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ON OVERRINGS OF GORENSTEIN DEDEKIND DOMAINS

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Abstract. In this paper, we mainly discuss Gorenstein Dedekind domains (G-Dedekind domains for short) and their overrings. Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. Then it is proved that R is a G-Dedekind domain if and only if for any prime ideal P of R which contains $(R :_K T)$, P is Gorenstein projective. We also give not only an example to show that G-Dedekind domains are not necessarily Noetherian Warfield domains, but also a definition for a special kind of domain: a 2-DVR. As an application, we prove that a Noetherian domain R is a Warfield domain if and only if for any maximal ideal M of R, R_M is a 2-DVR.

Introduction

Throughout this paper, all rings are commutative with identity element and all modules are unitary. Let R be a domain with quotient field K and J a fractional ideal of R. The definition of J^{-1} can be found in [14] as follows:

$$J^{-1} = \{x \mid x \in K,\ xJ \subset R\}.$$

The definition of divisorial ideals can be found in [10]: A fractional ideal J of R is said to be *divisorial* if $J_v = (J^{-1})^{-1} = J$.

Let R be a commutative ring. An R-module M is said to be *Gorenstein projective* (*G-projective* for short) if there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module. An R-module M is said to be *strongly Gorenstein projective* (*SG-projective* for short) if there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P \to P \to P \to P \to \cdots$$

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such that all these projective modules and all the morphisms of the exact sequence are the same and $M \cong \mathrm{Im}(P \to P)$ and also, $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module.

The concepts of G-hereditary rings and G-Dedekind domains were introduced in [16]. It can be seen from [12, Theorem 2.28] and [20, Theorem 6.3.4] that a one-dimensional Noetherian domain either is a G-Dedekind domain or has infinite Gorenstein global dimension. Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. It is proved in the first section that R is a G-Dedekind domain if and only if for any prime ideal P of R which contains $(R :_K T)$, P is G-projective.

An overring of a domain R is a ring between R and its quotient field K. It is well known that any overring of a Dedekind domain is still a Dedekind domain. It is natural to ask whether overrings of a G-Dedekind domain are G-Dedekind domains or not? In Section 2, we give an example which shows that this is not necessarily the case. In fact, it can be seen from [19, Theorem 6.3] and [19, Theorem 6.4] that a Noetherian domain has the property that each of its overring is a G-Dedekind domain if and only if it is a Warfield domain. In [5] and [6], Driss Bennis and Najib Mahdou introduced strongly Gorenstein projective modules and n-strongly Gorenstein projective modules respectively. In Section 2, we utilize these concepts as tools to investigate some properties of Noetherian Warfield domains.

The author in [19] gave a general introduction to Warfield domains. A domain R is called a *Warfield domain* if, given any submodule A of the field of quotients K, the A-torsionless $\mathrm{End}_R(A)$-modules of finite rank are A-reflexive. In Section 2, we not only provide an example to show that G-Dedekind domains are not necessarily Noetherian Warfield domains, but also give a definition for a special kind of domain: a 2-DVR. A Noetherian local domain is called a 2-*DVR* if its maximal ideal is strongly G-projective. As an application, we prove that a Noetherian domain R is a Warfield domain if and only if for any maximal ideal M of R, R_M is a 2-DVR.

1. One-dimensional Noetherian domains and Gorenstein Dedekind domains

We say that a module M has *Gorenstein projective dimension* at most a positive integer n, and we write $Gpd(M) \leqslant n$, if there exists an exact sequence of modules

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is G-projective.

A module M is said to be *Gorenstein injective*, if there exists an exact sequence of injective modules

$$\mathbf{I} = \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

such that $M \cong \mathrm{Im}(I_0 \to I^0)$ and such that $\mathrm{Hom}_R(E,-)$ leaves the sequence $\mathbf{I}$ exact whenever E is an injective module.

We say that a module M has *Gorenstein injective dimension* at most a positive integer n and we write $Gid(M) \leqslant n$, if there exists an exact sequence of modules

$$0 \to M \to G_0 \to \cdots \to G_n \to 0,$$

and each G_i is Gorenstein injective. The authors in [7] have studied global Gorenstein dimensions of a ring R, which are called *Gorenstein projective, injective, and weak dimensions of* R, denoted by $\mathrm{GPD}(R)$, $\mathrm{GID}(R)$, and $\text{G-}w\dim(R)$, respectively, defined as follows:

$$\begin{aligned} \mathrm{GPD}(R) &= \sup\{Gpd(M) \mid M\ R\text{-module}\} \\ \mathrm{GID}(R) &= \sup\{Gid(M) \mid M\ R\text{-module}\}, \text{ and} \\ \text{G-}w\dim(R) &= \sup\{Gfd_R(M) \mid M\ R\text{-module}\}. \end{aligned}$$

They proved that for any ring R, $\text{G-}w\dim(R) \leqslant \mathrm{GID}(R) = \mathrm{GPD}(R)$ [7, Theorems 3.2 and 4.2]. The common value of $\mathrm{GPD}(R)$ and $\mathrm{GID}(R)$ is called *Gorenstein global dimension* of R, and denoted by $G\text{-}gl\dim(R)$.

If R is a ring of finite global dimension (denoted by $gl.\dim(R) < \infty$), then every Gorenstein projective module over R is projective since its projective dimension is finite. Therefore $gl.\dim(R) = G\text{-}gl\dim(R)$.

A domain is called a *Gorenstein Dedekind domain* (*G-Dedekind domain* for short) if every submodule of a projective module is G-projective (i.e., $G\text{-}gl\dim(R) \leqslant 1$). Dedekind domains are G-Dedekind domains since their Gorenstein global dimensions are equal to their global dimensions and must be at most 1.

Recall that the *FPD dimension* of a ring R is defined as follows:

$$\mathrm{FPD}(R) = \sup\{pd_R(M) \mid M \text{ is an } R\text{-module such that } pd_R(M) < \infty\},$$

where $pd_R(M)$ denotes projective dimension of M. Likewise in [12], the *FGPD dimension* of a ring R is defined as follows:

$$\mathrm{FGPD}(R) = \sup\{Gpd_R(M) \mid M \text{ is an } R\text{-module such that } Gpd_R(M) < \infty\}.$$

The following theorem is [12, Theorem 2.28].

Theorem 1.1. *For any ring R there is an equality*: $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$.

For the relation between $\mathrm{FPD}(R)$ and Krull dimension $\dim(R)$, we have the following theorem [20, Theorem 6.3.4] (its proof can be seen in [11]):

Theorem 1.2. *For any Noetherian ring R there is an equality*: $\mathrm{FPD}(R) = \dim(R)$.

Therefore, for any Noetherian ring R, we surely have $\mathrm{FGPD}(R) = \dim(R)$. If R is a one-dimensional Noetherian domain (i.e., $\dim(R) = 1$), then $\mathrm{FGPD}(R) = 1$. Furthermore, if $G\text{-}gl\dim(R) < \infty$, then $Gpd_R(M) < \infty$ for any R-module M. Thus $Gpd_R(M) \leqslant \mathrm{FGPD}(R) = 1$. Hence $G\text{-}gl\dim(R) \leqslant 1$. In other words, a one-dimensional Noetherian domain either is a G-Dedekind domain or has infinite Gorenstein global dimension.

Example 1.3. Let $\mathbb{Q}$ be the field of rational numbers and X be an indeterminate. We consider the ring $R = \mathbb{Q} + X^3\mathbb{Q}[X]$. We prove that the ideal $I = (X^3, X^4, X^5)$ is not G-projective. Consider the short exact sequence

$$(*) \qquad 0 \longrightarrow I_1 \oplus I_2 \xrightarrow{\alpha} R \oplus R \oplus R \xrightarrow{\beta} I \longrightarrow 0$$

where $I_1 = I_2 = I$. The homomorphisms α and β are defined as follows: $\alpha((i_1, i_2)) = (i_1X, -i_1, 0) + (0, i_2X, -i_2)$ and $\beta((r_1, r_2, r_3)) = r_1X^3 + r_2X^4 + r_3X^5$.

Next, we show that $\mathrm{Ext}^1_R(I, R) \neq 0$ and this means that I is not G-projective. Let φ be a homomorphism from $I_1 \oplus I_2$ to R such that $\varphi((X^3, 0)) = X^3X^2 = X^5$, this is possible because $\mathrm{Hom}(I, R) \cong I^{-1} = \mathbb{Q}[X]$. If there exists a homomorphism θ from $R \oplus R \oplus R$ to R such that $\varphi = \theta\alpha$, then there must exist $r, r' \in R$ such that $X^5 = rX^4 - r'X^3$. But this is impossible since the right hand side of this equation will never have a nonzero term of X^5.

Since the ideal $I = (X^3, X^4, X^5)$ is not G-projective, R is not a G-Dedekind domain and $G\text{-}gl\dim(R) = \infty$. In fact, it can be seen from the short exact sequence $(*)$ that the Gorenstein projective dimension of I can not be finite.

Recall that a Noetherian domain is said to be a *1-Gorenstein domain* if its self-injective dimension is at most 1. The following theorem is [4, Theorem 3.4].

Theorem 1.4. *Let R be an integral domain. Then $G\text{-}gl\dim(R) \leqslant 1$ if and only if R is 1-Gorenstein.*

Therefore, a domain is a G-Dedekind domain if and only if it is 1-Gorenstein.

Proposition 1.5. *Let R be a domain with quotient field K such that $R \neq K$. Then the following statements are equivalent*:

(1) *K/R is an injective R-module.*
(2) *$id_R(R) = 1$, where $id_R(M)$ denotes injective dimension of M.*
(3) *For any submodule M of a free module, $\mathrm{Ext}^1_R(M, R) = 0$.*
(4) *For any ideal $I \subset R$, $\mathrm{Ext}^1_R(I, R) = 0$.*

Proof. (1)⇒(2) By [20, Theorem 3.2.6], the quotient field K of R is an injective R-module. So, it can be seen from the following short exact sequence

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0$$

that $id_R(R) \leqslant 1$. Since $R \neq K$, R is not a field and is not an injective module, therefore $id_R(R) = 1$.

(2)⇒(1) It can be seen from the above short exact sequence.

(1)⇒(3) Let M be a submodule of some free module F. Consider the following exact sequence:

$$0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$$

Because F is a projective module, it can be seen that

$$\mathrm{Ext}^1_R(M,R) \cong \mathrm{Ext}^2_R(F/M,R).$$

Consider also the exact sequence:

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0$$

Because K is an injective module, it can be seen that

$$\mathrm{Ext}^2_R(F/M,R) \cong \mathrm{Ext}^1_R(F/M,K/R).$$

Therefore, $\mathrm{Ext}^1_R(M,R) \cong \mathrm{Ext}^1_R(F/M,K/R)$, and $\mathrm{Ext}^1_R(M,R) = 0$ if and only if $\mathrm{Ext}^1_R(F/M,K/R) = 0$. So, when K/R is an injective module, $\mathrm{Ext}^1_R(M,R) = 0$.

(3)$\Rightarrow$(4) This is obvious.

(4)$\Rightarrow$(1) Let I be an ideal of R. Consider the following exact sequence:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

Because R is projective, it can be seen that $\mathrm{Ext}^1_R(I,R) \cong \mathrm{Ext}^2_R(R/I,R)$. Consider also the exact sequence:

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0$$

Since K is injective, it can be seen that $\mathrm{Ext}^2_R(R/I,R) \cong \mathrm{Ext}^1_R(R/I,K/R)$. So

$$\mathrm{Ext}^1_R(I,R) \cong \mathrm{Ext}^1_R(R/I,K/R)$$

and, $\mathrm{Ext}^1_R(I,R) = 0$ if and only if $\mathrm{Ext}^1_R(R/I,K/R) = 0$. Therefore, when $\mathrm{Ext}^1_R(I,R) = 0$ holds for any ideal I of R, $\mathrm{Ext}^1_R(R/I,K/R) = 0$ also holds for any ideal I of R. Thus K/R is an injective module. □

It follows from Proposition 1.5 that a Noetherian domain R with quotient field K is a G-Dedekind domain if and only if K/R is injective.

Lemma 1.6. *Let R be a domain with quotient field K and I be an ideal of R. Then* $\mathrm{Ext}^1_R(I,R) = 0$ (*for example, I is G-projective*) *if and only if* $\mathrm{Ext}^1_R(R/I,K/R) = 0$.

Proof. The proof is the same as that of (4)$\Rightarrow$(1) in Proposition 1.5. □

Proposition 1.7. *Let R be a Noetherian domain with quotient field K. If every prime ideal of R is G-projective, then R is a G-Dedekind domain.*

Proof. From the remark above, it will suffice to show that K/R is injective. If we can show that $\mathrm{Ext}^1_R(M,K/R) = 0$ for any finitely generated R-module M, then K/R must be injective. Since R is Noetherian and M is a finitely generated R-module, by [20, Theorem 6.5.12], there exists an ascending chain of submodules of M:

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that $M_{i+1}/M_i \cong R/P_{i+1}$ where $P_i's$ are prime ideals of R and $i = 0,1,\ldots,n-1$. Let P be any prime ideal of R. Since P is G-projective, by

Lemma 1.6, $\mathrm{Ext}^1_R(R/P, K/R) = 0$. Therefore $\mathrm{Ext}^1_R(M_{i+1}/M_i, K/R) = 0$ and inductively, $\mathrm{Ext}^1_R(M, K/R) = 0$. □

Lemma 1.8. *Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. Denote the ideal $(R :_K T)$ by I. If J is an ideal of R such that $J + I = R$, then J is invertible.*

Proof. In order to show that J is invertible, it will suffice to show that J_P is a principal ideal of R_P for any prime ideal P of R.

If P is a prime ideal containing I, then $J + P = R$ and J can not be inside P. So $J_P = R_P$ is principal.

Next we will show that R_P is a principal ideal domain when P is a prime ideal which doesn't contain I. Since P is a prime ideal which doesn't contain I, there exists an element $t \in I$ but t is not in P. Since $tT \subset R$, $T \subset \frac{1}{t}R \subset R_P$. Let $Q = PR_P \bigcap T$. Then Q is a prime ideal of T such that $Q \bigcap R = P$. Trivially, $R_P \subseteq T_Q$. Suppose $\frac{v}{s}$ is an element of T_Q such that $v \in T$ and $s \in T - Q$. Then $tv \in R$ and $ts \in R - P$, so $\frac{v}{s} = \frac{tv}{ts} \in R_P$. Thus, we have proved that $R_P = T_Q$. But T_Q is a discrete valuation domain since T is a Dedekind domain. Therefore $R_P = T_Q$ is a principal ideal domain. In this case, J_P is also a principal ideal of R_P. □

Remark. If the ideal $(R :_K T) = 0$, then the conclusion of Lemma 1.8 will become trivial. The following proposition shows that this will not happen when T is a finitely generated R-module.

Proposition 1.9. *Let R be a one-dimensional Noetherian domain with quotient field K and let T be an overring of R. Then $(R :_K T) \neq 0$ if and only if T is a finitely generated R-module.*

Proof. If T is a finitely generated R-module, then it can be assumed that T is generated by $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_n}{a_n}$ where a_i's and b_i's are elements of R. Let $s = a_1 a_2 \cdots a_n$. We have $s \neq 0$ and $s \in (R :_K T)$. This means that $(R :_K T) \neq 0$.

On the other hand, if $(R :_K T) \neq 0$, then there is some $b \neq 0$ and $b \in (R :_K T)$. Thus bT is an ideal of R. Since R is Noetherian, bT is in fact a finitely generated R-module. Furthermore, by the Krull-Akizuki theorem, T/bT is also a finitely generated R-module. Thus, it can be seen from the short exact sequence

$$0 \longrightarrow bT \longrightarrow T \longrightarrow T/bT \longrightarrow 0$$

that T is also a finitely generated R-module. □

Theorem 1.10. *Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. Denote the ideal $(R :_K T)$ by I. Then R is a G-Dedekind domain if and only if for any prime ideal P of R which contains I, P is G-projective.*

Proof. If R is a G-Dedekind domain, P is surely G-projective. For the "if" part, by Proposition 1.7, it will suffice to prove that every prime ideal of R is G-projective. If P is any prime ideal which doesn't contain $I = (R :_K T)$, then by Lemma 1.8, P is invertible and therefore G-projective. If P is any prime ideal which contains I, then P is G-projective from the hypothesis. □

Corollary 1.11. *Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. If $(R :_K T)$ is a G-projective prime ideal of R, then R is a G-Dedekind domain.*

Example 1.12. Let R be the ring of $\mathbb{Q}+X^2\mathbb{Q}[X]$ where $\mathbb{Q}$ is the field of rational numbers and X an indeterminate. We prove that the ideal $I = (X^2, X^3)$ is G-projective. Consider the short exact sequence

$$(**) \qquad 0 \longrightarrow I \xrightarrow{\alpha} R \oplus R \xrightarrow{\beta} I \longrightarrow 0$$

where $\alpha(i) = (iX, -i)$ and $\beta((r_1, r_2)) = r_1X^2 + r_2X^3$.

In the following, we will prove that $\mathrm{Ext}^1_R(I, R) = 0$ and this will show from [5, Proposition 2.9] that I is G-projective (in fact it is strongly G-projective).

Applying the functor $\mathrm{Hom}(-, R)$ to the short exact sequence $(**)$, the following exact sequence is obtained:

$$0 \longrightarrow \mathrm{Hom}(I, R) \xrightarrow{\beta^*} \mathrm{Hom}(R \oplus R, R) \xrightarrow{\alpha^*} \mathrm{Hom}(I, R) \longrightarrow \mathrm{Ext}^1_R(I, R) \longrightarrow 0$$

Note that $\mathrm{Ext}^1_R(I, R) = 0$ is equivalent to the fact that α^* is a surjective map. In other words, if for any homomorphism ψ from I to R, there exists a homomorphism θ from $R \oplus R$ to R such that $\psi = \theta\alpha$, then $\mathrm{Ext}^1_R(I, R) = 0$. Since $\mathrm{Hom}(I, R) \cong I^{-1} = \mathbb{Q}[X]$, we can assume that $\psi(i) = if(X) = iq_1X + ig(X)$ for some $f(X) = q_1X + g(X) \in \mathbb{Q}[X]$ and $g(X) \in R$. Now, let $\theta(r_1, r_2) = q_1r_1 - r_2g(X)$. Then

$$\theta\alpha(i) = \theta((iX, -i)) = q_1iX + ig(X) = iq_1X + ig(X) = if(X) = \psi(i).$$

Therefore $\mathrm{Ext}^1_R(I, R) = 0$.

Note that the integral closure of R in its field of quotients is $\mathbb{Q}[X]$ and $I = (R : \mathbb{Q}[X])$. We also notice that I is a prime ideal of R. Therefore, by Corollary 1.11, R is in fact a G-Dedekind domain. Since R is not integrally closed, R is not a Dedekind domain.

2. Overrings of G-Dedekind domains

It is well known that any overring of a Dedekind domain is still a Dedekind domain. It is natural to ask whether overrings of a G-Dedekind domain are G-Dedekind domains or not? The following example shows that this is not necessarily the case.

Example 2.1. Let $D = \mathbb{Q}[y, z]$, where y, z are two indeterminates and $\mathbb{Q}$ is the field of rational numbers. Then, the ring

$$R = \{f \in \mathbb{Q}[x] \mid f = q_0+q_3X^3+q_4X^4+q_6X^6+\cdots+q_kX^k, q_i \in \mathbb{Q}\} \cong D/(y^4-z^3)$$

is a G-Dedekind domain. The elements in R are those polynomials whose terms of X, X^2 and X^5 are zero. To see that R is a G-Dedekind domain, we just need to notice that $gl.\dim(D) = 2$ and R is not a QF-ring. Then, an application of [13, Corollary 2.6] gives the result. But the ring $\mathbb{Q} + X^3\mathbb{Q}[X]$ is an overring of R which is not a G-Dedekind domain. From this fact, we later show that R is not a Warfield domain.

In order to give some definitions, we need to consider the following evaluation map:

$$\Phi_M : M \longrightarrow M^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$$

where M is an R-module and R is a domain. The following definitions can be found in [19].

If the evaluation map Φ_M is injective, then M is said to be a *torsionless* module. If the evaluation map Φ_M is bijective, then M is said to be *reflexive*; R is called *reflexive* if all torsionless R-modules M of finite rank are reflexive; R is called *divisorial* if all the ideals (i.e., torsionless modules of rank 1) of R are reflexive; R is said to be *totally divisorial* if all the overrings of R are divisorial; R is said to be *totally reflexive* if all the overrings of R are reflexive.

The following theorem is [19, Theorem 3.1]. From this, it can be seen that a Noetherian domain is divisorial if and only if it is reflexive.

Theorem 2.2. *For a Noetherian domain R with quotient field K, the following conditions are equivalent*:

(1) *R is divisorial*;

(2) *K/R is an injective R-module*;

(3) *R has Krull dimension 1 and $(R : P)$ can be generated by 2 elements, for every maximal ideal P of R*;

(4) *R is reflexive.*

Since the statement (2) is equivalent to saying that R is a G-Dedekind domain for a Noetherian domain R, we also have that a Noetherian domain is divisorial if and only if it is a G-Dedekind domain. The following two theorems are [19, Theorem 6.3] and [19, Theorem 6.4], respectively.

Theorem 2.3. *A domain R is a Warfield domain if and only if it is totally reflexive.*

Theorem 2.4. *For a Noetherian domain R the following conditions are equivalent*:

(1) *R is a Warfield domain.*

(2) *all the ideals of R are 2-generated.*

If a one-dimensional Noetherian domain R is a Warfield domain, then any overring of R must be divisorial since R is totally reflexive. Furthermore, by the Krull-Akizuki theorem, any overring of R is also Noetherian. Therefore, any overring of R must be a G-Dedekind domain. Thus the ring we give at the beginning of this section is not a Warfield domain. On the other hand, if any overring of a G-Dedekind domain R is still a G-Dedekind domain, then R must be totally divisorial and further, totally reflexive since it is a Noetherian domain. Therefore, by the above two theorems or [3, Theorem 7.3], R must be a Warfield domain and all the ideals of R are 2-generated (the definition of a Warfield domain can be seen in [19]). Next, we will prove that the G-Dedekind domain $\mathbb{Q}+X^2\mathbb{Q}[X]$ is a Warfield domain. From this example and the example before, the following relations can be given:

Dedekind domain $\subsetneq$ Noetherian Warfield domain $\subsetneq$ G-Dedekind domain.

2.1. Ideals of the ring $\mathbb{Q} + X^2\mathbb{Q}[X]$

Proposition 2.5. *Let f and g be two elements of the ring $R = \mathbb{Q} + X^2\mathbb{Q}[X]$. If $f\mathbb{Q}[X] + g\mathbb{Q}[X] = \mathbb{Q}[X]$, then $fR + gR = R$.*

Proof. Suppose $fu + gv = 1$ where $u, v \in \mathbb{Q}[X]$. Then, we can assume that $u = u' + t_1X$ and $v = v' + t_2X$ where $u', v' \in R$ and $t_1, t_2 \in \mathbb{Q}$. So,

$$fu + gv = 1 \Rightarrow fu' + gv' = 1 - X(t_1f + t_2g).$$

Since $fu' + gv' \in R$, $1 - X(t_1f + t_2g) \in R$. So $1 + X(t_1f + t_2g) \in R$. Let $h = 1 + X(t_1f + t_2g)$. Then $fu'h + gv'h = 1 - X^2(t_1f + t_2g)^2$. A simple observation will give that $fR + gR = R$. □

Proposition 2.6. *Let J be an ideal of the ring $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ such that $J + I = R$ where $I = (X^2, X^3)$. Then J is invertible and can be factored uniquely into a product of prime ideals.*

Proof. Notice that $I = (R : \mathbb{Q}[X])$ and $\mathbb{Q}[X]$ is the integral closure of R. An application of Lemma 1.8 will give that J is invertible. For the second claim, we will prove that any ideal J with the property that $J+I = R$ can be factored uniquely into a product of prime ideals.

Suppose the set $\mathbb{M}$ of those ideals which do not have factorization is not empty. As R is Noetherian, $\mathbb{M}$ admits a maximal element L. This can not be a prime ideal and hence must be contained in a maximal ideal P. Since $L + I = R$, we have $P + I = R$ and P is invertible by Lemma 1.8. Thus P is divisorial. So we have $P^{-1} \supsetneq R$ and $L \neq P$. We have the following: $L \subseteq LP^{-1} \subseteq PP^{-1} = R$. From the invertibility of L and P we get that $LP^{-1} \neq L$ and $LP^{-1} \neq PP^{-1}$. Since LP^{-1} properly contains L, LP^{-1} can be uniquely factored and so L can be factored too. This contradiction shows that $\mathbb{M}$ is empty. Therefore every ideal which is prime to I can be factored into a product of prime ideals. Notice that prime factors are also relatively prime to I, and so we get the invertibility of these factors and the uniqueness of the factorization. □

Remark. Let R be a one-dimensional Noetherian domain with T its integral closure in the quotient field K. It can be seen from the above proof that, if J is an ideal of R such that $J + (R :_K T) = R$, then J is invertible and can be factored uniquely into a product of some prime ideals.

Example 2.7. This example is to show that invertible ideals of $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ are not necessarily principal. Let $J = (1 + X^2 + X^4, 1 + X^3)$. Then $J + I = R$ since $I = (X^2, X^3)$ is a maximal ideal of R and J is not contained in I. So, by Proposition 2.6, J is invertible. Since

$$\begin{aligned} J\mathbb{Q}[X] &= (1 + X^2 + X^4)\mathbb{Q}[X] + (1 + X^3)\mathbb{Q}[X] \\ &= (1 - X + X^2)((1 + X + X^2)\mathbb{Q}[X] + (1 + X)\mathbb{Q}[X]) \\ &= (1 - X + X^2)\mathbb{Q}[X], \end{aligned}$$

we have that $J \neq R$. Next, we show that J is not a principal ideal.

First, we prove that $1+X^2+X^4$ is an irreducible elements of R. To see this fact, we just need to notice that $1 + X^2 + X^4 = (1 - X + X^2)(1 + X + X^2)$ and the ring $\mathbb{Q}[X]$ is a UFD (uniquely factorization domain).

Second, we show that $1 + X^3$ is not in the ideal $(1 + X^2 + X^4)$. Otherwise, $1 + X^3 = (1 + X^2 + X^4)g(X) \Rightarrow 1 + X = (1 + X + X^2)g(X)$. But this is impossible since $1 + X$ and $1 + X + X^2$ are irreducible elements of the UFD $\mathbb{Q}[X]$.

If J is principal, then $J = (1 + X^2 + X^4)$ since $1 + X^2 + X^4$ is irreducible. But the second step shows that this can not happen.

Next, we will prove that every ideal of the ring $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ can be generated by two elements. This means that R is a Warfield domain.

Lemma 2.8. *Let $R' = \mathbb{Q}[X^2]$. Then R' is a principal ideal domain and $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ is a free R'-module of rank* 2.

Proof. Since $R' = \mathbb{Q}[X^2] \cong \mathbb{Q}[Y]$ where Y is an indeterminate and $\mathbb{Q}[Y]$ is a principal ideal domain, R' is also a principal ideal domain. For the second claim, we can find a basis of the R'-module R: 1 and X^3. Let f be any element of R. Then, the odd degree of X in terms of f must be larger than or equal to 3. So, $f = r'_1 + r'_2X^3$ for some $r'_1, r'_2 \in R'$. The uniqueness of this expression can be explained as follows: $f = 0$ if and only if both the sum of even degree terms and the sum of odd degree terms of f are zero. □

Theorem 2.9. *Every ideal of the ring $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ can be generated by two elements.*

Proof. Let $R' = \mathbb{Q}[X^2]$. Then, every ideal of the ring $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ is in fact an R'-module. By Lemma 2.8, $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ is a free R'-module of rank 2. Therefore, every ideal of R is in fact a submodule a free R'-module of rank 2. Again by Lemma 2.8, $R' = \mathbb{Q}[X^2]$ is a principal ideal domain. So any ideal J of R is also a free R'-module and the rank of J is at most 2. Since, as an R'-module, any ideal J of R can be generated by two elements, J can also be generated by two elements as an R-module. □

2.2. Some properties of Noetherian Warfield domains

In what follows, we will give a direct proof to show that the Krull dimension of any Noetherian Warfield domain is at most 1 and prove that any ideal of a Noetherian Warfield domain is 2-strongly Gorenstein projective.

Proposition 2.10. *Let R be a Noetherian local ring and M be its maximal ideal. If both M and M^2 can be generated by two elements, then the Krull dimension of R is at most* 1.

Proof. We prove that M is the minimal prime ideal over some principal ideal of R. Suppose that $M=(x_1,x_2)$, then $M^2=(x_1^2,x_1x_2,x_2^2)$. Since M^2 can be generated by two elements, the dimension of the vector space M^2/M^3 over the field R/M is at most two. Therefore, there exist $\bar{a},\bar{b},\bar{c}\in R/M$, not all of them be zero, such that $\bar{a}\bar{x_1}^2+\bar{b}\bar{x_1}\bar{x_2}+\bar{c}\bar{x_2}^2=0$ where $\bar{x_i}\bar{x_j}\in M^2/M^3$. This means that $ax_1^2+bx_1x_2+cx_2^2\in M^3$. Next we will prove that there exist two elements $\bar{a_1},\bar{a_2}\in R/M$ such that $\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}\bar{a_2}+\bar{c}\bar{a_2}^2\neq 0$.

Case 1: $|R/M|=2$. In this case, the symbol (u,v,w) means an ordered triple where $u,v,w\in R/M$. Considering the symmetry, we only need to discuss the following five situations:

$$(\bar{a},\bar{b},\bar{c})=(1,0,0);(0,1,0);(1,1,0);(1,0,1);(1,1,1).$$

If $(\bar{a},\bar{b},\bar{c})=(1,0,0)$ or $(1,1,0)$ or $(1,0,1)$, then we let $\bar{a_1}=1,\bar{a_2}=0$ and get that $\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}\bar{a_2}+\bar{c}\bar{a_2}^2=1\neq 0$. If $(\bar{a},\bar{b},\bar{c})=(0,1,0)$ or $(1,1,1)$, then we let $\bar{a_1}=1,\bar{a_2}=1$ and get that $\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}\bar{a_2}+\bar{c}\bar{a_2}^2=1\neq 0$.

Case 2: $|R/M|>2$. Let $\bar{a_2}=1$. Then $\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}\bar{a_2}+\bar{c}\bar{a_2}^2=\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}+\bar{c}$. Note that the quadratic equation $\bar{a}y^2+\bar{b}y+\bar{c}=0$ has at most two roots in R/M. So if we let $\bar{a_1}$ be an element other than these roots, we have that $\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}\bar{a_2}+\bar{c}\bar{a_2}^2\neq 0$.

Without loss of generality, we can assume that $\bar{a_2}=\bar{1}$. So, we have $\bar{a}\bar{a_1}^2+\bar{b}\bar{a_1}+\bar{c}\neq 0$. This means that $aa_1^2+ba_1+c$ is a unit of R. Let $x=x_1-a_1x_2$. We have

$$ax_1^2+bx_1x_2+cx_2^2=a(x+a_1x_2)^2+b(x+a_1x_2)x_2+cx_2^2=(aa_1^2+ba_1+c)x_2^2+rx$$

for some $r\in R$. This means that $x_2^2\in Rx+M^3$. Since

$$M^2=(x_1^2,x_1x_2,x_2^2)=((x+a_1x_2)^2,(x+a_1x_2)x_2,x_2^2)\subset Rx+Rx_2^2,$$

we have $M^2\subset Rx+M^3$, and hence $M^2\subset Rx+M^k$ for all integers $k\geqslant 3$. Therefore $M^2\subset\bigcap_{i=3}^{\infty}(Rx+M^i)=Rx$ by [15, p. 55, Exercise 7(d)]. Since some power of M is inside Rx, M must be the minimal prime ideal over the principal ideal Rx. Therefore, by [20, Theorem 6.6.2], the Krull dimension of R is at most 1. □

Definition 2.11 ([6, Definition 2.1])**.** A module M is said to be *n-strongly Gorenstein projective* (n-SG-projective), if there exists an exact sequence of modules $0\longrightarrow M\longrightarrow P_n\longrightarrow\cdots\longrightarrow P_1\longrightarrow M\longrightarrow 0$, where each P_i is projective such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective module.

Lemma 2.12. *Let R be a domain. If $I = (a, b)$ and $J = (c, d)$ are two fractional ideals of R such that $\frac{b}{a} = \frac{d}{c}$, then $I \cong J$.*

Proof. Denote $\frac{b}{a} = \frac{d}{c}$ by α. Then, $I = a(1, \alpha) \cong (1, \alpha) \cong c(1, \alpha) = J$. □

Lemma 2.13. *Let R be a domain. If $I = (a, b)$ is an ideal of R which is not principal, then there exists a short exact sequence*

$$0 \longrightarrow I^{-1} \longrightarrow R \oplus R \longrightarrow I \longrightarrow 0.$$

Proof. Since I is not principal, neither $\frac{a}{b}$ nor $\frac{b}{a}$ is inside R. So, by Lemma 2.12, we can assume that $I \cong (1, \beta)$ where $(1, \beta) \supset R$ is a fractional ideal of R. Since $(1, \beta) \supset R$, we have $(1, \beta)^{-1} \subset R$ is an ideal of R. Denote $(1, \beta)$ by J. Then $J^{-1} = \{x \in K \mid x(1, \beta) \in R\} = \{x \in R \mid x\beta \in R\}$. Now, we can construct the following sequence

$$0 \longrightarrow J^{-1} \xrightarrow{f} R \oplus R \xrightarrow{g} J \longrightarrow 0$$

where $f(x) = (x\beta, -x), x \in J^{-1}$ and $g((r_1, r_2)) = r_1 + r_2\beta, (r_1, r_2) \in R \oplus R$. The injectivity of f and the surjectivity of g are obvious. That $\mathrm{Im}f \subset \ker g$ can be seen from the construction. In order to prove the exactness of this sequence, we only need to confirm that $\ker g \subset \mathrm{Im}f$. Suppose $g((r_1, r_2)) = r_1 + r_2\beta = 0$. Then $r_2\beta = -r_1 \in R$ and $r_2 \in R$ and this means that $r_2 \in (1, \beta)^{-1} = J^{-1}$. Thus $(r_1, r_2) = (-r_2\beta, r_2) = f(-r_2) \in \mathrm{Im}f$.

Finally, since $I \cong J$, it can be seen that $I^{-1} \cong \mathrm{Hom}(I, R) \cong \mathrm{Hom}(J, R) \cong J^{-1}$. Thus, we have a short exact sequence

$$0 \longrightarrow I^{-1} \longrightarrow R \oplus R \longrightarrow I \longrightarrow 0.$$ □

Corollary 2.14. *Let R be a G-Dedekind domain and let I be an ideal of R which is generated by two elements. If $I^{-1} \cong I$, then I is strongly G-projective.*

Proof. If I is principal, it is surely strongly G-projective. In the case I is not principal, just apply Lemma 2.13. □

Example 2.15. Consider the ideal $I = (X^2, X^3)$ of the ring $R = \mathbb{Q} + X^2\mathbb{Q}[X]$. Since $I^{-1} = \mathbb{Q}[X] = (1, X)$, $I^{-1} \cong I$. By Corollary 2.14, I is strongly G-projective. Similarly, ideals of the form (X^n, X^{n+1}) $(n \geqslant 2)$ are also strongly G-projective.

Lemma 2.16. *Let R be a domain. If I is a divisorial ideal of R such that both I and I^{-1} can be generated by two elements, then I is G-projective.*

Proof. If I is principal, it is certainly G-projective. So, it can be assumed that I is not principal. By Lemma 2.13, there exists the following exact sequence:

$$0 \longrightarrow I^{-1} \xrightarrow{f} R \oplus R \xrightarrow{g} I \longrightarrow 0.$$

Without loss of generality, we can assume that $I = (1, a)$ and $f(x) = (xa, -x)$, $x \in I^{-1}$ and $g((r_1, r_2)) = r_1 + r_2 a, (r_1, r_2) \in R \oplus R$. Next, we prove that $\mathrm{Ext}^1_R(I, R) = 0$. Let α be any homomorphism from I^{-1} to R. If we can find a homomorphism β from $R \oplus R$ to R such that $\alpha = \beta f$, then the homomorphism $\mathrm{Hom}(R \oplus R, R) \longrightarrow \mathrm{Hom}(I^{-1}, R)$ is surjective and it can be seen from the long exact sequence

$$0 \to \mathrm{Hom}(I, R) \to \mathrm{Hom}(R \oplus R, R) \to \mathrm{Hom}(I^{-1}, R) \to \mathrm{Ext}^1_R(I, R) \to 0$$

that $\mathrm{Ext}^1_R(I, R) = 0$. Since $I = I_v \cong \mathrm{Hom}(I^{-1}, R)$, we can assume that $\alpha(x) = x(r'_1 + r'_2 a)$ for some $r'_1, r'_2 \in R$. Let $\beta((r_1, r_2)) = r_1 r'_2 - r_2 r'_1$. Then $\alpha(x) = x(r'_1 + r'_2 a) = r'_1 x + r'_2 a x = \beta((xa, -x)) = \beta f(x)$. Similarly, we also have a short exact sequence

$$0 \longrightarrow I \xrightarrow{f} R \oplus R \xrightarrow{g} I^{-1} \longrightarrow 0$$

and $\mathrm{Ext}^1_R(I^{-1}, R) = 0$. Combining these two short exact sequences, we get an exact sequence:

$$0 \longrightarrow I \longrightarrow R \oplus R \longrightarrow R \oplus R \xrightarrow{g} I \longrightarrow 0.$$

Note that the kernel of g is isomorphic to I^{-1}. Since $\mathrm{Ext}^1_R(I^{-1}, R) = 0$ and $\mathrm{Ext}^1_R(I, R) = 0$, I must be G-projective (in fact, it is 2-SG-projective). □

Corollary 2.17. *Let R be a domain in which every ideal can be generated by two elements. Then every divisorial ideal is G-projective.*

Remark. By Proposition 2.10, if R is a ring in which every ideal can be generated by two elements, the Krull dimension of R is at most 1. In fact, it had been proved in [8] that, if there exists a fixed upper bound for the number of generating set of any ideal of R, then R is Noetherian and the Krull dimension of R is at most 1. Therefore, if R is a domain in which every ideal can be generated by two elements, then, by [20, Theorem 8.1.5], every nonzero prime ideal of R is divisorial since it is minimal over some principal ideal. Thus, by Corollary 2.17, every prime ideal of R is G-projective. Furthermore, by Proposition 1.7, R is a G-Dedekind domain.

Lemma 2.18. *Let R be a G-Dedekind domain and let I be a nonzero ideal of R. Then I is principal if and only if I^{-1} is principal.*

Proof. If I^{-1} is principal, then $I^{-1} \cong R$. So $I = (I^{-1})^{-1} \cong \mathrm{Hom}(I^{-1}, R) \cong \mathrm{Hom}(R, R) \cong R$. This means that I must be principal. On the other hand, if $I = (x)$ is principal, then $I^{-1} = (x^{-1})$ is principal. □

Lemma 2.19. *Let R be a Noetherian Warfield domain and let $I = (a, b)$ be a fractional ideal of R which is not principal. Then, there exists a short exact sequence*

$$0 \longrightarrow I \longrightarrow R \oplus R \longrightarrow I^{-1} \longrightarrow 0.$$

Proof. Without loss of generality, we can assume that $I \subset R$ is an ideal of R. Since I is not principal, by Lemma 2.18, neither is I^{-1} principal. We can further assume that $I^{-1} \cong (1, \beta)$ where $(1, \beta) \supset R$ is a fractional ideal of R. By noticing that I is divisorial, the rest of the proof is just as that of Theorem 2.13. □

Theorem 2.20. *Let R be a Noetherian Warfield domain. Then every ideal of R is 2-SG-projective.*

Proof. Let I be an ideal of R. If I is principal, it is certainly projective and therefore 2-strongly G-projective. So, we can assume that I is not principal. In this case, neither I nor I^{-1} is principal by Lemma 2.18. Further, by Lemma 2.19, we have two exact sequences

$$0 \longrightarrow I \longrightarrow R \oplus R \longrightarrow I^{-1} \longrightarrow 0$$

and

$$0 \longrightarrow I^{-1} \longrightarrow R \oplus R \longrightarrow I \longrightarrow 0.$$

Combining these two exact sequences, we get an exact sequence:

$$(****) \qquad 0 \longrightarrow I \longrightarrow R \oplus R \longrightarrow R \oplus R \longrightarrow I \longrightarrow 0.$$

Since R is already a G-Dedekind domain, every ideal of R is G-projective and we have $\mathrm{Ext}^1_R(I, Q) = 0$ and $\mathrm{Ext}^1_R(I^{-1}, Q) = 0$. Thus, $\mathrm{Hom}(-, Q)$ leaves the sequence $(****)$ exact whenever Q is a projective module. Therefore, I is 2-SG-projective. □

2.3. 2-DVRs and Noetherian Warfield domains

In this subsection, we will give a characterization of Noetherian Warfield domains. For this purpose, we give a definition for a special kind of Noetherian local domain.

Definition 2.21. Let R be a Noetherian local domain. Then R is called a 2-DVR if the maximal ideal of R is strongly G-projective.

Remark. Comparing with the definition of a DVR (discrete valuation ring), a 2-DVR is a generalization of a DVR. A domain R is a Dedekind domain if and only if R_M is a DVR for any maximal ideal M. In the following, we will prove that a Noetherian domain R is a Warfield domain if and only if R_M is a 2-DVR for any maximal ideal M of R. In order to give a characterization of a 2-DVR, we need the following three lemmas. Some tricks are adopted from [17].

Lemma 2.22. *Let R be a domain with quotient field K and I be a nonzero ideal of R. If the ring $R_1 = (I :_K I)$ is divisorial, then I is a projective R_1-module.*

Proof. For any ideal J of R_1, we define $J^{-1_1} = \{x \in K \mid xJ \subset R_1\}$. If we can prove that $II^{-1_1} = R_1$, then I is a projective R_1-module. Suppose $II^{-1_1} \neq R_1$. Thus there exists a maximal ideal P of R_1 such that $II^{-1_1} \subset P$. Hence

$$P^{-1_1} II^{-1_1} \subset P^{-1_1} P \subset R_1,$$

and therefore $P^{-1_1}I \subset (I^{-1_1})^{-1_1}$. Since R_1 is divisorial, we have $(I^{-1_1})^{-1_1} = I$. Thus $P^{-1_1} \subset R_1$, and it follows that $P^{-1_1} = R_1$. But R_1 is divisorial, and therefore $(P^{-1_1})^{-1_1} = P$. This contradiction shows that I is a projective R_1-module. □

Lemma 2.23. *Let R be a Noetherian domain with quotient field K and I be a nonzero ideal of R. Then the ring $R_1 = (I :_K I)$ is a finitely generated R-module.*

Proof. Take a nonzero element $a \in I$. Then aR_1 is a nonzero ideal of R which is contained in I. Since R is Noetherian, aR_1 is a finitely generated R-module. Thus R_1 is also a finitely generated R-module. □

Lemma 2.24. *Let R be a local Noetherian domain with quotient field K and I be an ideal of R. If the ring $R_1 = (I :_K I)$ is divisorial, then R_1 is isomorphic to I as R-modules.*

Proof. Since, by Lemma 2.23, R_1 is a finitely generated module over the local domain R, R_1 is a semi-local domain, and finitely generated projective R_1-modules are free by [9, Chapter VI, Theorem 1.11]. Because, by Lemma 2.22, I is a projective R_1-module, I must be a principal ideal of R_1. This means that $I = R_1 a$ for some $a \in I$. Therefore $I \cong R_1$ as R-modules. □

Theorem 2.25. *Let R be a local Noetherian domain with quotient field K and M be the maximal ideal of R. The following statements are equivalent*:

(1) *R is a 2-DVR.*

(2) *Every ideal of R can be generated by two elements.*

(3) *M can be generated by two elements and both R and $R_1 = (M :_K M)$ are divisorial domains.*

Proof. (1) ⇒ (2) If M is projective, then M must be principal since R is local. This means that every ideal of R is also principal. So we can assume that M is not projective. Denote the ring $(M :_K M)$ by R_1. Then $M^{-1} = R_1$.

First, we prove that, as an R_1-ideal, M is principal. Since M is a strongly G-projective R-module, there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0.$$

Since R is a local ring, the middle projective module P of rank 2 is a free module by [1, Corollary 26.7]. Thus we surely have a short exact sequence

$$0 \longrightarrow M \longrightarrow R \oplus R \longrightarrow M \longrightarrow 0.$$

It can be seen from this short exact sequence that M is generated by two elements. Thus, by Lemma 2.13, there also exists the following short exact sequence

$$0 \longrightarrow M^{-1} \longrightarrow R \oplus R \longrightarrow M \longrightarrow 0.$$

Therefore, there exists the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{-1} & \longrightarrow & R \oplus R & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & R \oplus R & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

So considering the mapping cone complex or by Schanuel's Lemma, we have $M^{-1} \oplus R \oplus R \cong M \oplus R \oplus R$. Since R is a local ring, by [20, Theorem 2.10.10], $M^{-1} \cong M$. Let $\psi : M^{-1} \to M$ be the isomorphism. We will prove that $M = M^{-1}\psi(1)$ ($1 \in M^{-1}$ follows from the fact that $M^{-1} = (M :_K M)$ is a ring). It is obvious that $M^{-1}\psi(1) \subset M^{-1}M = M$. Let $x \in M$ be any element of M. Then $x = \psi(\frac{r_2}{r_1})$ for some $\frac{r_2}{r_1} \in M^{-1}$. It can be seen that $x = \frac{r_2}{r_1}\psi(1)$. So $M \subset M^{-1}\psi(1)$. This means that $M = M^{-1}\psi(1) = R_1\psi(1)$ is a principal ideal of R_1.

We have just shown that there exists an element $a \in M$ such that $M = R_1 a$. Therefore we have the relations

$$M^2 = Ma \subsetneq Ra \subsetneq M \subsetneq R.$$

Thus a is a system of parameters, and so $\dim(R) = 1$ by [18, Theorem 14.1].

Note that R is a one-dimensional Noetherian domain. Thus if I is a nonzero ideal of R, R/I has finite length, which we shall denote by $l(R/I)$. Since M is generated by two elements, $l(M/M^2) = 2$. Thus $l(R/M^2) = 3$. It can be seen from above relations that $l(R/Ra) = 2$.

Next we prove that any non-principal ideal I of R is also an ideal of R_1. It can be seen by Proposition 1.7 that R is a G-Dedekind domain and therefore every ideal of R is divisorial. Since I is not invertible, $II^{-1} \subset M$. Therefore

$$R_1 II^{-1} \subset R_1 M = M.$$

Hence $R_1 I \subset (I^{-1})^{-1} = I$.

Let I be a non-principal ideal of R. Then I is also an ideal of R_1 and $MI = R_1 aI = aI$. Hence $Ra/MI = Ra/Ia \cong R/I$. Considering the following two short exact sequences

$$0 \longrightarrow Ra/MI \longrightarrow R/MI \longrightarrow R/Ra \longrightarrow 0$$

and

$$0 \longrightarrow I/MI \longrightarrow R/MI \longrightarrow R/I \longrightarrow 0,$$

it can be seen that $l(I/MI) = l(R/Ra) = 2$. This means that I can be generated by two elements.

(2) $\Rightarrow$ (3) By [19, Theorem 6.4], R is a Noetherian Warfield domain.

(3) $\Rightarrow$ (1) If M is invertible, then M is projective and therefore surely strongly G-projective. So we can assume that M is not invertible. Since $M \subset MM^{-1} \subset R$ and M is the maximal ideal, $M = MM^{-1}$ and $M^{-1} = (M :_K M)$. Thus, by Lemma 2.24, $M^{-1} \cong M$. Therefore, by Corollary 2.14, M is a strongly G-projective R-module. $\square$

If M is an R-module, we let $\mu_R(M)$ denote the least number of elements required to generate M, and $\mu^*(R) = \sup\{\mu_R(I)\}$ where I ranges over all finitely generated ideals of R.

Lemma 2.26 ([2, Proposition 1.4])**.** *If R is a Noetherian integral domain for which $\mu^*(R_M) \leqslant k$ for all maximal ideals M, then* $\dim(R) \leqslant 1$ *and* $\mu^*(R) \leqslant \max\{2, k\}$.

Theorem 2.27. *Let R be a Noetherian domain. Then R is a Warfield domain if and only if, for any maximal ideal M of R, R_M is a 2-DVR.*

Proof. First, we assume that R is a Noetherian Warfield domain and M is any maximal ideal of R. Then R_M is also a Noetherian Warfield domain. So every ideal of R_M can be generated by two elements. Thus, by Theorem 2.25, R_M is a 2-DVR.

Second, we assume that R_M is a 2-DVR for any maximal ideal M of R. Then by Lemma 2.25, $\mu^*(R_M) \leqslant 2$ for all maximal ideals M. Thus, by Lemma 2.26, every ideal of R can be generated by two elements. Therefore R is a Warfield domain. □

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