1(ii), we obtain an interesting relation

$$B_{n_1,n_2}^{(\alpha_1,\alpha_2-1;\gamma)}(x)-B_{n_1,n_2}^{(\alpha_1-1,\alpha_2;\gamma)}(x) = x\left(n_1B_{n_1-1,n_2}^{(\alpha_1+1,\alpha_2;\gamma)}(x) - n_2B_{n_1,n_2-1}^{(\alpha_1,\alpha_2+1;\gamma)}(x)\right).$$

Now, we will give a differential-difference relation for multiple Bessel polynomials. In order to obtain the relation, we need a differential equation of generating function.

Lemma 6.2. *The generating function $G^{(\alpha_1,\alpha_2;\gamma)}(x,\mathbf{t})$ of multiple Bessel polynomials satisfies*

$$x^2G_x + [(\alpha_1+1)x-\gamma]G + t_1xG_{t_1} = G_{t_1}^{(\alpha_1-1,\alpha_2-1;\gamma)} \tag{6.1}$$

and

$$x^2G_x + [(\alpha_2+1)x-\gamma]G + t_2xG_{t_2} = G_{t_2}^{(\alpha_1-1,\alpha_2-1;\gamma)}, \tag{6.2}$$

where $G = G^{(\alpha_1,\alpha_2;\gamma)}(x,\mathbf{t})$, $G_x = \frac{dG}{dx}$, $G_{t_1} = \frac{dG}{dt_1}$, and $G_{t_2} = \frac{dG}{dt_2}$.

Proof. For convenience, we let $A = 1-t_1z$, $B = 1-t_2z$, and $\Phi = AB - t_1zB - t_2zA$. Then we have

$$x = zAB \quad \text{and} \quad G^{(\alpha_1,\alpha_2;\alpha)}(x,\mathbf{t}) = A^{-\alpha_1}B^{-\alpha_2}e^{\gamma(\frac{1}{z}-\frac{1}{x})}.$$

A direct calculation similar to the proof of Lemma 6.2 shows

$$G_{t_1} = \left(\alpha_1 z(B - t_2 z) + \alpha_2 t_2 z^2 - \gamma B + 2zB - t_2 z^2 - \frac{z^2 B}{\Phi}\frac{\partial \Phi}{\partial z}\right)\frac{G}{\Phi}$$

and

$$G_x = \frac{\gamma}{x^2}G + \left(\frac{\alpha_1 t_1}{A} + \frac{\alpha_2 t_2}{B} - \frac{\gamma}{z^2} - \frac{1}{\Phi}\frac{\partial \Phi}{\partial z}\right)\frac{G}{\Phi}$$

so that

$$xG_x - \frac{\gamma}{x}G - \frac{A}{z}G_{t_1} = -\alpha_1 G + (t_2 zA - 2AB)\frac{G}{\Phi}.$$

Hence, we have

$$\begin{aligned} xG_x - \frac{\gamma}{x}G + t_1 G_{t_1} + \alpha_1 G &= \frac{1}{z}G_{t_1} + (t_2 zA - 2AB)\frac{G}{\Phi} \\ &= \frac{1}{x}(ABG)_{t_1} + (B + t_2 zA - 2AB)\frac{G}{\Phi} \\ &= \frac{1}{x}G_{t_1}^{(\alpha_1 - 1, \beta_1 - 1;\gamma)} - G \end{aligned}$$

which is (6.1). The equation (6.2) can be proved by the same method. □

Theorem 6.3. *The multiple Bessel polynomial* $\{B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\}_{n_1+n_2=0}^{\infty}$ *satisfies a differential-difference equation*

$$x^2\frac{d}{dx}B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) + [(\alpha_1 + 1 + n_1)x - \gamma]B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) = B_{n_1+1,n_2}^{(\alpha_1-1,\alpha_2-1;\gamma)}(x) \tag{6.3}$$

and

$$x^2\frac{d}{dx}B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) + [(\alpha_2 + 1 + n_2)x - \gamma]B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) = B_{n_1,n_2+1}^{(\alpha_1-1,\alpha_2-1;\gamma)}(x). \tag{6.4}$$

Proof. Since

$$\begin{aligned} t_1 G_{t_1} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \\ G_{t_1} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} B_{n_1+1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \\ G_x &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{d}{dx}B_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \end{aligned}$$

we prove the equation (6.3) from the equation (6.1). The equation (6.4) can be proved by the same method using the equation (6.2) in Lemma 6.2. □

The equations (6.3) and (6.4) are kinds of raising operators for multiple Bessel polynomials that is very useful to find a differential equation for orthogonal polynomials. It is well known that the monic Bessel polynomial

$\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ satisfies a differential-difference equation

$$x^2\frac{d}{dx}B_n^{(\alpha,\beta)}(x) = \left(nx - \frac{\beta n}{2n+\alpha-2}\right)B_n^{(\alpha,\beta)}(x) + \frac{n(n+\alpha-2)\beta^2}{(2n+\alpha-3)(2n+\alpha-2)^2}B_{n-1}^{(\alpha,\beta)}(x),$$

where $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is orthogonal with respect to $w(x) = x^\alpha e^{\frac{\beta}{x}}$ on the unit circle in complex plane. Hence, the result of Theorem 6.3 is a generalization of the relation for classical Bessel polynomials.

Subtracting the equation (6.4) from (6.3) in Theorem 6.3 gives

$$(\alpha_1-\alpha_2+n_1-n_2)xB_{n_1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) = B_{n_1+1,n_2}^{(\alpha_1-1,\alpha_2-1;\alpha)}(x) - B_{n_1,n_2+1}^{(\alpha_1-1,\alpha_2-1;\alpha)}(x)$$

and subtracting the equation (6.4) of the case (n_1, n_2-1) from (6.3) of the case (n_1-1, n_2), we get

$$\begin{aligned} x^2\,\frac{d}{dx}\left(B_{n_1-1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) - B_{n_1,n_2-1}^{(\alpha_1,\alpha_2;\gamma)}(x)\right) &= \gamma\left(B_{n_1-1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) - B_{n_1,n_2-1}^{(\alpha_1,\alpha_2;\gamma)}(x)\right) \\ &\quad - x\left((\alpha_1+n_1)B_{n_1-1,n_2}^{(\alpha_1,\alpha_2;\gamma)}(x) - (\alpha_2+n_2)B_{n_1,n_2-1}^{(\alpha_1,\alpha_2;\gamma)}(x)\right), \end{aligned}$$

which seems to be a new differential-difference equation for multiple Bessel polynomials.

Acknowledgements. The author thanks the referee for his/her careful reading to improve this paper. This research was supported by Kyungpook National University Research Fund, 2012.

References

[1] A. Angelesco, *Sur l'approximatiön simultaneé de plusieurs integrales definies*, C. R. Paris, **167** (1918), 629–631.

[2] A. I. Aptekarev, *Multiple orthogonal polynomials*, J. Comput. Appl. Math. **99** (1998), no. 1-2, 423–447.

[3] A. I. Aptekarev, A. Branquinho, and W. Van Assche, *Multiple orthogonal polynomials for classical weights*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 3887–3914.

[4] A. I. Aptekarev, V. Kalyagin, G. López Lagomasino, and I. A. Rocha, *On the limit behavior of recurrence coefficients for multiple orthogonal polynomials*, J. Approx. Theory **139** (2006), no. 1-2, 346–370.

[5] B. Beckermann, J. Coussement, and W. Van Assche, *Multiple Wilson and Jacobi-Pineiro polynomials*, J. Approx. Theory **132** (2005), no. 2, 155–181.

[6] P. M. Bleher and A. B. J. Kuijlaars, *Large n limit of Gaussian random matrices with external source. I*, Comm. Math. Phys. **252** (2004), no. 1-3, 43–76.

[7] ———, *Integral representations for multiple Hermite and multiple Laguerre polynomials*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 6, 2001–2014.

[8] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[9] P. Desrosiers, *Duality in random matrix ensembles for all β*, Nuclear Phys. B **817** (2009), no. 3, 224–251.

[10] P. Desrosiers and P. J. Forrester, *Asymptotic correlations for Gaussian and Wishart matrices with external source*, Int. Math. Res. Not. (2006), Art. ID 27395, 43p.

[11] ———, *A note on biorthogonal ensembles*, J. Approx. Theory **152** (2008), no. 2, 167–187.

[12] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, in Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, 2005.

[13] D. W. Lee, *Properties of multiple Hermite and multiple Laguerre polynomials by the generating function*, Integral Transforms Spec. Funct. **18** (2007), no. 11-12, 855–869.

[14] ———, *Generating functions and multiple orthogonal polynomials*, In 5th Asian Mathematical Conference Proceedings, Vol. II, (Yahya Abu Hasan et al., ed.), 44–51, 2009.

[15] V. Lysov and F. Wielonsky, *Strong asymptotics for multiple Laguerre polynomials*, Constr. Approx. **28** (2008), no. 1, 61–111.

[16] E. M. Nikishin and V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, 92. American Mathematical Society, Providence, RI, 1991.

[17] E. D. Rainville, *Special Functions*, Chelsea Publishing Company, New York, 1960.

[18] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. vol 23., 4th ed., Amer. Math. Soc., Providence, RI, 1975.

[19] W. Van Assche, *Multiple orthogonal polynomials, irrationality and transcendence*, Continued fractions: from analytic number theory to constructive approximation (Columbia, MO, 1998), 325–342, Contemp. Math., 236, Amer. Math. Soc., Providence, RI, 1999.

[20] ———, *Nearest neighbor recurrence relations for multiple orthogonal polynomials*, J. Approx. Theory **163** (2011), no. 10, 1427–1448.

[21] W. Van Assche and E. Coussement, *Some classical multiple orthogonal polynomials*, J. Comput. Appl. Math. **127** (2001), no. 1-2, 317–347.

Department of Mathematics
Teachers College
Kyungpook National University
Daegu 702-701, Korea
E-mail address: dongwon@mail.knu.ac.kr