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KaiFangShu in SanHak JeongEui

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This paper is a sequel to the paper [8], where we discussed the connection between ShiShou KaiFangFa originated from JiuZhang SuanShu and ZengCheng KaiFangFa. Investigating KaiFangShu in a Chosun mathemtics book, SanHak JeongEui and ShuLi JingYun, we show that its authors, Nam ByungGil and Lee SangHyuk clearly understood the connection and gave examples to show that the KaiFangShu in the latter is not exact. We also show that Chosun mathematicians were very much selective when they brought in Chinese mathematics.

Keywords: SanHak JeongEui(算學正義, 1867), Nam ByungGil(南秉吉, 1820–1869), Lee SangHyuk(李尙爀, 1810–?), KaiFangShu(開方術), ShiShou KaiFanfFa(釋鎖開方法), ZengCheng KaiFangFa(增乘開方法), ShuLi JingYun(數理精蘊).

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0 Introduction

Theory of equations is one of the most important subjects in mathematics. It is well known that TianYuanShu had provided the main tool for constructing equations throughout in Chosun mathematics. Although the names, ZengCheng KaiFang-Fa(增乘開方法) and ShiShou KaiFangFa(釋鎖開方法) were not known in Chosun, Chosun mathematicians fully understood their mathematical structures and employed them to solve polynomial equations. We briefly recall their structures [8].

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and α a first guess of a solution of the equation $p(x) = 0$ called a ChuShang(初商). For the remaining part y, CiShang(次商), i.e., $x = y + \alpha$, one has $a_n(y+\alpha)^n + a_{n-1}(y+\alpha)^{n-1} + \cdots + a_1(y+\alpha) + a_0 = 0$.

In ShaoGuang(少廣) chapter of JiuZhang SuanShu(九章算術) which deals with

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the extractions of square and cube roots, the terms $(y+\alpha)^k$ were expanded by geometrical reasoning. Extending these processes of expansions by Jia Xian(賈憲, ca. 11th C.)'s triangle, also known as the Pascal(1623–1662)'s triangle, one has the equation $b_n y^n + b_{n-1} y^{n-1} + \cdots + b_1 y + b_0 = 0$ for y. This method is the ShiShou KaiFangFa [3].

Since $y = x - \alpha$, the above equation for y is as follows:

$$b_n(x-\alpha)^n + b_{n-1}(x-\alpha)^{n-1} + \cdots + b_1(x-\alpha) + b_0 = 0.$$

In this case, $b_0, b_1, \ldots, b_{n-1}, b_n$ can be obtained by successive synthetic divisions of $p(x)$ by $x-\alpha$. Synthetic divisions involve only elementary operations. This method is ZengCheng KaiFangFa.

We note that $p(x) = b_n(x-\alpha)^n + b_{n-1}(x-\alpha)^{n-1} + \cdots + b_1(x-\alpha) + b_0$ is precisely the Taylor series of the polynomial function $p(x)$ at $x = \alpha$, in particular $b_1 = p'(\alpha)$.

Reversing from ZengCheng KaiFangFa to ShiShou KaiFangFa, one can have the ZiaXian triangle for $p(x) = x^n$ at $x = 1$ (see [5, 8]). Indeed, $x^n = \sum_{k=0}^{n} b_k(x-1)^k$ implies $(x+1)^n = \sum_{k=0}^{n} b_k x^k$. Since $b_k = p^{(k)}(1)/k!$, one could have also $b_k = n(n-1)\ldots(n-k+1)/k! = \binom{n}{k}$.

We note that ShuLi JingYun(數理精蘊, 1723, [2]) was brought into Chosun in the mid 18th century and had a great influence on Chosun mathematics and astronomy. In particular, King JungJo(正祖, r. 1777–1800) included it as a subject for the national examination for astronomy(陰陽科) in 1791 [9].

For the sake of completeness, we recall the method of solving equaitons in ShuLi JingYun. Since the other problems of equations of degrees 3 to 6 in Chapter 33 in its second book(下卷) follow exactly the same method as the first problem, we take the first problem $x^3 + 8x = 1,824$. First 10 was chosen as ChuShang(定初商) and find the constant term 744 of the equation $q(y) = 0$ for CiShang by substituting it in the equation and $1824 - (10^3 + 8 \times 10) = 744$(次商積). The coefficient of the linear term of $q(y)$ was calculated by $3 \times 10^2 + 8 \times 1 = 308$ which was obtained by expanding $(y+10)^3 + 8(y+10)$(次商廉法). Dividing 744 by 308(以除次商積足二倍), they guess 2 as CiShang(定次商). Finally they add ChuShang and CiShang $12 = 10 + 2$(合初商共一十二尺) and substituting 12 into the original equation, they verify 12 is the solution of the equation. Finally, the process is explained by geometrical reasoning(此法以積計之爲一正方體及八根之共數 以邊計之 則所得每根之數 卽正方體之每一邊因正方體之外多八根 故成一磬折體 而非正方體亦非長方體也). They put this geometrical explanation only for cubic equations. We note that in the above example, $y = 308(x-10) - 744$ is the equation of the tangent line of $p(x) = x^3 + 8x - 1824$ at 10 and hence $10 + 744/308$ is the solution of $308(x-10) - 744 = 0$.

We give an example that above process is useless for guessing CiShang. Let $p(x) =$

$x^3 - 13x^2 - 13x - 14$, then the equation $p(x) = 0$ has a unique real solution 14. Furthermore, the equation of the tangent line of $p(x)$ at 10 is $y = 27(x - 10) - 444$ and $444/27$ is too big to guess 4 as CiShang.

In this process, one can also easily discern that they don't find the equation $q(y) = 0$ for the CiShang. We just put it for the convenience. We note that they proceed with the same method as above when they have to find the next part of CiShang (see the first problem of LiFang(立方) of Chapter 32 and also compare this with the method of extraction of cube roots in Chapter 23). Thus, the method in ShuLi JingYun is neither ShiShou nor ZengCheng KaiFangFa and is very much complicated than the latter.

Investigating KaiFangFa in Chosun mathematics book SanHak JeongEui(算學正義, 1867, [1]) written by Nam ByungGil(南秉吉, 1820–1869) and Lee SangHyuk(李尙爀, 1810–?), we show that the authors did understand the connection between ShiShou and ZengCheng KaiFangFa [8] and that pointed out the deficiency of the KaiFangFa in ShuLi JingYun. So far, we have not found any Chosun mathematics book which used the KaiFangFa in ShuLi JingYun. Noting that KaiFangFa is the most important subject in Chosun Mathematics, we conclude that Chosun mathematicians were very much selective when they brought in Chinese mathematics.

1 KaiFangShu in SanHak JeongEui

In the first page of the main body of SanHak JeongEui, it is said that Nam ByungGil compiled SanHak JeongEui and Lee SangHyuk corrected it. But we quote the following from its preface written by Nam and then conclude that Nam and Lee are coauthors of SanHak JeongEui.

余於養痾之餘 采集諸書 李君志叟釐正編修彙成一書曰算學正義

As mentioned in [5], Nam and Lee had a complete collection of books published by four great mathematicians, Li Ye(李冶, 1192–1279), Qin JiuShao(秦九韶), Yang Hui(楊輝) and Zhu ShiJie(朱世傑) in Song - Yuan era, their commentaries published by Qing mathematicians including Luo ShiLin(羅士琳, 1774–1853)'s SiYuan YuJian XiCao(四元玉鑑細艸, 1835, [4]) and books by Qing Mathematicians. Furthermore, they served as officials in the national observatory, GwanSangGam(觀象監) and studied ShuLi JingYun thoroughly. Thus they could notice that algebra of Song - Yuan era is much more superior than western algebra in ShuLi JingYun and that the deductive reasoning in ShuLi JingYun is rather advanced than in Chinese mathematics. Combining advantages from the two mathematics, they compiled SanHak JeongEui. In particular, they adopted freely the format and terminology from ShuLi JingYun.

KaiFangShu in SanHak JeongEui was introduced directly after basic operations of the field of rational numbers in the first book(上編). The usual traditional mathematics was put in the second book(中編). Finally, the authors discussed surveying, TianYuanShu up to SiYuanShu(四元術) and Qin's DaYanShu(大衍術).

We now return to the detail of KaiFangShu in SanHak JeongEui. Although the authors used general terminology and theory from Chapter 11, 23, 24, 25, 33 in ShuLi JingYun, they completely disregarded methods in the book but discussed the usual ZengCheng KaiFangFa. We note that they studied Hong JungHa(洪正夏, 1684–?)'s GuIlJib(九一集, 1724, [1]) and found the similarity between their KaiFangShu.

Using the second problem in the section JeSeungBangBub(諸乘方法), we show the structure of Nam and Lee's KaiFangShu. The problem is to solve an equation $2x^6-3x^5+5x^4-30x^3+41^2-500x-5,277,216=0$. It is interesing to note that they call coefficients of the equation from the linear term to the sixth degree term JongSaSeungYumBub(縱四乘廉法), JongSamSeungYumBub(縱三乘廉法), JongIbBangYumBub(縱立方廉法), JongPyungBangYumBub(縱平方廉法), JongJangYumBub(縱長廉法) and WooBub(隅法) in turn. Except the cubed term, their notions are quite confusing. First 10 was taken as ChuShang and then using the synthetic divisions in ZengCheng KaiFangFa, they have the equation $2y^6+117y^5+2855y^4+37,170y^3+272,141y^2+1,061,320y-3,558,116=0$. Although their calculation was carried out by the synthetic divisions by $x-10$, they stated the method to get the digits of solutions as they added "一退, 二退, ⋯, 六退". As usual, they first get the constant term for the equation. For the other coefficients of the equation for CiShang, they added the following:

> b_1 : 即初商數之四乘積二段六倍 立方積五段四倍 商數四十一段二倍
> 共數內減 三乘積三段五倍 平方積三十段三倍 及五百算之餘數也;
> b_2 : 即初商數之三乘積二段十五倍 平方積五段六倍 及四十一算
> 共數內減 立方積三段十倍 商數三十段三倍之餘數也;
> b_3 : 即初商數之立方積二段二十倍 商數五段四倍
> 共數內減 平方積三段十倍 及三十算之餘數也;
> b_4 : 即初商數之平方積二段十五倍 及五算
> 共數內減 商數三段五倍之餘數也
> b_5 : 即初商數二段六倍 內減 三算之餘數也

In the following, the left sides of the original equation and that for CiShang will be denoted by $p(x), q(y)$ respectively. In the above quote, we add b_k which denote the coefficients of $q(y)$. Then one can easily see that b_k's are obtained by expanding each term of $p(y+10)$. Indeed, for b_1, n in "n段" is precisely the corresponding coefficient of $p(x)$ and m in "m倍" is the corresponding coefficient from the Jia Xian's triangle. One has the other b_k as b_1. In the end, using the synthetic division by $y-2$ as usual, they have the solution 12 for $p(x) = 0$.

SanHak JeongEui is the unique book which indicates ShiShou and ZengCheng KaiFangFa side by side so that it reveals the connection between the two KaiFang-Shu.

Further, in the process to solve the above equation, there are many places where FanJian(飜減) has occurred and then they introduced Qin JiuShao's terminology, HuanGu(換骨) and TouTai(投胎) along with FanJi(飜積) and YiJi(益積).

Moreover, they gave three equations, $2x^3 + 200x^2 + 60,000x - 3,750,000 = 0$, $x^3+8,000x^2-392,500x-500,000 = 0$, and $16x^3+8,000x^2+8,000x-22,400,000 = 0$ with the same solution 50. There are clearly no common relation between the linear terms and constant terms. Further, they consider an equation $x^3 - 392x^2 - 3,185x - 6,000 = 0$ which has the solution 400. For the above examples, every coefficient from WooBub effects in turn for the given numbers to be the solutions as they can be verified by synthetic divisions. By these examples, they pointed out the flaw in ShuLi JingYun discussed in the previous section.

Finally, the authors discussed the case where CiShang may be a negative number or multiple digits by the following example. For an equation $x^3 - 600x^2 + 50,000x - 498,456 = 0$, they took 90 as ChuShang and then had the equation $x^3 - 330x^2 - 33,700x - 129,456 = 0$ which has the negative solution -4. Thus they conclude that $90-4 = 86$ is the solution for the given equation. This is the first incident in Chosun mathematics which deals with equations with negative solutions. Furthermore, for the same equation, they took ChuShang 70 and then got the CiShang 16 so that they arrive at the same solution. In the appendix by YiZhiHan(易之瀚) in Luo's SiYuan YuJian XiCao, Yi dealt with problems as above. We note that Nam and Lee studied the XiCao [6].

2 Conclusion

Theory of equations had been the most important subject in Chosun mathematics. Chosun mathematicians built the theory first based on SuanXue QiMeng(1299, [2]) and YangHui SuanFa(1274–1275, [2]). ShuLi JingYun gave a great impact on Chosun mathematics since the mid 18th century. For the theory of equations, Chosun mathematicians paid attention to the method JieGenFang(借根方) in ShuLi JingYun to represent polynomials and equations using the equal sign. But they did go back to TianYuanShu shortly because they noticed much more deeper methods in SiYuan YuJian and Li Ye's TianYuanShu [7]. As shown in the first section, the method of solving polynomial equations in ShuLi JingYun is rather complicated and has a deficiency. Noting these, Chosun mathematicians completely disregarded the method.

Nam ByungGil and Lee SangHyuk even pointed out the defects. Furthermore, they showed the connection between ShiShou and ZengCheng KaiFangFa. Except that they didn't treat TianYuanShu and KaiFangFa together, they completed perfectly the theory of equations including systems of equations of higher degrees.

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