

POSET METRICS ADMITTING ASSOCIATION SCHEMES AND A NEW PROOF OF MACWILLIAMS IDENTITY

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ABSTRACT. It is known that being hierarchical is a necessary and sufficient condition for a poset to admit MacWilliams identity. In this paper, we completely characterize the structures of posets which have an association scheme structure whose relations are indexed by the poset distance between the points in the space. We also derive an explicit formula for the eigenmatrices of association schemes induced by such posets. By using the result of Delsarte which generalizes the MacWilliams identity for linear codes, we give a new proof of the MacWilliams identity for hierarchical linear poset codes.

1. Introduction

Let \mathbb{F}_q be the finite field with q elements and \mathbb{F}_q^n be the n -dimensional vector space over \mathbb{F}_q . The Hamming weight $w_H(u)$ of a vector $u \in \mathbb{F}_q^n$ is the number of its nonzero coordinates. The Hamming distance $d_H(u, v)$ between two vectors $u, v \in \mathbb{F}_q^n$ is $d_H(u, v) = w_H(u - v)$. Let \mathbf{P} be a partially ordered set (poset) on the underlying set $[n] = \{1, 2, \dots, n\}$ of coordinate positions of vectors in \mathbb{F}_q^n with the partial order relation denoted by \leq as usual. A subset I of \mathbf{P} is called an ideal if $a \in I$ and $b \leq a$, then $b \in I$. For $u = (u_1, u_2, \dots, u_n) \in \mathbb{F}_q^n$, the support $\text{supp}(u)$ and \mathbf{P} -weight $w_{\mathbf{P}}(u)$ of u are defined to be

$$\text{supp}(u) = \{i \mid u_i \neq 0\} \text{ and } w_{\mathbf{P}}(u) = |\langle \text{supp}(u) \rangle|,$$

where $\langle \text{supp}(u) \rangle$ is the smallest ideal containing the support of u . For any $u, v \in \mathbb{F}_q^n$, $d_{\mathbf{P}}(u, v) := w_{\mathbf{P}}(u - v)$ is a metric on \mathbb{F}_q^n [5]. The metric $d_{\mathbf{P}}$ is called the \mathbf{P} -metric on \mathbb{F}_q^n . Let \mathbb{F}_q^n be endowed with the \mathbf{P} -metric. Then a subset (resp., subspace) \mathcal{C} of \mathbb{F}_q^n is called a \mathbf{P} -code (resp., linear \mathbf{P} -code) of length n . The \mathbf{P} -weight enumerator of a linear code \mathcal{C} is defined by

$$(1) \quad W_{\mathcal{C}, \mathbf{P}}(x) = \sum_{u \in \mathcal{C}} x^{w_{\mathbf{P}}(u)} = \sum_{i=0}^n A_{i, \mathbf{P}} x^i,$$

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where $A_{i, \mathbf{P}} = |\{u \in \mathcal{C} \mid w_{\mathbf{P}}(u) = i\}|$.

If \mathbf{P} is an antichain, then the \mathbf{P} -metric is equal to the Hamming metric, and $d_{\mathbf{P}}$ is denoted by d_H . In this case, the \mathbf{P} -weight enumerator of a linear code \mathcal{C} becomes the Hamming weight enumerator of \mathcal{C} .

Coding theory may be considered as the study of \mathbb{F}_q^n when \mathbb{F}_q^n is endowed with the Hamming distance d_H . Several attempts have been made to generalize the classical problems in coding theory by introducing a new non-Hamming metric on \mathbb{F}_q^n [16, 17, 18]. New non-Hamming metrics called the poset metric [5] and the Rosenbloom and Tsfasman metric [19] have been introduced. The latter is equivalent to the poset metric over the disjoint union of chains of the same cardinality. Many subsequent attempts have been made to find the MacWilliams-type identities about linear codes for such metrics corresponding to the MacWilliams identity about linear codes for the Hamming distance (see [8, 9, 10, 11, 12, 20]).

The MacWilliams identity about a linear code over \mathbb{F}_q is one of the most important identities in coding theory, and it expresses the Hamming weight enumerator of the dual code $\mathcal{C}^\perp (= \{y \in \mathbb{F}_q^n \mid x \cdot y = 0 \text{ for all } x \in \mathcal{C}\})$, where \cdot is the usual inner product on \mathbb{F}_q^n of a linear code \mathcal{C} in terms of the Hamming weight enumerator of \mathcal{C} . Essentially, what enables us to obtain the MacWilliams identity for the Hamming distance is that

Property (*) the Hamming weight enumerator of the dual code is uniquely determined by that of a linear code \mathcal{C} .

Unfortunately, however, Property (*) does not generally hold for a poset code; for any poset \mathbf{P} , \mathbf{P} -weight enumerators of the dual codes of linear codes \mathcal{C}_1 and \mathcal{C}_2 may be different even if the \mathbf{P} -weight enumerators of \mathcal{C}_1 and \mathcal{C}_2 are the same [9, 10, 11].

Kim and Oh [12] derived the MacWilliams identity about poset linear codes for a certain type of poset by using the discrete Poisson formula: For a poset \mathbf{P} , by introducing the concept of the dual poset $\overline{\mathbf{P}}$ (defined in Section 3), they classified all poset structures which admit MacWilliams identity (i.e., all poset structures \mathbf{P} such that the $\overline{\mathbf{P}}$ -weight enumerator of the dual code \mathcal{C}^\perp of a linear code \mathcal{C} is uniquely determined by the \mathbf{P} -weight enumerator of \mathcal{C} for all linear codes \mathcal{C} over \mathbb{F}_q). By applying the identity of Kim and Oh to antichain and chain, one can obtain the same result as the MacWilliams identity for the Hamming distance and the result of Skrikanov [20] on the Rosenbloom-Tsfasman metric, respectively.

The MacWilliams identity for the Hamming distance is also generalized by Delsarte using the theory of association schemes: For a finite abelian group X under addition and a subgroup C (called an additive code) of X , Delsarte [6] relates the weight distribution of an additive code C in X to that of its dual code $C' = \{\chi \in X^* \mid \chi(y) = 1 \text{ for all } y \in C\}$, where X^* is the character group of X . In the case of \mathbb{F}_q^n with the usual inner product, it is possible to identify

C' with C^\perp for a linear code C of \mathbb{F}_q^n and Delsarte's result specializes to the MacWilliams identity about a linear code over a finite field (see [4]).

Martin and Stinson [15] introduced the kernel scheme and constructed the ordered Hamming scheme as the s -fold product of the kernel scheme (kernel scheme is a special case of the ordered Hamming scheme). Furthermore, by applying Delsarte's theory to ordered Hamming scheme, they derived the MacWilliams-type identity for linear ordered codes. When we take a poset as the disjoint union of s chains of the same cardinality, the relations of the ordered Hamming scheme are the same as those which are indexed by the shape (not poset distance) and the MacWilliams-type identity for linear ordered codes becomes the MacWilliams identity for shape enumerator (not poset weight enumerator) about poset linear codes. When we consider a chain poset, it turns out that the relations of the kernel scheme are the same as those which are indexed by the poset distances and the MacWilliams identity for shape enumerator on the kernel scheme is the same as the MacWilliams identity for poset weight enumerator.

We can deduce from the Hamming scheme and the kernel scheme that antichain and chain are posets which admit association schemes whose relations are indexed by poset distance. So it is natural to attempt to classify all poset structures which admit association schemes with relations which are indexed by poset distance and to derive the MacWilliams identity for poset weight enumerators corresponding to such poset codes. This gives the motivation of our investigation.

Section 2 gives a necessary and sufficient condition for a poset \mathbf{P} to admit an association scheme with relations which are indexed by poset distance. In Section 3, the primitive idempotent basis and eigenmatrices of association schemes on such posets, called poset schemes, are explicitly expressed. By applying Delsarte's theory [6] to poset schemes, we will derive an explicit formula for the MacWilliams identity on such poset codes, which coincides with the result in [12].

2. Association schemes for poset metrics

The theory of association schemes is an important subject in algebraic combinatorics. The concept of association schemes was first introduced by Bose and Shimamoto [3] and the corresponding algebraic structure was developed by Bose and Mesner [2]. In this section, we will prove that being hierarchical is a necessary and sufficient condition for a poset \mathbf{P} to admit an association scheme with relations which are indexed by poset distance.

We first introduce some information on association schemes, which will be needed in our investigation. We refer to [1, 4, 6, 7] for more information on association schemes.

Definition 2.1. Let X be a non-empty finite set and $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ be a family of relations on X , that is, $d + 1$ subsets of the Cartesian product

X^2 . The ordered pair (X, \mathcal{R}) is a symmetric association scheme if the following conditions hold:

- (i) The set \mathcal{R} is a partition of X^2 and $R_0 = \{(x, x) \mid x \in X\}$ is the identity relation.
- (ii) Each R_i is symmetric: $(x, y) \in R_i \Rightarrow (y, x) \in R_i$.
- (iii) If $(x, y) \in R_k$, then the number of $z \in X$ such that $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant c_{ij}^k that depends on i, j, k but not on the particular choices of x and y .

The parameter c_{ij}^k is called the intersection number of the association scheme.

Throughout this paper, an association scheme means a symmetric association scheme. Let (X, \mathcal{R}) be an association scheme, where $\mathcal{R} = \{R_0, \dots, R_d\}$. For $0 \leq i \leq d$, let D_i be the adjacency matrix of R_i . Then a set of $d + 1$ symmetric $(0, 1)$ -matrices exists which satisfies the following conditions:

- (a) $D_0 = I$.
- (b) $\sum_{i=0}^d D_i = J$, the all-ones matrix.
- (c) For $0 \leq i, j \leq d$, $D_i D_j = \sum_{k=0}^d c_{ij}^k D_k = D_j D_i$.

Let \mathbb{A} denote the vector space over the complex number field \mathbb{C} spanned by $\mathcal{D} = \{D_0, D_1, \dots, D_d\}$. Condition (c) implies that \mathbb{A} is closed under matrix multiplication, and hence forms an algebra. This is called the Bose-Mesner algebra of the association scheme. The algebra \mathbb{A} has a unique basis of idempotents E_0, E_1, \dots, E_d . Since $\frac{1}{|X|}J$ is a primitive idempotent, we always choose $E_0 = \frac{1}{|X|}J$. These are $|X| \times |X|$ matrices satisfying: for $0 \leq i \neq j \leq d$,

$$(2) \quad E_i^2 = E_i, \quad E_i E_j = 0, \quad \text{and} \quad \sum_{i=0}^d E_i = I.$$

The transition matrices between the two bases $\{D_0, \dots, D_d\}$ and $\{E_0, \dots, E_d\}$ of \mathbb{A} play an important role in the theory of association schemes. The first and second eigenmatrices P and Q of the association scheme are defined by the equations

$$D_i = \sum_{j=0}^d P_{ji} E_j \quad \text{and} \quad E_j = \frac{1}{|X|} \sum_{i=0}^d Q_{ij} D_i, \quad 0 \leq i, j \leq d.$$

The eigenmatrices P and Q satisfy $PQ = QP = |X|I$. For each i , P_{ji} 's are the eigenvalues of D_i and the columns of E_j are eigenvectors of D_i .

The following, called the Hamming scheme, is an important example in coding theory.

Example 2.2 (Hamming scheme). Let \mathbf{F} be a finite set of cardinality $q \geq 2$ and let $X = \mathbf{F}^n$ be the set of n -tuples over \mathbf{F} . We make X a metric space by the Hamming distance $d_H(x, y) := |\{i \mid 1 \leq i \leq n, x_i \neq y_i\}|$ between two

words $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of X . Define the distance relations R_0, R_1, \dots, R_n in an obvious way:

$$R_i = \{(x, y) \in X^2 \mid d_H(x, y) = i\}.$$

Then (X, \mathcal{R}) is a symmetric association scheme for $\mathcal{R} = \{R_0, R_1, \dots, R_n\}$. We call (X, \mathcal{R}) the Hamming scheme of length n over \mathbf{F} , and denote it by $H(n, q)$.

We now introduce a hierarchical poset as the ordinal sum of antichains.

Let n_1, \dots, n_t be positive integers with $n_1 + \dots + n_t = n$. We define the poset $\mathbb{H}(n; n_1, \dots, n_t)$ on the set $\{(i, j) \mid 1 \leq i \leq t, 1 \leq j \leq n_i\}$ whose order relation is given by

$$(i, j) < (l, m) \Leftrightarrow i < l.$$

The poset $\mathbb{H}(n; n_1, \dots, n_t)$ is called a hierarchical poset with t levels and n elements.

Let \mathbf{P} be a poset of cardinality n and \mathbb{F}_q^n be endowed with the \mathbf{P} -metric. For $0 \leq k \leq n$, define a relation $R_{k, \mathbf{P}}$ on \mathbb{F}_q^n as follows:

$$(x, y) \in R_{k, \mathbf{P}} \text{ if and only if } d_{\mathbf{P}}(x, y) = k.$$

Definition 2.3. A poset \mathbf{P} is said to admit an association scheme if $(\mathbb{F}_q^n, \mathcal{R}_{\mathbf{P}})$ is an association scheme, where $\mathcal{R}_{\mathbf{P}} = \{R_{0, \mathbf{P}}, R_{1, \mathbf{P}}, \dots, R_{n, \mathbf{P}}\}$.

Let $\mathbb{A}\mathbb{C}_n$ denote the antichain of cardinality n . The classical Hamming scheme shows that $\mathbb{A}\mathbb{C}_n$ admits an association scheme.

For an arbitrary poset \mathbf{P} , conditions (i) and (ii) of Definition 2.1 are automatically satisfied since $d_{\mathbf{P}}$ is a metric on \mathbb{F}_q^n . Therefore, to check whether a poset \mathbf{P} admits an association scheme or not, it suffices to check condition (iii) of Definition 2.1. Since a poset metric is translation-invariant, i.e., $d_{\mathbf{P}}(x, y) = d_{\mathbf{P}}(x + z, y + z)$ for all $x, y, z \in \mathbb{F}_q^n$, the following, which modifies condition (iii), is easily obtained.

Lemma 2.4. *Let \mathbf{P} be a poset with cardinality n and \mathbb{F}_q^n be endowed with the \mathbf{P} -metric. Then the following are equivalent:*

- (1) *Condition (iii) of Definition 2.1 is satisfied.*
- (2) *For each $x \in \mathbb{F}_q^n$ with $w_{\mathbf{P}}(x) = k$, $|\{z \in \mathbb{F}_q^n \mid d_{\mathbf{P}}(x, z) = i, w_{\mathbf{P}}(z) = j\}|$ is a constant c_{ij}^k that depends on i, j, k but not on x .*

To prove that a poset which admits an association scheme is hierarchical, some lemmas are required. For a poset \mathbf{P} , we define $\min(\mathbf{P}) = \{i \in \mathbf{P} \mid i \text{ is minimal in } \mathbf{P}\}$ to be the set of minimal elements in \mathbf{P} and put $\mathbf{P}' := \mathbf{P} \setminus \min(\mathbf{P})$. Clearly, \mathbf{P}' is also a poset under the partial order relation induced by \mathbf{P} .

Lemma 2.5. *Let \mathbf{P} be a poset which admits an association scheme. Then each element of \mathbf{P}' is greater than any element of $\min(\mathbf{P})$.*

Proof. This lemma can be proved simply by showing that each element of $\min(\mathbf{P}')$ is greater than any element of $\min(\mathbf{P})$. Suppose that there exists $i_0 \in \min(\mathbf{P}')$ such that $|\langle i_0 \rangle| < 1 + |\min(\mathbf{P})|$. Let $y \in \mathbb{F}_q^n$ be a vector such that $\text{supp}(y) = \{i_0\}$. Then $w_{\mathbf{P}}(y) = |\langle i_0 \rangle|$. Since $|\langle i_0 \rangle| < 1 + |\min(\mathbf{P})|$, we can choose a vector $y' \in \mathbb{F}_q^n$ such that $|\langle \text{supp}(y') \rangle| = w_{\mathbf{P}}(y)$ and $\text{supp}(y') \subseteq \min(\mathbf{P})$. Since $\min(\mathbf{P})$ is an antichain, $w_{\mathbf{P}}(y') = w_H(y') = |\langle i_0 \rangle| = w_{\mathbf{P}}(y)$. Since \mathbf{P} admits an association scheme, the equation

$$(3) \quad \begin{aligned} & |\{z \in \mathbb{F}_q^n \mid d_{\mathbf{P}}(y, z) = i, w_{\mathbf{P}}(z) = j\}| \\ &= |\{z \in \mathbb{F}_q^n \mid d_{\mathbf{P}}(y', z) = i, w_{\mathbf{P}}(z) = j\}| \end{aligned}$$

holds true for each i and j . Put $j = 1$ and $i = w_{\mathbf{P}}(y') - 1 (= w_{\mathbf{P}}(y) - 1 = w_H(y') - 1)$. Then the left-hand side of (3) is 0, while the right-hand side of (3) is $w_H(y')$. This yields a contradiction to the fact that \mathbf{P} admits an association scheme. \square

Lemma 2.6. *Let \mathbf{P} be a poset which admits an association scheme. Then \mathbf{P}' also admits an association scheme.*

Proof. Let $|\mathbf{P}| = n$ and $|\min(\mathbf{P})| = n_1$. Without loss of generality, we may assume that $\mathbf{P} = \{1, \dots, n\}$ and $\min(\mathbf{P}) = \{1, \dots, n_1\}$. If $n = n_1$, then the proof is complete. Hence we assume that $n_1 < n$. Since \mathbf{P}' induces a poset metric on $\mathbb{F}_q^{n-n_1}$, it suffices to show that $(\mathbb{F}_q^{n-n_1}, \mathcal{R}_{\mathbf{P}'})$ satisfies condition (iii) of Definition 2.1. Note that every vector $x \in \mathbb{F}_q^n$ can be written as $x = (x_1, x_2)$, $x_1 \in \mathbb{F}_q^{n_1}$, $x_2 \in \mathbb{F}_q^{n-n_1}$ in an obvious way. For $x' \in \mathbb{F}_q^{n-n_1}$ with $w_{\mathbf{P}'}(x') = k$, consider the set

$$A' = \{z' \in \mathbb{F}_q^{n-n_1} \mid w_{\mathbf{P}'}(z') = j \text{ and } d_{\mathbf{P}'}(x', z') = i\}.$$

We can choose a vector $x \in \mathbb{F}_q^n$ such that $x = (a, x')$, where a is an arbitrary vector in $\mathbb{F}_q^{n_1}$, and consider the set

$$A = \{z \in \mathbb{F}_q^n \mid w_{\mathbf{P}}(z) = n_1 + j \text{ and } d_{\mathbf{P}}(x, z) = n_1 + i\}.$$

It is clear that $|A| = q^{n_1}|A'|$. Since \mathbf{P} admits an association scheme, $|A|$ is a constant not depending on the choice of x . Hence $|A'|$ is also a constant not depending on the choice of x' . Therefore, \mathbf{P}' admits an association scheme. \square

The following is immediately obtained from Lemmas 2.5 and 2.6 and an inductive argument:

Theorem 2.7. *If \mathbf{P} admits an association scheme, then \mathbf{P} is a hierarchical poset.*

We now will derive a sufficient condition for a poset which admits an association scheme. We start from a well-known result on the Hamming scheme.

If \mathbf{P} is an antichain of cardinality n , then \mathbf{P} gives the Hamming scheme $H(n, q)$ on \mathbb{F}_q^n . Recall that for each $x \in \mathbb{F}_q^n$,

$$(4) \quad |\{z \in \mathbb{F}_q^n \mid d_H(x, z) = i\}| = (q-1)^i \binom{n}{i}.$$

For the sake of convenience, we denote by $c_{ij}^k(H(n, q))$ the intersection number in $H(n, q)$.

Lemma 2.8. *Let $\mathbf{P} = \mathbb{H}(n; n_1, \dots, n_t)$ be a hierarchical poset. Then for each $x \in \mathbb{F}_q^n$ with $w_{\mathbf{P}}(x) = k$, $|A_{x,i,j}^k|$ is a constant depending on i, j , and k , but not on the choice of x , where $A_{x,i,j}^k = \{z \in \mathbb{F}_q^n \mid d_{\mathbf{P}}(x, z) = i, w_{\mathbf{P}}(z) = j\}$.*

Proof. If one of i, j , and k is 0, then it can be easily shown that the result is true. Hence we may assume that $0 < i, j, k \leq n$ and $x \in \mathbb{F}_q^n$ with $w_{\mathbf{P}}(x) = k$. We will prove it case by case. For convenience, we put $\widehat{n}_s = n - (n_1 + n_2 + \dots + n_s)$.

CASE I : $n - \widehat{n}_s < i, j, k \leq n - \widehat{n}_{s+1}$.

From the definition of \mathbf{P} -metric, for each $x \in \mathbb{F}_q^n$ with $w_{\mathbf{P}}(x) = k$,

$$|A_{x,i,j}^k| = q^{n-\widehat{n}_s} c_{i'j'}^{k'}(H(n_{s+1}, q)),$$

where $i' = i - (n - \widehat{n}_s)$, $j' = j - (n - \widehat{n}_s)$, and $k' = k - (n - \widehat{n}_s)$.

CASE II : $n - \widehat{n}_s < i, j \leq n - \widehat{n}_{s+1}$, and $n - \widehat{n}_l < k \leq n - \widehat{n}_{l+1}$, where $l \neq s$.

If $w_{\mathbf{P}}(z) = j$, then either $d_{\mathbf{P}}(x, z) = j$ when $l < s$ or $d_{\mathbf{P}}(x, z) = k$ when $l > s$. Hence if $z \in A_{x,i,j}^k$, then $l < s$ and $i = j$. From Equation (4),

$$|A_{x,i,j}^k| = \begin{cases} q^{n-\widehat{n}_s} (q-1)^{i'} \binom{n_{s+1}}{i'} & \text{if } l < s \text{ and } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $i' = i - (n - \widehat{n}_s)$.

CASE III : $n - \widehat{n}_s < i \leq n - \widehat{n}_{s+1}$, $n - \widehat{n}_l < j \leq n - \widehat{n}_{l+1}$ with $s < l$, and $n - \widehat{n}_m < k \leq n - \widehat{n}_{m+1}$.

If $w_{\mathbf{P}}(z) = j$, then $d_{\mathbf{P}}(x, z) > n - \widehat{n}_l$ when either $m \neq l$ or $j \neq k$. Hence if $z \in A_{x,i,j}^k$, then $m = l$ and $j = k$ since $s < l$. From Equation (4),

$$|A_{x,i,j}^k| = \begin{cases} q^{n-\widehat{n}_s} (q-1)^{i'} \binom{n_{s+1}}{i'} & \text{if } m = l \text{ and } j = k \\ 0 & \text{otherwise,} \end{cases}$$

where $i' = i - (n - \widehat{n}_s)$.

CASE IV : $n - \widehat{n}_s < i \leq n - \widehat{n}_{s+1}$, $n - \widehat{n}_l < j \leq n - \widehat{n}_{l+1}$ with $s > l$, and $n - \widehat{n}_m < k \leq n - \widehat{n}_{m+1}$.

If $w_{\mathbf{P}}(z) = j$ and $d_{\mathbf{P}}(x, z) = i$, then $w_{\mathbf{P}}(x) = i$ since $s > l$. Hence if $z \in A_{x,i,j}^k$, then $i = k$. From Equation (4),

$$|A_{x,i,j}^k| = \begin{cases} q^{n-\widehat{n}_l} (q-1)^{j'} \binom{n_{l+1}}{j'} & \text{if } m = s \text{ and } i = k \\ 0 & \text{otherwise,} \end{cases}$$

where $j' = j - (n - \widehat{n}_l)$.

From CASES I - IV, $|A_{x,i,j}^k|$ is a constant that depends on i, j , and k , but not on the choice of x . □

By combining Theorem 2.7 with Lemma 2.8, we obtain:

Theorem 2.9. *Let \mathbf{P} be a poset of cardinality n . Then $(\mathbb{F}_q^n, \mathcal{R}_{\mathbf{P}})$ is an association scheme if and only if \mathbf{P} is a hierarchical poset.*

By applying Theorem 2.9 to antichain and chain, the Hamming and kernel schemes are obtained (see [14, 15]).

Corollary 2.10. *Let \mathbf{P} be a poset of cardinality n . Then*

- (i) $(\mathbb{F}_q^n, \mathcal{R}_{\mathbf{P}})$ is a Hamming scheme if $\mathbf{P} = \mathbb{H}(n; n, 0, \dots, 0)$,
- (ii) $(\mathbb{F}_q^n, \mathcal{R}_{\mathbf{P}})$ is a kernel scheme if $\mathbf{P} = \mathbb{H}(n; 1, 1, \dots, 1)$.

3. Poset scheme and MacWilliams duality

In Section 2, it is proved that being hierarchical is a necessary and sufficient condition for a poset to admit an association scheme. An association scheme that arises in this way is called a poset scheme. In this section, we will derive explicit formulas for the primitive idempotent basis and eigenmatrices of poset schemes. Then we will derive the MacWilliams identity for linear poset codes using Delsarte’s theory [6]. We briefly introduce a translation scheme, its dual scheme, and their properties. We borrow the terminology and some results from [4].

Let X be a finite abelian group under addition, and (X, \mathcal{R}) be an d -class association scheme. If $(x, y) \in R$ implies $(x + z, y + z) \in R$ for all $z \in X$ and all relation $R \in \mathcal{R}$, then the scheme (X, \mathcal{R}) is said to be a translation scheme. It is clear that a poset scheme is a translation scheme. For a d -class translation scheme (X, \mathcal{R}) , we define

$$(5) \quad N_i = \{x \in X \mid (0, x) \in R_i\}.$$

Note that $N_0 = \{0\}$. Then the group X can be partitioned into $d + 1$ blocks $N_i, 0 \leq i \leq d$, and the relation R_i can be recovered from the block N_i as follows:

$$(6) \quad R_i = \{(x, y) \in X^2 \mid y - x \in N_i\}.$$

To define dual scheme of a translation scheme, we denote the character group of a finite abelian group X by X^* . For a given d -class translation scheme (X, \mathcal{R}) , we define the dual scheme (X^*, \mathcal{R}^*) by $\mathcal{R}^* = \{R_0^*, R_1^*, \dots, R_d^*\}$, where

$$(7) \quad \begin{aligned} (\chi, \psi) \in R_i^* &\Leftrightarrow E_i \eta = \eta, \eta = \chi^{-1} \psi \\ &\Leftrightarrow \eta = \chi^{-1} \psi \in N_i^*. \end{aligned}$$

Here $\{E_0, \dots, E_d\}$ denotes the basis of primitive idempotents of (X, \mathcal{R}) and $N_i^* = \{\eta \in X^* \mid E_i \eta = \eta\}$.

The following establishes a relation between a translation scheme and its dual scheme.

Theorem 3.1 ([4, 6, 21]). *For any d -class translation scheme (X, \mathcal{R}) with eigenmatrices P and Q , the dual scheme (X^*, \mathcal{R}^*) is also a translation scheme with eigenmatrices $P^* = Q$ and $Q^* = P$.*

Let (X, \mathcal{R}) be a translation scheme. If Y is a subgroup of X , then it is called an additive code in X . For an element x of X , the weight $w(x)$ of x is defined as the number i if $x \in N_i$. The weight distribution of an additive code Y is the vector $a = (a_0, a_1, \dots, a_n)$, where a_i is defined as follows:

$$(8) \quad a_i = |Y \cap N_i| \quad (i = 0, 1, \dots, n).$$

The dual code Y' of an additive code Y is the subgroup $\{\chi \in X^* \mid \chi(y) = \chi_y = 1 \text{ for all } y \in Y\}$. The following establishes a relation between the weight distribution of a code Y and that of its dual code, which generalizes the MacWilliams identity for linear codes over finite fields.

Theorem 3.2 ([4, 6]). *Let a and a' be the weight distributions of an additive code Y and its dual code Y' , respectively. Then we have the following relations:*

$$(9) \quad a' = \frac{1}{|Y|} a Q, \quad a = \frac{|Y|}{|X|} a' P,$$

where P and Q are eigenmatrices of the translation scheme (X, \mathcal{R}) .

We now describe explicitly the basis of primitive idempotents and eigenmatrices of a poset scheme. We briefly introduce the Krawtchouk polynomials, additive characters of \mathbb{F}_q^n , and their relations.

Definition 3.3. For any prime power q and a positive integer n , the Krawtchouk polynomial $P_k(x : n)$ is defined by

$$P_k(x : n) = \sum_{j=0}^k (-1)^j \gamma^{k-j} \binom{x}{j} \binom{n-x}{k-j}, \quad k = 0, 1, \dots, n,$$

where $\gamma = q - 1$.

It is known in [14] that $P_k(x : n)$ has the generating function

$$(1 + \gamma x)^{n-i} (1 - x)^i = \sum_{k=0}^n P_k(i : n) x^k, \quad 0 \leq i \leq n.$$

We now define additive characters on \mathbb{F}_q^n . Let χ be a nontrivial additive character on \mathbb{F}_q . For $z \in \mathbb{F}_q^n$, define a homomorphism χ_z from the $(\mathbb{F}_q^n, +)$ into \mathbb{C} as follows:

$$(10) \quad \chi_z : \mathbb{F}_q^n \rightarrow \mathbb{C} \quad \text{via} \quad \chi_z(a) = \chi(a \cdot z),$$

where \cdot means the usual inner product. Note that χ_z is an additive character on \mathbb{F}_q^n . χ_z is sometimes considered to be the character vector, i.e., the column vector of length q^n whose a -th component is $\chi_z(a)$. We give the following lemma about additive characters on \mathbb{F}_q^n without proof (see [13], [14] for detailed discussion on additive characters).

Lemma 3.4. For $z, z_1,$ and z_2 in \mathbb{F}_q^n , we have the following relations:

$$(i) \quad \chi_{z_1}^T \chi_{z_2} = \sum_{a \in \mathbb{F}_q^n} \chi_{z_1+z_2}(a) = \begin{cases} q^n & \text{if } z_1 + z_2 = 0 \\ 0 & \text{if } z_1 + z_2 \neq 0. \end{cases}$$

$$(ii) \quad \sum_{w_H(z)=k} \chi_z(a) = P_k(w_H(a) : n),$$

where $P_k(i : n)$ is the Krawtchouk polynomial.

We first describe the basis of primitive idempotents of a poset scheme. We introduce a simple terminology, which is needed in the sequel.

For a given poset \mathbf{P} , we define the poset $\overline{\mathbf{P}}$ as follows:

$$\mathbf{P} \text{ and } \overline{\mathbf{P}} \text{ have the same underlying set, and}$$

$$x \leq y \text{ in } \overline{\mathbf{P}} \Leftrightarrow y \leq x \text{ in } \mathbf{P}.$$

The poset $\overline{\mathbf{P}}$ is called the dual poset of \mathbf{P} .

Let $\mathbf{P} = \mathbb{H}(n; n_1, \dots, n_t)$ be a hierarchical poset and $D_{i,\mathbf{P}}$ be the adjacency matrix of the relation $R_{i,\mathbf{P}}$, where $R_{i,\mathbf{P}} = \{(x, y) \mid d_{\mathbf{P}}(x, y) = i\}$, $0 \leq i \leq n$. The vector space \mathbb{A} spanned by $\{D_{i,\mathbf{P}}\}_{i=0, \dots, n}$ over \mathbb{C} forms the Bose-Mesner algebra of the poset scheme. The algebra \mathbb{A} has a unique basis of primitive idempotents $E_{0,\mathbf{P}}, E_{1,\mathbf{P}}, \dots, E_{n,\mathbf{P}}$.

Theorem 3.5. Let $\mathbf{P} = \mathbb{H}(n; n_1, \dots, n_t)$ be a hierarchical poset with t levels and n elements. Then the primitive idempotents of the poset scheme are as follows:

$$(11) \quad E_{i,\mathbf{P}} = \frac{1}{q^n} \sum_{w_{\overline{\mathbf{P}}}(z)=i} \chi_z \chi_{-z}^T, \quad 0 \leq i \leq n.$$

Furthermore, $E_{0,\mathbf{P}} = \frac{1}{q^n} J$, and for $1 \leq k \leq n_{i+1}$, $0 \leq i \leq t - 1$,

$$(12) \quad E_{n_t + \dots + n_{i+2} + k, \mathbf{P}} = \frac{q^{\widehat{n}_{i+1}}}{q^n} \left(\sum_{j=0}^{n_1 + \dots + n_i} \gamma^k \binom{n_{i+1}}{k} D_{j,\mathbf{P}} + \sum_{j=1}^{n_{i+1}} P_k(j : n_{i+1}) D_{n_1 + \dots + n_i + j, \mathbf{P}} \right),$$

where $\widehat{n}_i = n - (n_1 + \dots + n_i)$ and $\gamma = q - 1$.

Proof. Equation (11) follows readily from [4].

Put $E_{i,\mathbf{P}} := \frac{1}{q^n} \sum_{w_{\overline{\mathbf{P}}}(z)=i} \chi_z \chi_{-z}^T$. Note that $E_{0,\mathbf{P}} = \frac{1}{q^n} J$. For convenience, we denote the (a, b) entry of $E_{i,\mathbf{P}}$ by $(E_{i,\mathbf{P}})_{(a,b)}$. Since $n = n_1 + \dots + n_t$ and $\mathbb{F}_q^n = \mathbb{F}_q^{n_1} \oplus \mathbb{F}_q^{n_2} \oplus \dots \oplus \mathbb{F}_q^{n_t}$, for $z \in \mathbb{F}_q^n$, we can write $z = (z_1, z_2, \dots, z_t)$, where $z_i \in \mathbb{F}_q^{n_i}$. Note that

$$(13) \quad w_{\overline{\mathbf{P}}}(z) = n_t + \dots + n_{i+2} + k \iff \begin{cases} w_H(z_{i+1}) = k, \\ z_1 = z_2 = \dots = z_i = 0, \\ (z_{i+2}, \dots, z_t) \in \mathbb{F}_q^{\widehat{n}_{i+1}}. \end{cases}$$

For $1 \leq k \leq n_{i+1}$ and $0 \leq i \leq t - 1$, by Lemma 3.4 and Equation (13), we obtain:

$$\begin{aligned} & (E_{n_t+\dots+n_{i+2}+k, \mathbf{P}})_{(a,b)} \\ &= \frac{1}{q^n} \sum_{w_{\overline{\mathbf{P}}}(z)=n_t+\dots+n_{i+2}+k} \chi_z(a) \chi_{-z}^T(b) = \frac{1}{q^n} \sum_{w_{\overline{\mathbf{P}}}(z)=n_t+\dots+n_{i+2}+k} \chi((a-b) \cdot z) \\ &= \frac{1}{q^n} \sum_{w_H(z_{i+1})=k} \chi((a_{i+1} - b_{i+1}) \cdot z_{i+1}) \\ & \quad \times \sum_{(z_{i+2}, \dots, z_t) \in \mathbb{F}_q^{\widehat{n}_{i+1}}} \chi(((a_{i+2}, \dots, a_t) - (b_{i+2}, \dots, b_t)) \cdot (z_{i+2}, \dots, z_t)) \\ &= \frac{q^{\widehat{n}_{i+1}}}{q^n} \begin{cases} 0 & \text{if } (a_{i+2}, \dots, a_t) \neq (b_{i+2}, \dots, b_t), \\ \sum_{w_H(z_{i+1})=k} \chi((a_{i+1} - b_{i+1}) \cdot z_{i+1}) & \text{otherwise.} \end{cases} \end{aligned}$$

By Lemma 3.4, we obtain

$$(14) \quad (E_{n_t+\dots+n_{i+2}+k, \mathbf{P}})_{(a,b)} = \frac{q^{\widehat{n}_{i+1}}}{q^n} \begin{cases} 0 & \text{if } (a_{i+2}, \dots, a_t) \neq (b_{i+2}, \dots, b_t), \\ P_k(w_H(a_{i+1} - b_{i+1}) : n_{i+1}) & \text{otherwise.} \end{cases}$$

Since $P_k(0 : n) = \gamma^k \binom{n}{k}$, Equation (14) can be written as follows:

$$(15) \quad (E_{n_t+\dots+n_{i+2}+k, \mathbf{P}})_{(a,b)} = \frac{q^{\widehat{n}_{i+1}}}{q^n} \begin{cases} 0 & \text{if } d_{\mathbf{P}}(a, b) > n_1 + \dots + n_{i+1}, \\ \gamma^k \binom{n_{i+1}}{k} & \text{if } d_{\mathbf{P}}(a, b) \leq n_1 + \dots + n_i, \\ P_k(w_H(a_{i+1} - b_{i+1}) : n_{i+1}) & \text{otherwise.} \end{cases}$$

Equation (12) follows immediately from Equation (15). □

Remark. If we replace $w_{\overline{\mathbf{P}}}(z) = i$ by $w_{\mathbf{P}}(z) = i$ in the summand of Equation (11), then $E_{i, \overline{\mathbf{P}}} := \frac{1}{q^n} \sum_{w_{\overline{\mathbf{P}}}(z)=i} \chi_z \chi_{-z}^T$ satisfies Equation (2). However, it does not belong to the Bose-Mesner algebra \mathbb{A} spanned by $\{D_{i, \mathbf{P}}\}$, but belongs to the Bose-Mesner algebra spanned by $\{D_{i, \overline{\mathbf{P}}}\}$.

By applying Theorem 3.5 to special posets, say antichain and chain, we obtain the following results.

Corollary 3.6 (Hamming scheme [14]). *In a poset $\mathbf{P} = \mathbb{H}(n; n, 0, \dots, 0)$, for each $0 \leq i \leq n$, $E_{i, \mathbf{P}}$ is given by*

$$E_{i, \mathbf{P}} = E_i = \frac{1}{q^n} \sum_{w_H(z)=i} \chi_z \chi_{-z}^T.$$

Therefore, for $1 \leq k \leq n$,

$$E_{k, \mathbf{P}} = E_k = \frac{1}{q^n} \sum_{j=0}^n P_k(j : n) D_{j, \mathbf{P}}.$$

In the case of chain poset, this is the same result as in [15].

Corollary 3.7 (Kernel scheme [15]). *Let $\mathbf{P} = \mathbb{H}(n; 1, 1, \dots, 1)$ be a chain with n -elements. Then for each $0 \leq i \leq n$, $E_{i, \overline{\mathbf{P}}}$ is given by*

$$E_{i, \overline{\mathbf{P}}} = \frac{1}{q^n} \sum_{w_{\mathbf{P}}(z)=i} \chi_z \chi_{-z}^T = \frac{1}{q^n} \sum_{top(z)=i} \chi_z \chi_{-z}^T,$$

where $top(z) = \max\{i \mid z_i \neq 0\}$, $z = (z_1, z_2, \dots, z_n)$.

We now compute the transition matrices between the bases $\{E_{i, \mathbf{P}}\}_{i=0, \dots, n}$ and $\{D_{j, \mathbf{P}}\}_{j=0, \dots, n}$ of the Bose-Mesner algebra of poset schemes. Recall that the first eigenmatrix P and the second eigenmatrix Q of an association scheme are defined by the relations:

$$(16) \quad D_{j, \mathbf{P}} = \sum_{i=0}^n P_{ij} E_{i, \mathbf{P}}, \quad E_{k, \mathbf{P}} = \frac{1}{q^n} \sum_{i=0}^n Q_{ik} D_{i, \mathbf{P}}.$$

It is well-known that the characters $\chi_z, z \in \mathbb{F}_q^n$, give a basis for \mathbb{F}_q^n diagonalizing the Bose-Mesner algebra, and that the eigenvalue of $D_{j, \mathbf{P}}$ corresponding to χ_z is given by

$$\sum_{a \in N_j} \chi_z(a),$$

where $N_j = \{a \in \mathbb{F}_q^n \mid w_{\mathbf{P}}(a) = j\}$ (see [4]).

It follows from Lemma 3.4, Theorem 3.5, and Equation (16) that P_{ij} can be written as:

$$(17) \quad P_{ij} = \sum_{w_{\mathbf{P}}(a)=j} \chi_z(a),$$

where $w_{\overline{\mathbf{P}}}(z) = i$.

Let $i = n_t + \dots + n_{s+1} + i_0$ and $j = n_1 + \dots + n_l + j_0$, where $1 \leq i_0 \leq n_s$ and $1 \leq j_0 \leq n_{l+1}$. For $z \in \mathbb{F}_q^n$, we can write $z = (z_1, \dots, z_t)$, where $z_i \in \mathbb{F}_q^{n_i}$. Since

$$(18) \quad \begin{cases} w_{\mathbf{P}}(a) = j, \\ w_{\overline{\mathbf{P}}}(z) = i \end{cases} \iff \begin{cases} w_H(a_{l+1}) = j_0 \text{ and } w_H(z_s) = i_0, \\ a_{l+2} = a_{l+1} = \dots = a_t = 0 = z_1 = z_2 = \dots = z_{s-1}, \\ (a_1, \dots, a_l) \in \mathbb{F}_q^{n_1 + \dots + n_l} \text{ and } (z_{s+1}, \dots, z_t) \in \mathbb{F}_q^{n_s}, \end{cases}$$

we have the following equation:

$$(19) \quad \sum_{w_{\mathbf{P}}(a)=j} \chi_z(a) = \sum_{\tilde{a} \in \mathbb{F}_q^{n_1 + \dots + n_l}} \chi_{\tilde{z}}(\tilde{a}) \sum_{w_H(a_{l+1})=j_0} \chi_{z_{l+1}}(a_{l+1}),$$

where $\tilde{a} = (a_1, \dots, a_l)$ and $\tilde{z} = (z_1, \dots, z_l)$. By Lemma 3.4, the first eigenmatrix P is given as follows: for $i = n_t + \dots + n_{s+1} + i_0$ and $j = n_1 + \dots + n_l + j_0$,

$$(20) \quad P_{ij} = \begin{cases} 0 & \text{if } (z_1, \dots, z_l) \neq 0 \Leftrightarrow s \leq l, \\ q^{j-j_0} \gamma^{j_0} \binom{n_{l+1}}{j_0} & \text{if } (z_1, \dots, z_{l+1}) = 0 \Leftrightarrow s > l + 1, \\ q^{j-j_0} P_{j_0}(w_H(z_{l+1}) : n_{l+1}) = q^{j-j_0} P_{j_0}(i_0 : n_{l+1}) & \text{if } (z_1, \dots, z_l) = 0, z_{l+1} \neq 0 \Leftrightarrow s = l + 1. \end{cases}$$

From Theorem 3.5, the second eigenmatrix Q can be easily described: For $j = n_t + \dots + n_{l+2} + k$, $1 \leq k \leq n_{l+1}$,

$$(21) \quad Q_{ij} = \begin{cases} q^{\hat{n}_{l+1}} \gamma^k \binom{n_{l+1}}{k} & \text{if } 0 \leq i \leq n_1 + \dots + n_l, \\ q^{\hat{n}_{l+1}} P_k(i_0 : n_{l+1}) & \text{if } n_1 + \dots + n_l < i \leq n_1 + \dots + n_{l+1}, \\ 0 & \text{if } n_1 + \dots + n_{l+1} < i, \end{cases}$$

where $i_0 = i - (n_1 + \dots + n_l)$.

We now apply Theorem 3.2 to poset schemes. Let $\mathbf{P} = \mathbb{H}(n; n_1, \dots, n_t)$ be a hierarchical poset with t levels and n elements and $X = \mathbb{F}_q^n$. Since the dual poset $\overline{\mathbf{P}}$ is also a hierarchical poset, clearly $(X, \mathcal{R}_{\mathbf{P}})$ and $(X, \mathcal{R}_{\overline{\mathbf{P}}})$ are translation schemes. For $x \in N_{i, \overline{\mathbf{P}}} = \{x \in X \mid w_{\overline{\mathbf{P}}}(x) = i\}$, it follows from Lemma 3.4 that $E_{i, \mathbf{P}} \chi_x = \chi_x$. Hence, by Equation (7), we have

$$(22) \quad x \in N_{i, \overline{\mathbf{P}}} \Leftrightarrow \chi_x \in N_{i, \mathbf{P}}^*, \quad N_{i, \mathbf{P}}^* = \{\eta \in X^* \mid E_{i, \mathbf{P}} \eta = \eta\}.$$

By an isomorphism $f : X \rightarrow X^*$ defined by $a \mapsto \chi_a (a \in X)$, the following is easily obtained.

Theorem 3.8. *Let \mathbf{P} be a hierarchical poset with t levels and n elements. Then $(\mathbb{F}_q^n, \mathcal{R}_{\overline{\mathbf{P}}})$ and $(\mathbb{F}_q^{n^*}, \mathcal{R}_{\mathbf{P}}^*)$ are isomorphic as association schemes.*

The following, which is the MacWilliams identity for hierarchical poset codes, is an easy consequence of Theorems 3.2, 3.5, and 3.8. It is the same as in [12].

Theorem 3.9. *Let $\mathbf{P} = \mathbb{H}(n; n_1, \dots, n_t)$ be a hierarchical poset with t levels and n elements and \mathcal{C} be a linear code of \mathbb{F}_q^n . Then the \mathbf{P} -weight distribution a of \mathcal{C} is as follows:*

$$(23) \quad a^\perp = \frac{1}{|\mathcal{C}|} a Q, \quad a = \frac{|\mathcal{C}^\perp|}{|\mathcal{C}|} a^\perp P,$$

where a^\perp is $\overline{\mathbf{P}}$ -weight distribution of the dual code \mathcal{C}^\perp and P, Q are the eigenmatrices of the association scheme $(\mathbb{F}_q^n, \mathcal{R}_{\mathbf{P}})$.

In other words, if $a = (A_{0, \mathbf{P}}, \dots, A_{n, \mathbf{P}})$ and $a^\perp = (A'_{0, \overline{\mathbf{P}}}, \dots, A'_{n, \overline{\mathbf{P}}})$, then, for each $0 \leq i \leq t - 1$, $1 \leq k \leq n_{i+1}$,

$$A'_{n_t + \dots + n_{i+2} + k, \overline{\mathbf{P}}}$$

$$= \frac{q^{\widehat{n_{i+1}}}}{|\mathcal{C}|} \left(\sum_{j=1}^{n_{i+1}} P_k(j : n_{i+1}) A_{n_1+\dots+n_i+j, \mathbf{P}} + \sum_{j=0}^{n_1+\dots+n_i} \gamma^k \binom{n_{i+1}}{k} A_{j, \mathbf{P}} \right).$$

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