

ON A GENERALIZATION OF MCCOY RINGS

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ABSTRACT. Rege-Chhawchharia, and Nielsen introduced the concept of right McCoy ring, based on the McCoy's theorem in 1942 for the annihilators in polynomial rings over commutative rings. In the present note we concentrate on a natural generalization of a right McCoy ring that is called a *right nilpotent coefficient McCoy ring* (simply, a *right NC-McCoy ring*). The structure and several kinds of extensions of right NC-McCoy rings are investigated, and the structure of minimal right NC-McCoy rings is also examined.

Throughout this paper R denotes an associative ring with identity unless otherwise stated. Let $N(R)$ be the set of all nilpotent elements in R . We use $R[x]$ to denote the polynomial ring with an indeterminate x over R . Let $C_{f(x)}$ denote the set of all coefficients of $f(x) \in R[x]$. Denote the n by n full matrix ring over R by $\text{Mat}_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$. Use E_{ij} for the matrix with (i, j) -entry 1 and elsewhere 0. By \mathbb{Z}_n we mean the ring of integers modulo n .

McCoy [27] showed that if two polynomials annihilate each other over a commutative ring, then each polynomial has a nonzero annihilator in the base ring. Weiner [16] showed this fact fails in non-commutative rings. Based on this result, Nielsen [29] and Rege-Chhawchharia [30] each called a non-commutative ring R *right McCoy* (resp., *left McCoy*) if whenever any nonzero polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ (resp., $cg(x) = 0$) for some nonzero $c \in R$, and a ring R is called McCoy if it is both left and right McCoy. Rege-Chhawchharia also called R an *Armendariz ring* [30, Definition 1.1] if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for each $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Any reduced ring (i.e., it has no nonzero nilpotent elements) is Armendariz by [4, Lemma 1]. Armendariz rings are clearly McCoy but the converse does not hold by [30, Remark 4.3]. A ring is called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [2, Theorem 6].

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There exist several generalizations of a reduced ring. Cohn [8] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Due to Narbonne [28], a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Nielsen developed and extended the concept of a McCoy ring. In particular, he showed that any reversible ring is McCoy [29, Theorem 2] and gave an example that is a semicommutative ring but not McCoy [29, Section 3]. The concept of a McCoy ring is generalized in [10] to a *weak McCoy* ring, but to have the terminology be more expressive we will call this ring a *nilpotent coefficient McCoy* ring, or an *NC-McCoy* ring for short. In this paper, we study the structure of NC-McCoy rings. Several kinds of extensions of NC-McCoy rings are investigated and some well-known results are extended. The structure of minimal right NC-McCoy rings is also examined.

Let $N_*(R)$ and $N^*(R)$ denote the prime radical and the upper nilradical (i.e., the sum of all nil two-sided ideals) of a ring R , respectively. A generalization of a semicommutative ring is the 2-primal condition. A ring R (possibly without identity) is called *2-primal* [5] if $N_*(R) = N(R)$. Note that a ring R is 2-primal if and only if $R/N_*(R)$ is reduced. In [26], a ring R (possibly without identity) is called *NI* if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms a two-sided ideal if and only if $R/N^*(R)$ is reduced. It is obvious that 2-primal rings are NI, but the converse need not hold by Hwang et al. [15, Example 1.2] or Marks [26, Example 2.2]. But if R is an NI ring of bounded index of nilpotency, then R is 2-primal by [15, Proposition 1.4].

On the other hand, Nielsen gave an example of a semicommutative ring which is not one-sided McCoy [29, Section 3] and proved that for any $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x]$ over a semicommutative ring R , if $f(x)g(x) = 0$, then $a_i b_0^{i+1} = 0$ for all $i \in \{0, 1, \dots, m\}$ [29, Lemma 1]. Moreover we get $0 = a_i b_0 \in N(R)$ for any i since $R/N(R)$ is reduced. Now let us add a condition $g(x) \neq 0$ to [29, Lemma 1]. Then we may assume that $b_0 \neq 0$ and put k be the largest integer among $\{0, 1, \dots, m\}$ such that $b_0^{k+1} \neq 0$, and thus $a_i b_0^{k+1} \in N(R)$ for some $b_0^{k+1} \neq 0$ since $N(R)$ is a two-sided ideal of R . This leads us take the following weak McCoy condition which is a generalization of McCoy condition.

Definition 1. A ring R is called *right nilpotent coefficient McCoy* (simply, *right NC-McCoy*) if for any nonzero polynomials $f(x)$ and $g(x)$ in $R[x]$, $f(x)g(x) = 0$ implies $f(x)c \in N(R)[x]$ for some $0 \neq c \in R$, equivalently, there exists $0 \neq c \in R$ such that $ac \in N(R)$ for any $a \in C_{f(x)}$. Left NC-McCoy rings are defined analogously, and a ring R is called *NC-McCoy* if it is both left and right NC-McCoy ([10, Definition 2.1]).

It is obvious that every right McCoy ring is right NC-McCoy, but not conversely by Example 4 to follow. NI rings are NC-McCoy by [10, Proposition 2.7], but the converse does not hold by Example 4. The concepts of right McCoy rings and NI rings are independent of each other. Over an NI ring A ,

$U_2(A)$ is evidently NI, but not right McCoy by Example 4. While, there exists an Armendariz (hence McCoy) ring that is not NI by [3, Example 4.8].

Recall that an element u of a ring R is *left regular* if $ru = 0$ implies $r = 0$ for $r \in R$. The *right regular* is defined similarly, and *regular* means both left and right regular (hence not a zero divisor). NC-McCoy condition is not left-right symmetric by the following example.

Example 2. (1) Let K be a field and $A = K\langle a_0, b_0, a_1, b_1 \rangle$ be the free K -algebra generated by the noncommuting indeterminates a_0, b_0, a_1, b_1 . Let I be the ideal of A generated by

$$a_0b_0, a_0b_1 + a_1b_0, a_1b_1, b_s b_t$$

for $s, t \in \{0, 1\}$ and let $R = A/I$. We identify a_i and b_j with their images in R for simplicity.

By the construction of R , we have $(a_0 + a_1x)(b_0 + b_1x) = 0$ while $a_0 + a_1x$ and $b_0 + b_1x$ are nonzero polynomials over R . Assume by way of contradiction that there exists $0 \neq \alpha \in R$ such that $a_0\alpha, a_1\alpha \in N(R)$. A computation using the reduced forms for elements in R shows that a_0R and a_1R contain no nonzero idempotents. Thus $a_0\alpha = 0 = a_1\alpha$, which quickly implies $\alpha = 0$, a contradiction. This yields that R is not right NC-McCoy.

Next we show that R is left NC-McCoy. We will use $-a_0b_1$ in place of a_1b_0 when writing monomials in reduced form. Let $f(x)$ and $g(x)$ be nonzero in $R[x]$ with $f(x)g(x) = 0$. Note that $f(x), g(x) \notin K$. We can express $g(x)$ by

$$g(x) = k + a_0g_1(x) + a_1g_2(x) + b_0g_3(x) + b_1g_4(x),$$

where $k \in K$ and $g_i(x) \in R[x]$ for all i . Here we claim

$$g(x) = b_0g_3(x) + b_1g_4(x).$$

To see this, set S be the multiplicative semigroup generated by a_0, a_1, b_0, b_1 . Notice that nonzero monomials in S can be embedded into the set of natural numbers through the corresponding

$$a_0 \rightarrow 1, a_1 \rightarrow 2, b_0 \rightarrow 3, b_1 \rightarrow 4.$$

This corresponding is due to a method in [14, Example 14]. Then S is a totally ordered set with the inequalities $a_0 < a_1 < b_0 < b_1$, only subject to $14 = 23$ (since $a_0b_1 = -a_1b_0$). For example,

$$\begin{aligned} a_0 < a_1 < a_0^2 < a_0a_1 < a_0b_0 < a_0b_1 (= a_1b_0) < a_1a_0 < a_1^2 < a_1b_1 \\ < b_0a_0 < \dots < b_0b_1 < b_1a_0 < \dots < b_1^2 < a_0^3 \end{aligned}$$

because

$$\begin{aligned} 1 < 2 < 11 < 12 < 13 < 14(= 23) < 21 < 22 < 23 \\ < 31 < \dots < 34 < 41 < \dots < 44 < 111. \end{aligned}$$

So $f(x)$ can be expressed by $f(x) = \sum_{i=1}^m k_i(x)h_i$ with $0 \neq k_i(x) \in K[x]$ and $0 \neq h_i \in S$ for all i such that $h_1 < \dots < h_m$ where m is a positive integer.

But $k_1(x)h_1k = k_1(x)kh_1$ is unique in the expansion of $f(x)g(x)$ since h_1 is smallest in the set

$$\{p \in S \mid p \text{ occurs in the coefficients of the expansion of } f(x)g(x)\},$$

and so $k_1(x)h_1k$ must be zero since $f(x)g(x) = 0$. This entails $k = 0$, obtaining $g(x) = a_0g_1(x) + a_1g_2(x) + b_0g_3(x) + b_1g_4(x)$.

Next we can express $g_n(x)$ (for $n = 1, 2, 3, 4$) by $g_n(x) = \sum_{j=1}^{l_n} t(n)_j(x)v(n)_j$ with $t(n)_j(x) \in K[x]$ and $0 \neq v(n)_j \in S$ for all j such that $v(n)_1 < \dots < v(n)_{l_n}$ where l_n 's are positive integers. Note that $h_1a_0v(1)_1$ is smallest in the set

$$\{q \in S \mid q \text{ occurs in the coefficients of the expansion of } f(x)a_0g_1(x)\}.$$

Here letting $h_1 = v$ and $v(n)_j = w_{n_j}$ for simplicity, we have

$$h_1a_0v(1)_1 = v1w_{1_1}, h_1a_1v(2)_j = v2w_{2_j}, h_1b_0v(3)_j = v3w_{3_j}, h_1b_1v(4)_j = v4w_{4_j}.$$

But these are distinct of each other, and hence

$$(k_1(x)h_1)(a_0t(1)_1(x)v(1)_1) = (k_1(x)t(1)_1(x))(h_1a_0v(1)_1)$$

is unique in the expansion of $f(x)g(x)$, and so $a_0t(1)_1(x)v(1)_1$ must be zero since $(k_1(x)h_1)(a_0t(1)_1(x)v(1)_1) \neq 0$ when $a_0(t(1)_1(x)v(1)_1) \neq 0$. Inductively we obtain $a_0t(1)_j(x)v(1)_j = 0$ for $j = 2, \dots, l_1$, entailing $a_0g_1(x) = 0$. We also get $a_1g_2(x) = 0$ through a similar method. These yield

$$g(x) = b_0g_3(x) + b_1g_4(x).$$

Now we have $b_jg(x) = 0 \in N(R)[x]$ for $j = 0, 1$, concluding that R is left NC-McCoy.

(2) Let K be a field and $A = K\langle a_0, b_0, a_1, b_1 \rangle$ be the free K -algebra generated by the noncommuting indeterminates a_0, b_0, a_1, b_1 . Let I be the ideal of A generated by

$$a_0b_0, a_0b_1 + a_1b_0, a_1b_1, a_s a_t$$

for $s, t \in \{0, 1\}$ and $R = A/I$.

Note $(a_0 + a_1x)(b_0 + b_1x) = 0$. Assume that $0 \neq \beta \in R$ such that $\beta(b_0 + b_1x) \in N(R)[x]$. Then we obtain $\beta b_0 = 0$ and $\beta b_1 = 0$ through a similar method to one of (1), noting that βb_j is right regular when $\beta b_j \neq 0$. So β must be zero, a contradiction. Thus R is not left NC-McCoy.

Let $f(x)$ and $g(x)$ be nonzero in $R[x]$ with $f(x)g(x) = 0$. Then we have

$$f(x) = f_1(x)a_0 + f_2(x)a_1 \text{ for some } f_1(x), f_2(x) \in R[x]$$

by a similar method to one of (1). Thus $f(x)a_i = 0 \in N(R)[x]$ for $i = 0, 1$, concluding that R is right NC-McCoy.

In the following note, we find all cases of $f(x)$ and $g(x)$ with $f(x)g(x) = 0$ in Example 2.

Note. (1) We can rewrite $f(x)$ by

$$f(x) = h + f_1(x)a_0 + f_2(x)a_1 + f_3(x)b_0 + f_4(x)b_1,$$

where $h \in K$ and $f_i(x) \in R[x]$ for all i . Then we have

$$\begin{aligned} 0 = f(x)g(x) &= f(x)(b_0g_3(x) + b_1g_4(x)) \\ &= h(b_0g_3(x) + b_1g_4(x)) + f_1(x)a_0b_1g_4(x) + f_2(x)a_1b_0g_3(x). \end{aligned}$$

This entails

$$(hb_0 + f_2(x)a_1b_0)g_3(x) = -(hb_1 + f_1(x)a_0b_1)g_4(x).$$

Assume $h \neq 0$. If $g_3(x) = 0$, then $(hb_1 + f_1(x)a_0b_1)g_4(x) = 0$ and so $g_4(x) = 0$ since $hb_1 + f_1(x)a_0b_1 \neq 0$. This yields $g(x) = 0$, a contradiction. Thus we must have $h = 0$ and $f(x) = f_1(x)a_0 + f_2(x)a_1 + f_3(x)b_0 + f_4(x)b_1$. Then

$$0 = f(x)g(x) = f_1(x)a_0b_1g_4(x) + f_2(x)a_1b_0g_3(x).$$

This equality gives the following cases.

If $f_1(x)a_0 = f_2(x)a_1 = 0$ (i.e., $f(x) = f_3(x)b_0 + f_4(x)b_1$), then $f(x)g(x) = 0$ obviously.

If $f_1(x)a_0 \neq 0$ and $f_2(x)a_1 = 0$, then we must have $g(x) = b_0g_3(x)$ since $f(x)g(x) = f_1(x)a_0b_0g_3(x) + f_1(x)a_0b_1g_4(x) = f_1(x)a_0b_1g_4(x)$.

If $f_1(x)a_0 = 0$ and $f_2(x)a_1 \neq 0$, then we must have $g(x) = b_1g_4(x)$ since $f(x)g(x) = f_2(x)a_1b_0g_3(x) + f_2(x)a_1b_1g_4(x) = f_2(x)a_1b_0g_3(x)$.

Suppose $f_1(x)a_0 \neq 0$ and $f_2(x)a_1 \neq 0$. Then we must have $f_1(x) = f_2(x)$ and $g_3(x) = g_4(x)$ because the right hand side of the preceding equality must be of the form

$$\begin{aligned} 0 = s(x)(a_0b_1 + a_1b_0)t(x) &= s(x)(a_0b_1 + a_1b_0)xt(x) \\ &= s(x)(a_0 + a_1x)(b_0 + b_1x)t(x) \end{aligned}$$

or

$$0 = s(x)(a_0b_1 + a_1b_0)t(x) = s(x)(a_0 + a_1)(b_0 + b_1)t(x)$$

for some $s(x), t(x) \in R[x]$.

Summarizing, $f(x)$ and $g(x)$ have one of the following cases:

$$f(x) = c(x)(a_0 + a_1x) + \sum_{i=0}^1 h_i(x)b_i, \quad g(x) = (b_0 + b_1x)d(x);$$

$$f(x) = h(x)(a_0 + a_1) + \sum_{i=0}^1 h_i(x)b_i, \quad g(x) = (b_0 + b_1)k(x);$$

$$f(x) = \sum_{i=0}^1 h_i(x)b_i, \quad g(x) = \sum_{j=0}^1 b_jk_j(x);$$

$$f(x) = r(x)a_0 + \sum_{i=0}^1 h_i(x)b_i, \quad g(x) = b_0s(x);$$

$$f(x) = u(x)a_1 + \sum_{i=0}^1 h_i(x)b_i, \quad g(x) = b_1v(x),$$

where $c(x), d(x), h(x), h_i(x), k(x), k_j(x), r(x), s(x), u(x), v(x) \in R[x]$.

(2) We have one of the following cases through a similar method to the preceding one:

$$f(x) = c(x)(a_0 + a_1x), \quad g(x) = \sum_{i=0}^1 h_i(x)a_i + (b_0 + b_1x)d(x);$$

$$f(x) = h(x)(a_0 + a_1), \quad g(x) = \sum_{i=0}^1 h_i(x)a_i + (b_0 + b_1)k(x);$$

$$f(x) = \sum_{i=0}^1 h_i(x)a_i, \quad g(x) = \sum_{j=0}^1 a_jk_j(x);$$

$$f(x) = r(x)a_0, \quad g(x) = \sum_{i=0}^1 h_i(x)a_i + b_0s(x);$$

$$f(x) = u(x)a_1, \quad g(x) = \sum_{i=0}^1 h_i(x)a_i + b_1v(x),$$

where $c(x), d(x), h(x), h_i(x), k(x), k_j(x), r(x), s(x), u(x), v(x) \in R[x]$.

The following gives us basic examples of NC-McCoy rings.

Proposition 3. *For a ring R , we have the following:*

(1) *If R contains a nonzero nil one-sided ideal, then R is an NC-McCoy ring.*

(2) *Every ring R with $N^*(R) \neq 0$ is a NC-McCoy ring. Hence, every non-semiprime ring is an NC-McCoy ring.*

(3) *Let R be a ring with a nonzero central nilpotent element. Then $\text{Mat}_n(R)$ is an NC-McCoy ring for $n \geq 2$.*

(4) *$U_n(R)$ is an NC-McCoy ring for $n \geq 2$.*

$$(5) \quad D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\} \text{ is an NC-McCoy ring for}$$

$n \geq 2$.

$$(6) \quad V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, \dots, a_n \in R \right\} \cong R[x]/(x^n) \text{ is an}$$

NC-McCoy ring for $n \geq 2$, where (x^n) is a two-sided ideal of $R[x]$ generated by x^n for $n \geq 2$.

(7) *Let R and S be rings. For a nonzero bimodule ${}_R M_S$, $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is an NC-McCoy ring.*

Proof. (1) The hypothesis is left-right symmetric, and if I is a nil left ideal of R , then c in Definition 1 can be any nonzero element of I . Parts (2) through (7) are all trivial consequence of part (1). \square

Example 4. (1) Let $R = U_2(A)$ over any ring A . Then R is NC-McCoy by Proposition 3(4). For

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} x, \quad g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \in R[x],$$

we have $f(x)g(x) = 0$. But there cannot exist nonzero $c \in R$ such that $f(x)c = 0$, and thus R is not right McCoy.

(2) Let $F\langle X, Y \rangle$ be the free algebra on X, Y over a field F and I denote the ideal $(X^2)^2$ of $F\langle X, Y \rangle$, where (X^2) is the two-sided ideal of $F\langle X, Y \rangle$ generated by X^2 . Consider the ring $R = F\langle X, Y \rangle / I$. Then $0 \neq N_*(R) = N^*(R) \subsetneq N(R)$ by [13, Example 3], showing that R is not an NI ring. However, R is NC-McCoy by Proposition 3(2).

(3) Let R be a ring with a nonzero central nilpotent element. Then $\text{Mat}_n(R)$ ($n \geq 2$) is an NC-McCoy ring by Proposition 3(3). However $\text{Mat}_n(R)$ cannot be an NI ring as can be seen by the two nilpotent matrix units E_{12} and E_{21} .

From Proposition 3, one may conjecture that the $n \times n$ full matrix ring over any ring is NC-McCoy for $n \geq 2$, but the possibility is erased by the following.

Theorem 5. *Let R be a reduced ring. Then $\text{Mat}_n(R)$ is neither right nor left NC-McCoy for $n \geq 2$.*

Proof. Note that $\text{Mat}_n(R)[x] \cong \text{Mat}_n(R[x])$ for $n \geq 2$. Consider nonzero polynomials

$$f(x) = \begin{pmatrix} 1 & x & \cdots & x^{n-1} \\ x^n & x^{n+1} & \cdots & x^{2n-1} \\ \vdots & \vdots & \cdots & \vdots \\ x^{n(n-1)} & x^{n(n-1)+1} & \cdots & x^{n^2-1} \end{pmatrix} = E_{11} + E_{12}x + \cdots + E_{nn}x^{n^2-1}$$

and

$$g(x) = \begin{pmatrix} x & x & \cdots & x \\ -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

in $\text{Mat}_n(R)[x]$ with $f(x)g(x) = 0$. Assume to the contrary that $\text{Mat}_n(R)$ is right NC-McCoy. Then there exists nonzero $C = (c_{ij}) \in \text{Mat}_n(R)$ such that $f(x)C \in N(\text{Mat}_n(R))[x]$, say $(E_{ij}C)^{k_{ij}} = 0$ for any i and j . Put $k = \max\{k_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$. Then $(E_{ij}C)^k = 0$, and so $c_{ji} = 0$ for any i and j by a simple computation, since R is reduced. This implies $C = 0$; which is a contradiction. Thus $\text{Mat}_n(R)$ is not right NC-McCoy. Similarly, we can see that $\text{Mat}_n(R)$ is not left NC-McCoy either. \square

The following example shows that the condition “ R is a reduced ring” in Theorem 5 cannot be weakened to the condition “ R is a semiprime ring”.

Example 6. Let S be a reduced ring. For a positive integer n , put R_n be the $2^n \times 2^n$ upper triangular matrix ring over S . Define a map $\sigma : R_n \rightarrow R_{n+1}$ by $\sigma(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Set $R = \varinjlim R_n$ be the direct limit of the direct system (R_n, σ_{ij}) over $\{1, 2, \dots\}$, where $\sigma_{ij} = \sigma^{j-i}$ for $i \leq j$. Then it is proved that R is a semiprime ring, by using the same argument as in [15, Example 1.2]. For a two-sided ideal $I = \{A \in R \mid \text{the diagonal entries of } A \text{ are zero}\}$ of R , it can be easily checked that $\text{Mat}_n(I)$ is a nil two-sided ideal of $\text{Mat}_n(R)$. Thus $\text{Mat}_n(R)$ is NC-McCoy for $n \geq 2$ by Proposition 3(1).

Notice that the $n \times n$ full matrix ring $S = \text{Mat}_n(R)$ over a reduced ring R is not one-sided NC-McCoy by Theorem 5, but the ring $U_2(S)$ is NC-McCoy by Proposition 3(4). Moreover, if R is the ring of quaternions with integer coefficients, then R is a domain, and so NC-McCoy; while for any odd prime integer q , we have $R/qR \cong \text{Mat}_2(\mathbb{Z}_q)$ by the argument in [11, Exercise 2A], and thus the factor ring R/qR is not NC-McCoy by Theorem 5.

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Note that both NI rings and right McCoy rings are directly finite by [15, Proposition 2.7(1)] and [6, Theorem 5.2], respectively. However, there exists an NC-McCoy ring which is not directly finite.

Example 7. Let R be the ring of column finite countable matrices over a field F . Let $a \in R$ be the matrix with $(i, i + 1)$ -entry 1 for all $i \geq 1$ and zero elsewhere, and $b \in R$ be the $(i + 1, i)$ -entry 1 for all $i \geq 1$ and zero elsewhere. Then $ab = 1$, but $ba \neq 1$. Consider the $n \times n$ upper triangular matrix ring $U_n(R)$ for $n \geq 2$. Then $U_n(R)$ is NC-McCoy by Proposition 3(4). But $AB = 1$ and $BA \neq 1$ with the help of the computation above, where $A, B \in U_n(R)$ are scalar matrices with diagonals a and b , respectively. Hence $U_n(R)$ is not directly finite.

Theorem 8. (1) For a ring R , if $R[x]$ is right NC-McCoy, then so is R .
 (2) Assume that $N(R)[x] \subseteq N(R[x])$ for a ring R . If R is a right NC-McCoy ring, then so is $R[x]$.

Proof. (1) Suppose that $R[x]$ is right NC-McCoy. Let $f(x)g(x) = 0$ for nonzero polynomials $f(x) = a_0 + a_1x + \dots + a_mx^m$ and $g(x) = b_0 + b_1x + \dots + b_nx^n$ in $R[x]$. Then let $f(y) = a_0 + a_1y + \dots + a_my^m$, $g(y) = b_0 + b_1y + \dots + b_ny^n \in (R[x])[y]$, where $(R[x])[y]$ is the polynomial ring with an indeterminate y over $R[x]$. Then $f(y)$ and $g(y)$ are nonzero since $f(x)$ and $g(x)$ are nonzero. Moreover $f(y)g(y) = 0$. So there exists a nonzero $c(x) = c_0 + c_1x + \dots + c_kx^k \in R[x]$ such that $f(y)c(x) \in N(R[x])[y]$ since $R[x]$ is right NC-McCoy. Then $a_ic(x) \in N(R[x])$ for any $0 \leq i \leq m$. Since $c(x)$ is nonzero, there exists the smallest positive integer l such that $c_l \neq 0$. Then $a_ic_l \in N(R)$ for all $0 \leq i \leq m$, and so $f(x)c_l \in N(R)[x]$. Therefore R is right NC-McCoy.

(2) Suppose that R is right NC-McCoy and $f(y)g(y) = 0$ for nonzero polynomials $f(y) = f_0 + f_1y + \dots + f_my^m$ and $g(y) = g_0 + g_1y + \dots + g_ny^n$ in $(R[x])[y]$. Take the positive integer k with $k = \sum_{i=0}^m \deg(f_i) + \sum_{j=0}^n \deg(g_j)$ where the degree of the zero polynomial is taken to be 0. Then $f(x^k)$ and $g(x^k)$ are nonzero polynomials in $R[x]$ and $f(x^k)g(x^k) = 0$, since the set of coefficients of the f_i 's and g_j 's coincides with the set of coefficients of $f(x^k)$ and $g(x^k)$. Since R is right NC-McCoy, there exists a nonzero $c \in R$ such that $f(x^k)c \in N(R)[x]$. Hence, $ac \in N(R)$ for any $a \in C_{f_i(x)}$, and so $f_i c \in N(R)[x] \subseteq N(R[x])$ for each $0 \leq i \leq m$. Thus $R[x]$ is right NC-McCoy. \square

Birkenmeier et al. [5, Proposition 2.6] proved that the polynomial ring $R[x]$ over a 2-primal ring R is 2-primal. Thus if R is a 2-primal ring, then both R and $R[x]$ are NC-McCoy, but the polynomial ring over an NI ring need not be NI with the help of Smoktunowicz [31, Corollary 13].

Corollary 9. (1) *If R is a right NC-McCoy ring such that $N(R[x])$ is a subring of $R[x]$, then $R[x]$ is right NC-McCoy.*

(2) *Suppose that an NI ring R satisfies either of the following conditions: (i) $N(R)[x] \subseteq N(R[x])$; (ii) $R[x]$ is a nil-Armendariz ring. Then both R and $R[x]$ are right NC-McCoy rings.*

(3) *If R_0 is a nil algebra over an uncountable field K , then both $K + R_0$ and $(K + R_0)[x]$ are right NC-McCoy rings.*

Proof. (1) Assume that $N(R[x])$ is a subring of $R[x]$. For any $a \in N(R)$ and nonnegative integer t , ax^t is nilpotent. Thus $ax^t \in N(R[x])$, and so $N(R)[x] \subseteq N(R[x])$ as the latter is closed under addition. The proof is completed by Theorem 8(2).

(2) Part (i) comes immediately from Theorem 8(2). For part (ii), recall that a ring R is nil-Armendariz if whenever $f(x)g(x) \in N(R)[x]$, then $C_{f(x)}C_{g(x)}$ is nil. If $R[x]$ is nil-Armendariz, then $N(R[x])$ is a subring of $R[x]$ by [3, Theorem 3.2]. We are done by (1).

(3) It was shown by Amitsur [1] and Krempa [19] that if R_0 is a nil algebra over an uncountable field, then $R_0[x]$ is nil as well. Letting $R = K + R_0$, we have that R is an NI ring with $N(R) = R_0$ and $N(R[x]) = R_0[x] = N(R)[x]$. By Theorem 8(2), we are done. \square

Remark 10. (1) If R is an NI ring, then $N(R[x]) \subseteq N(R)[x]$, and so by the hypothesis in Corollary 9(2)(i) we actually have $N(R[x]) = N(R)[x]$.

(2) There exists an NI ring R with $N(R[x]) = N(R)[x]$, but R is not 2-primal. Consider the NI ring R in Example 6. Notice $N(R) = I = \{A \in R \mid \text{the diagonal entries of } A \text{ are zero}\}$. So it is obvious that $N(R[x]) = N(R)[x]$, but R is not 2-primal by [15, Example 1.2].

(3) Let R be a ring with the nonzero nilpotent $N^*(R)$. Then $N^*(R)$ and $N^*(R)[x]$ are nil ideals of R and $R[x]$, respectively. Thus both R and $R[x]$ are NC-McCoy by Proposition 3(1). Notice that every ring which satisfies ascending chain condition on both right and left annihilators, every right Goldie

ring, and every ring with right Krull dimension (in [12]) imply that $N^*(R)$ is nilpotent by [7, Theorem 1.34], [23] and [24], respectively.

Proposition 11. *If R is a ring of bounded index with a nonzero nil two-sided ideal of bounded index, then both R and $R[x]$ are NC-McCoy rings.*

Proof. Let I be a nonzero nil two-sided ideal of R . Since $0 \neq I \subseteq N^*(R)$, $N^*(R)$ contains a nonzero two-sided nilpotent ideal N of R by [18, Lemma 5]. Then $N[x]$ is a nonzero two-sided nilpotent ideal of $R[x]$, and thus R and $R[x]$ are NC-McCoy rings by Proposition 3(1). \square

Recall that a ring R is called *weak Armendariz* [25, Definition 2.1] if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Any Armendariz ring is weak Armendariz and every weak Armendariz ring is NC-McCoy; while the converses do not hold by [25, Example 2.4] and Example 12(2) to follow. The concepts of weak Armendariz rings and right McCoy rings are independent of each other by the following.

Example 12. (1) For a field F , $U_2(F)$ is weak Armendariz by [25, Proposition 2.2], but not right McCoy by Example 4(1).

(2) Let K be a field and $A = K\langle e, a_0, a_1, b_0, b_1, y, z \rangle$ be the free K -algebra generated by noncommuting indeterminates $e, a_0, a_1, b_0, b_1, y, z$. Let I be the ideal of A generated by the relations $e^2 = e, a_0b_0 = 0, a_0b_1 + a_1b_0 = 0, a_1b_1 = 0, ea_i = a_ie = a_i, eb_i = b_ie = b_i, ey = 0, ye = y, ze = 0, ez = z, y^2 = yz = zy = z^2 = 0, a_iy = ya_i = b_iy = yb_i = 0, a_iz = za_i = b_iz = zb_i = 0$ and set $R = A/I$. Then R is left and right McCoy by [6, Example 10.4]. Consider a polynomial $f(x) = b_0a_0 + (b_0a_1 + b_1a_0)x + b_1a_1x^2 \in R[x]$. Then $f(x)^2 = 0$ since $a_0b_0 = 0, a_0b_1 + a_1b_0 = 0, a_1b_1 = 0$, but $(b_0a_1 + b_1a_0)^2$ is not nilpotent. This implies that R is not weak Armendariz.

Recall that a ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. When R is a regular ring, we have that R is reduced if and only if R is Abelian if and only if R is Armendariz if and only if R is weak Armendariz if and only if R is right (left) McCoy by [21, Theorem 19] and [17, Theorem 13]. So one may conjecture that R is Abelian if and only if R is right NC-McCoy when R is a regular ring. But the following erases the possibility.

Example 13. Let D be a division ring and $R_n = \text{Mat}_{2^n}(D)$ for any positive integer n . Define a map $\sigma : R_n \rightarrow R_{n+1}$ by $\sigma(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Set $R = \varinjlim R_n$ be the direct limit of the direct system (R_n, σ_{ij}) over $\{1, 2, \dots\}$, where $\sigma_{ij} = \sigma^{j-i}$ for $i \leq j$.

Let $a \in R$. Then $a \in R_n$ for some n . Since each R_n is regular, there exists $b \in R_n \subset R$ such that $a = aba$. Thus R is regular. Clearly R is non-Abelian.

Next let $f(x) = \sum_{i=0}^l a_i x^i$ be any nonzero polynomial in $R[x]$. Then $f(x) \in R_n[x]$ for some n . But then

$$(a_i E_{1(2^{n+1})})^{2^n} = 0 \text{ for any } i,$$

where a_i and $E_{1(2^{n+1})}$ are considered as elements in $R_{n+1} = \text{Mat}_{2^{n+1}}(D)$. This yields $f(x)E_{1(2^{n+1})} \in N(R)[x]$ and so R is right NC-McCoy.

In the following arguments, we characterize the class of minimal right NC-McCoy rings for the cases of with identity and without identity. Here by minimal we mean having smallest cardinality.

Proposition 14. *Let R be a right NC-McCoy ring with identity. If R is a minimal right NC-McCoy ring, then R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$.*

Proof. Let R be a minimal right NC-McCoy ring with identity. Then $|R| \geq 2^3$ by [9, Theorem]. If $|R| = 2^3$, then R is isomorphic to $U_2(\mathbb{Z}_2)$ by [9, Proposition]. But $U_2(\mathbb{Z}_2)$ is a right NC-McCoy ring by Proposition 3(4). This yields that R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$. \square

Next we observe the structure of minimal right NC-McCoy rings without identity. The Jacobson radical of a ring R is denoted by $J(R)$.

Example 15. Let $A = \mathbb{Z}_2\langle a, b \rangle$ be the free \mathbb{Z}_2 -algebra generated by the non-commuting indeterminates a, b and B be the subalgebra of polynomials with zero constant terms in A .

Let I_1, I_2 , and I_3 be the ideals of B generated by the subsets

$$\{a^2 - a, b^2, ba, ab - b\}, \{a^2 - a, b^2, ab, ba - b\}, \text{ and } \{a^2 - a, b^2, ab, ba\},$$

respectively. Next set $R_i = B/I_i$ for $i = 1, 2, 3$. We identify a and b with their images in R_i for simplicity. Note that every R_i is a ring without identity such that

$$R_i = \{0, a, b, a + b\}, J(R_i) = \{0, b\} = N^*(R_i) = N(R) \text{ and } R_i/J(R_i) \cong \mathbb{Z}_2$$

for all i . Thus every R_i is NI and hence NC-McCoy.

Given a ring R , R^+ means the additive Abelian group $(R, +)$. The characteristic of R is denoted by $Ch(R)$.

Proposition 16. *Let R be a ring without identity. If R is a minimal right NC-McCoy ring, then R is of order 4 and is isomorphic to R_i for some $i \in \{1, 2, 3\}$, where R_i 's are the rings in Example 15.*

Proof. Let R be a minimal right NC-McCoy ring without identity. If $|R| \leq 3$, then R must be commutative, and so $|R| \geq 2^2$. Then $|R| = 2^2$ by considering the rings in Example 15. If $Ch(R) = 4$, then R is commutative, and so $Ch(R)$ must be 2. So R is an algebra over \mathbb{Z}_2 .

Assume that R is nil. Note that $J(R) = N(R) = N^*(R) = N_*(R) = R$ and R is nilpotent. If R^+ is cyclic, then R is commutative clearly. If R^+ is

non-cyclic, then R is also commutative by [20, Theorem 2.3.3]. Thus R must be non-nil, entailing that $J(R) = 0$ or $|J(R)| = 2$.

Assume $J(R) = 0$. Since $Ch(R) = 2$, we can consider an extension ring $E = \mathbb{Z}_2 + R$ of R . Then $|E| = 8$ and $Ch(E) = 2$. We also get $J(E) = 0$ since $J(R) = 0$ and $J(E) \subseteq R$, entailing that E is semiprimitive Artinian. If E is non-reduced, then $8 = |E| \geq 2^4$ by the Wedderburn-Artin theorem, a contradiction. This yields that E is reduced such that $E = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. But then E (hence R) is commutative, which is also a contradiction. Thus we must have $J(R) \neq 0$.

Consequently we reduce to the case when $|J(R)| = 2$. Say $J(R) = \{0, b\}$. Here $J(R)^2 = 0$ (i.e., $b^2 = 0$) since $b^2 \neq 0$ means $b^2 = b$ (then $b \notin J(R)$.) Since $R/J(R) \cong \mathbb{Z}_2$, there exists an idempotent, say a , by [22, Proposition 3.6.2]. Then $R = \{0, a, b, a + b\}$. Now it suffices to compute ab and ba . Since ab and ba are contained in $J(R)$, it is obvious that we have one of the following three cases:

$$(ab = b, ba = 0), (ab = 0, ba = b), \text{ and } (ab = 0, ba = 0).$$

For the first case, R is isomorphic to the ring R_1 in Example 15. For the second case, R is isomorphic to the ring R_2 in Example 15. For the last case, R is isomorphic to the ring R_3 in Example 15. \square

Note that $U_2(\mathbb{Z}_2)$ and the rings in Example 15 are all NI. So we also obtain the following by Propositions 14 and 16.

Corollary 17. *Let R be a ring (possibly without identity). Then R is a minimal right NC-McCoy ring if and only if R is a minimal noncommutative NI ring if and only if R is a minimal left NC-McCoy ring if and only if R is a minimal noncommutative NC-McCoy ring.*

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