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# MODULAR TRIBONACCI NUMBERS BY MATRIX METHOD

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Abstract. In this work we study the tribonacci numbers. We find a tribonacci triangle which is an analog of Pascal triangle. We also investigate an efficient method to compute any $n$th tribonacci numbers by matrix method, and find periods of the sequence by taking modular tribonacci number.

## 1. Introduction

The study of Fibonacci sequence $F_n$ $(n \geq 0)$ has a long history since Lucas, 1885. The research has been extended to algebraic aspects, such as Fibonacci group([9], [4]) and Fibonacci ring[2], etc. It is also generalized to higher-order sequences including tribonacci[5], quatranacci, $k$-step Fibonacci sequences[1]. The 3-step Fibonacci sequence usually called the tribonacci sequence $T_n$ is the sum of the preceding three terms having initial values $0, 0, 1$. Hence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with $T_{-1} = T_0 = 0$ and $T_1 = 1$, so the first some numbers are $\{T_n\}: \; 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \cdots$.

The purpose of this work is to study the tribonacci numbers. We construct a tribonacci triangle which is an analog of Pascal triangle so that every tribonacci number appears in the triangle. We find an efficient method to compute any $n$th tribonacci numbers by matrix method, and investigate periods of the sequence by taking modular tribonacci number.

## 2. Tribonacci Numbers with Binomial Coefficients

For the Fibonacci sequence $F_n$, it is known that if $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then $M^n = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ thus $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$. Fibonacci sequence

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is to $M$ what tribonacci sequence is to $N = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, in fact

$$\begin{bmatrix} T_{n+1} \\ T_n \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_n \\ T_{n-1} \\ T_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix}.$$

**Theorem 2.1.** *Let $N$ be the matrix as above.*

*(1)* $N = \begin{bmatrix} T_2 & 1 & T_1 \\ T_1 & 0 & T_0 \\ T_0 & 1 & T_{-1} \end{bmatrix}$ *and* $N^n = \begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}$.

*(2)* $1 = T_0^2 + T_1^2 - T_1 T_{-1} - T_2 T_0$

*(3)* $$\begin{aligned} T_{n-1}^3 - 1 &= 2T_{n-2}T_{n-1}T_n + T_{n-3}T_{n-1}T_{n+1} - T_{n-2}^2 T_{n+1} - T_{n-3}T_n^2 \\ &= T_{n-2}(2T_{n-1}T_n - T_{n+1}) + T_{n-3}(T_n^2 - T_{n-1}T_{n+1}). \end{aligned}$$

*Proof.* Since $N^2 = \begin{bmatrix} T_3 & T_2 + T_1 & T_2 \\ T_2 & T_1 + T_0 & T_1 \\ T_1 & T_0 + T_{-1} & T_0 \end{bmatrix}$, (1) follows by induction. Moreover since

$$\begin{aligned} 1 = \det(N) &= T_0^2 + T_1^2 - T_1T_{-1} - T_2T_0 \\ = \det(N^n) &= \begin{vmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{vmatrix} = - \begin{vmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \\ T_{n-1} & T_{n-2} & T_{n-3} \end{vmatrix}, \end{aligned}$$

we have

$$T_{n+1}T_{n-2}^2 - T_{n+1}T_{n-1}T_{n-3} + T_n^2 T_{n-3} - 2T_nT_{n-2}T_{n-1} + T_{n-1}^3 = 1,$$

hence $T_{n-1}^3 - 1 = T_{n-3}(T_n^2 - T_{n-1}T_{n+1}) + T_{n-2}(2T_{n-1}T_n - T_{n+1})$. □

Next theorem is about the tribonacci numbers $T_n$ for negative $n$.

**Theorem 2.2.** $T_{-n} = \begin{vmatrix} T_{n-1} & T_n \\ T_{n-2} & T_{n-1} \end{vmatrix}$ *so* $T_{-n} \equiv T_{n-1}^2 \equiv (T_{n-2} + T_{n-3})^2 \pmod{T_n}$.

*Proof.* Since $N^{-n} = (N^n)^{-1}$, it follows that

$$\begin{bmatrix} T_{-n+1} & T_{-n} + T_{-n-1} & T_{-n} \\ T_{-n} & T_{-n-1} + T_{-n-2} & T_{-n-1} \\ T_{-n-1} & T_{-n-2} + T_{-n-3} & T_{-n-2} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \end{vmatrix} & -\begin{vmatrix} T_{n-1} & T_n \\ T_{n-3} & T_{n-2} \end{vmatrix} & \begin{vmatrix} T_{n-1} & T_n \\ T_{n-2} & T_{n-1} \end{vmatrix} \\ -\begin{vmatrix} T_n & T_{n-1} \\ T_{n-1} & T_{n-2} \end{vmatrix} & \begin{vmatrix} T_{n+1} & T_n \\ T_{n-1} & T_{n-2} \end{vmatrix} & -\begin{vmatrix} T_{n+1} & T_n \\ T_n & T_{n-1} \end{vmatrix} \\ \begin{vmatrix} T_n & T_{n-1} + T_{n-2} \\ T_{n-1} & T_{n-2} + T_{n-3} \end{vmatrix} & -\begin{vmatrix} T_{n+1} & T_n + T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} \end{vmatrix} & \begin{vmatrix} T_{n+1} & T_n + T_{n-1} \\ T_n & T_{n-1} + T_{n-2} \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \end{vmatrix} & -\begin{vmatrix} T_{n-1} & T_{n} \\ T_{n-3} & T_{n-2} \end{vmatrix} & \begin{vmatrix} T_{n-1} & T_{n} \\ T_{n-2} & T_{n-1} \end{vmatrix} \\ -\begin{vmatrix} T_{n} & T_{n-1} \\ T_{n-1} & T_{n-2} \end{vmatrix} & \begin{vmatrix} T_{n+1} & T_{n} \\ T_{n-1} & T_{n-2} \end{vmatrix} & -\begin{vmatrix} T_{n+1} & T_{n} \\ T_{n} & T_{n-1} \end{vmatrix} \\ \begin{vmatrix} T_{n} & T_{n+1} \\ T_{n-1} & T_{n} \end{vmatrix} & -\begin{vmatrix} T_{n+1} & T_{n+2} \\ T_{n-1} & T_{n} \end{vmatrix} & \begin{vmatrix} T_{n+1} & T_{n+2} \\ T_{n} & T_{n+1} \end{vmatrix} \end{bmatrix}$$

Hence we have $T_{-n} = \begin{vmatrix} T_{n-1} & T_n \\ T_{n-2} & T_{n-1} \end{vmatrix}$ for all $n$. Therefore, by mod $T_n$

$$T_{-n} = T_{n-1}^2 - T_n T_{n-2} \equiv T_{n-1}^2 \equiv (T_n - T_{n-2} - T_{n-3})^2 \equiv (T_{n-2} + T_{n-3})^2 \ . \qquad \square$$

**Example 1.** It is easy to see $T_{-10} \equiv T_9^2 \pmod{T_{10}} = 81^2 \pmod{149} \equiv 5$. Precisely $T_{-10} = \begin{vmatrix} T_9 & T_{10} \\ T_8 & T_9 \end{vmatrix} = \begin{vmatrix} 81 & 149 \\ 44 & 81 \end{vmatrix} = 5$.

Furthermore comparing Fibonacci and tribonacci, we can write

$$T_{n-1}^2 - T_n T_{n-2} = T_{-n} \quad \text{while} \quad F_{n-1}^2 - F_n F_{n-2} = (-1)^{n-1}.$$

The Fibonacci sequence is known for its connection to the Pascal's triangle.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | |
| | 1 | 1 | | | | | | | | |
| | | 1 | 2 | 1 | | | | | | |
| | | | 1 | 3 | 3 | 1 | | | | |
| | | | | 1 | 4 | 6 | 4 | 1 | | |
| | | | | | 1 | 5 | 10 | 10 | 5 | $\cdots$ |
| | $\cdots$ | | | | | | | | $\cdots$ | |

Upon using the binomial coefficients $C_n^k = \binom{n}{k}$, this table can be written as

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_0^0$ | | | | | | | | | | | |
| | $C_1^0$ | $C_1^1$ | | | | | | | | | |
| | | $C_2^0$ | $C_2^1$ | $C_2^2$ | | | | | | | |
| | | | $C_3^0$ | $C_3^1$ | $C_3^2$ | $C_3^3$ | | | | | |
| | | | | $C_4^0$ | $C_4^1$ | $C_4^2$ | $C_4^3$ | $C_4^4$ | | | |
| | | | | | $C_5^0$ | $C_5^1$ | $C_5^2$ | $C_5^3$ | $C_5^4$ | $\cdots$ | |
| + | | $\vdots$ | | | | | | | | $\vdots$ | |
| $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $\cdots$ | | | |

it thus follows that $F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} C_{n-i}^i = \sum_{i=0}^{n} C_{n-i}^i$.

An analog of tribonacci numbers and binomial coefficients is as follows.

**Theorem 2.3.** *If $n \geq 0$ is even, then*

$$T_n = C_{n-1}^0 C_0^0 + [C_{n-2}^1 \ C_{n-3}^1] \begin{bmatrix} C_1^0 \\ C_1^1 \end{bmatrix} + [C_{n-3}^2 \ C_{n-4}^2 \ C_{n-5}^2] \begin{bmatrix} C_2^0 \\ C_2^1 \\ C_2^2 \end{bmatrix}$$

$$+[C^3_{n-4}\ C^3_{n-5}\ C^3_{n-6}\ \ C^3_{n-7}]\begin{bmatrix} C^0_3 \\ C^1_3 \\ C^2_3 \\ C^3_3 \end{bmatrix} + \cdots + \begin{bmatrix} C^{\frac{n}{2}-2}_{\frac{n}{2}+1} C^{\frac{n}{2}-2}_{\frac{n}{2}} C^{\frac{n}{2}-2}_{\frac{n}{2}-1} C^{\frac{n}{2}-2}_{\frac{n}{2}-2} \end{bmatrix} \begin{bmatrix} C^0_{\frac{n}{2}-2} \\ C^1_{\frac{n}{2}-2} \\ C^2_{\frac{n}{2}-2} \\ C^3_{\frac{n}{2}-2} \end{bmatrix}$$

$$+\begin{bmatrix} C^{\frac{n}{2}-1}_{\frac{n}{2}}\ C^{\frac{n}{2}-1}_{\frac{n}{2}-1} \end{bmatrix} \begin{bmatrix} C^0_{\frac{n}{2}-1} \\ C^1_{\frac{n}{2}-1} \end{bmatrix}$$

$$= C^0_{n-1}C^0_0 + \sum_{i=0}^{1} C^1_{n-2-i}C^i_1 + \sum_{i=0}^{2} C^2_{n-3-i}C^i_2 + \cdots + \sum_{i=0}^{3} C^{\frac{n}{2}-2}_{\frac{n}{2}+1-i}C^i_{\frac{n}{2}-2} + \sum_{i=0}^{1} C^{\frac{n}{2}-1}_{\frac{n}{2}-i}C^i_{\frac{n}{2}-1}.$$

*If $n > 0$ is odd, then*

$$T_n = C^0_{n-1}C^0_0 + [C^1_{n-2}\ \ C^1_{n-3}]\begin{bmatrix} C^0_1 \\ C^1_1 \end{bmatrix} + [C^2_{n-3}C^2_{n-4}C^2_{n-5}]\begin{bmatrix} C^0_2 \\ C^1_2 \\ C^2_2 \end{bmatrix}$$

$$+[C^3_{n-4}C^3_{n-5}C^3_{n-6}C^3_{n-7}]\begin{bmatrix} C^0_3 \\ C^1_3 \\ C^2_3 \\ C^3_3 \end{bmatrix} + \cdots + \begin{bmatrix} C^{\lfloor\frac{n}{2}\rfloor-1}_{\lfloor\frac{n}{2}\rfloor+1} & C^{\lfloor\frac{n}{2}\rfloor-1}_{\lfloor\frac{n}{2}\rfloor} & C^{\lfloor\frac{n}{2}\rfloor-1}_{\lfloor\frac{n}{2}\rfloor-1} \end{bmatrix} \begin{bmatrix} C^0_{\lfloor\frac{n}{2}\rfloor-1} \\ C^1_{\lfloor\frac{n}{2}\rfloor-1} \\ C^2_{\lfloor\frac{n}{2}\rfloor-1} \end{bmatrix}$$

$$+C^{\lfloor\frac{n}{2}\rfloor}_{\lfloor\frac{n}{2}\rfloor}C^0_{\lfloor\frac{n}{2}\rfloor}$$

$$= C^0_{n-1}C^0_0 + \sum_{i=0}^{1} C^1_{n-2-i}C^i_1 + \sum_{i=0}^{2} C^2_{n-3-i}C^i_2 + \cdots + \sum_{i=0}^{3} C^{\lfloor\frac{n}{2}\rfloor-1}_{\lfloor\frac{n}{2}\rfloor+1-i}C^i_{\lfloor\frac{n}{2}\rfloor-1} + C^{\lfloor\frac{n}{2}\rfloor}_{\lfloor\frac{n}{2}\rfloor}C^0_{\lfloor\frac{n}{2}\rfloor}.$$

*Proof.* As an analog of the Pascal triangle, we consider the following table. Then we can see that the sum of each column produces tribonacci numbers as:

$T_1 = C^0_0C^0_0 = 1 = C^0_1C^0_0 = T_2,$

$T_3 = C^0_2C^0_0 + C^1_1C^0_1 = 2,$

$T_4 = C^0_3C^0_0 + C^1_2C^0_1 + C^1_1C^1_1 = 4,$

$T_5 = C^0_4C^0_0 + C^1_3C^0_1 + C^1_2C^1_1 + C^2_2C^0_2 = 7,$

$T_6 = C^0_5C^0_0 + C^1_4C^0_1 + C^1_3C^1_1 + C^2_3C^0_2 + C^2_2C^1_2 = 13,$

$T_7 = C^0_6C^0_0 + C^1_5C^0_1 + C^1_4C^1_1 + C^2_4C^0_2 + C^2_3C^1_2 + C^2_2C^2_2 + C^3_3C^0_3 = 24,$

$T_8 = C^0_7C^0_0 + C^1_6C^0_1 + C^1_5C^1_1 + C^2_5C^0_2 + C^2_4C^1_2 + C^2_3C^2_2 + C^3_4C^0_3 + C^3_3C^1_3 = 44,$

$T_9 = C^0_8C^0_0 + C^1_7C^0_1 + C^1_6C^1_1 + C^2_6C^0_2 + C^2_5C^1_2 + C^2_4C^2_2 + C^3_5C^0_3 + C^3_4C^1_3 + C^3_3C^2_3 + C^4_4C^0_4.$

Moreover these identities can be expressed by matrices that

$$T_4 = C^0_3C^0_0 + [C^1_2\ C^1_1]\begin{bmatrix} C^0_1 \\ C^1_1 \end{bmatrix}, \qquad T_5 = C^0_4C^0_0 + [C^1_3\ C^1_2]\begin{bmatrix} C^0_1 \\ C^1_1 \end{bmatrix} + C^2_2C^0_2,$$

$$T_6 = C^0_5C^0_0 + [C^1_4\ C^1_3]\begin{bmatrix} C^0_1 \\ C^1_1 \end{bmatrix} + [C^2_3\ C^2_2]\begin{bmatrix} C^0_2 \\ C^1_2 \end{bmatrix} + C^2_2C^1_2,$$

$$T_7 = C^0_6C^0_0 + [C^1_5\ C^1_4]\begin{bmatrix} C^0_1 \\ C^1_1 \end{bmatrix} + [C^2_4\ C^2_3\ \ C^2_2]\begin{bmatrix} C^0_2 \\ C^1_2 \\ C^2_2 \end{bmatrix} + C^3_3C^0_3,$$

$$T_8 = C_7^0 C_0^0 + [C_6^1\ C_5^1] \begin{bmatrix} C_1^0 \\ C_1^1 \end{bmatrix} + [C_5^2\ C_4^2\ \ C_3^2] \begin{bmatrix} C_2^0 \\ C_2^1 \\ C_2^2 \end{bmatrix} + [C_4^3\ C_3^3] \begin{bmatrix} C_3^0 \\ C_3^1 \end{bmatrix},$$

$$T_9 = C_8^0 C_0^0 + [C_7^1 C_6^1] \begin{bmatrix} C_1^0 \\ C_1^1 \end{bmatrix} + [C_6^2 C_5^2 C_4^2] \begin{bmatrix} C_2^0 \\ C_2^1 \\ C_2^2 \end{bmatrix} + [C_5^3 C_4^3 C_3^3] \begin{bmatrix} C_3^0 \\ C_3^1 \\ C_3^2 \end{bmatrix} + C_4^4 C_4^0.$$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_0^0C_0^0$ | $C_1^0C_0^0$ | $C_1^1C_1^0$ | $C_1^1C_1^1$ | | | | | | | |
| | | $C_2^0C_0^0$ | $C_2^1C_1^0$ | $C_2^1C_1^1$<br>$C_2^2C_2^0$ | $C_2^2C_2^1$ | $C_2^2C_2^2$ | | | | |
| | | | $C_3^0C_0^0$ | $C_3^1C_1^0$ | $C_3^1C_1^1$<br>$C_3^2C_2^0$ | $C_3^2C_2^1$<br>$C_3^3C_3^0$ | $C_3^2C_2^2$<br>$C_3^3C_3^1$ | $C_3^3C_3^2$ | $C_3^3C_3^3$ | |
| | | | | $C_4^0C_0^0$ | $C_4^1C_1^0$ | $C_4^1C_1^1$<br>$C_4^2C_2^0$ | $C_4^2C_2^1$<br>$C_4^3C_3^0$ | $C_4^2C_2^2$<br>$C_4^3C_3^1$<br>$C_4^4C_4^0$ | $C_4^3C_3^2$<br>$C_4^4C_4^1$ | $C_4^3C_3^3$<br>$C_4^4C_4^2$ |
| | | | | | $C_5^0C_0^0$ | $C_5^1C_1^0$ | $C_5^1C_1^1$<br>$C_5^2C_2^0$ | $C_5^2C_2^1$<br>$C_5^3C_3^0$ | $C_5^2C_2^2$<br>$C_5^3C_3^1$<br>$C_5^4C_4^0$ | $C_5^3C_3^2$<br>$C_5^4C_4^1$<br>$C_5^5C_5^0$ |
| | | | | | | $C_6^0C_0^0$ | $C_6^1C_1^0$ | $C_6^1C_1^1$<br>$C_6^2C_2^0$ | $C_6^2C_2^1$<br>$C_6^3C_3^0$ | $C_6^2C_2^2$<br>$C_6^3C_3^1$<br>$C_6^4C_4^0$ |
| + | $\vdots$ | | | | | | $C_7^0C_0^0$ | $C_7^1C_1^0$ | $C_7^1C_1^1$<br>$C_7^2C_2^0$<br>$\vdots$ | $C_7^2C_2^1$<br>$C_7^3C_3^0$<br>$\vdots$ |
| 1<br>$T_1$ | 1<br>$T_2$ | 2<br>$T_3$ | 4<br>$T_4$ | 7<br>$T_5$ | 13<br>$T_6$ | 24<br>$T_7$ | 44<br>$T_8$ | 81<br>$T_9$ | 149<br>$T_{10}$ | 274<br>$T_{11}\ \cdots$ |

Furthermore it can be written by

$$T_4 = \sum_{i=0}^{0} C_{3-i}^0 C_0^i + \sum_{i=0}^{1} C_{2-i}^1 C_1^i, \qquad T_5 = \sum_{i=0}^{0} C_{4-i}^0 C_0^i + \sum_{i=0}^{1} C_{3-i}^1 C_1^i + \sum_{i=0}^{0} C_{2-i}^2 C_2^i,$$

$$T_6 = \sum_{i=0}^{0} C_{5-i}^0 C_0^i + \sum_{i=0}^{1} C_{4-i}^1 C_1^i + \sum_{i=0}^{1} C_{3-i}^2 C_2^i,$$

$$T_7 = \sum_{i=0}^{0} C_{6-i}^0 C_0^i + \sum_{i=0}^{1} C_{5-i}^1 C_1^i + \sum_{i=0}^{2} C_{4-i}^2 C_2^i + \sum_{i=0}^{0} C_{3-i}^3 C_3^i,$$

$$T_8 = \sum_{i=0}^{0} C_{7-i}^0 C_0^i + \sum_{i=0}^{1} C_{6-i}^1 C_1^i + \sum_{i=0}^{2} C_{5-i}^2 C_2^i + \sum_{i=0}^{1} C_{4-i}^3 C_3^i,$$

$$T_9 = \sum_{i=0}^{0} C_{8-i}^{0} C_0^i + \sum_{i=0}^{1} C_{7-i}^{1} C_1^i + \sum_{i=0}^{2} C_{6-i}^{2} C_2^i + \sum_{i=0}^{2} C_{5-i}^{3} C_3^i + \sum_{i=0}^{0} C_{4-i}^{4} C_4^i.$$

Hence this can be proved for every $n$, so that the theorem holds. □

The author would like to thank the referee for providing information about new research for binomial expression of the tribonacci numbers [7].

## 3. Tribonacci Table and Tribonacci Matrix

In this section, we display the tribonacci sequence in a rectangle form with $k > 0$ columns. We call the rectangle composed of tribonacci numbers the $k$ columns tribonacci table. We begin to consider the 4 columns tribonacci table

| | | | |
|---|---|---|---|
| 1 | 1 | 2 | 4 |
| 7 | 13 | 24 | 44 |
| 81 | 149 | 274 | 504 |
| 927 | 1705 | 3136 | $\cdots$ |

Then we find that, for instance

$T_{19} = (11)3136 + (5)274 + 24 = (3T_4 - 1)T_{15} + (T_4 + 1)T_{11} + T_7 = 35890.$

Similarly from the 5 columns tribonacci table

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 7 |
| 13 | 24 | 44 | 81 | 149 |
| 274 | 504 | 927 | 1705 | 3136 |
| 5768 | 10609 | 19513 | 35890 | $\cdots$ |

we also can see that, for instance

$T_{23} = (21)19513 + 927 + 44 = (3T_5)T_{18} + T_{13} + T_8 = 410744.$

Moreover from the 6 columns tribonacci table

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 7 | 13 |
| 24 | 44 | 81 | 149 | 274 | 504 |
| 927 | 1705 | 3136 | 5768 | 10609 | 19513 |
| 35890 | 66012 | 121415 | 223317 | 410744 | $\cdots$ |

it can be seen that, for instance

$T_{22} = (39)5768 - (13 - 2)149 + 4 = (3T_6)T_{16} - (T_6 - 2)T_{10} + T_4 = 223317.$

**Theorem 3.1.** *Let $n = kt + r$ $(1 \le r \le k)$. Assume $4 \le k \le 10$. Then*

*(1) $T_n = T_{kt+r} = \mu_1 T_{k(t-1)+r} + \mu_2 T_{k(t-2)+r} + \mu_3 T_{k(t-3)+r}$,*

*where the coefficients $(\mu_1, \mu_2, \mu_3)$ depending on $k$ are as follows*

| | $k = 4$ | $k = 5$ | $k = 6$ |
|---|---|---|---|
| $(\mu_1, \mu_2, \mu_3)$ | $(3T_4 - 1, T_4 + 1, 1)$ | $(3T_5, 1, 1)$ | $(3T_6, -T_6 + 2, 1)$ |
| $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
| $(3T_7 - 1, 15, 1)$ | $(3T_8 - 1, -3, 1)$ | $(3T_9 - 2, -23, 1)$ | $(3T_{10} - 4, 41, 1)$ |

*(2) $T_{kt+r}$ is a linear sum of $T_r$ in the 1st row, $T_{k+r}$ in the 2nd row and $T_{2k+r}$ in the 3rd row of the table, and these belong to the same $r$th column.*

*Proof.* It is due to the above observations and mathematical induction. □

In [6], the identity $T_{4(n+1)} = 11T_{4n} + 5T_{4(n-1)} + T_{4(n-2)}$ was proved. This is the case only for $k = 4$. Theorem 3.1 gives the identities for all $4 \le k \le 10$.

**Corollary 3.2.** *For $5 \le k \le 10$, $\{T_{kn}\}$ are as follows.*
$T_{5(n+1)} = 21T_{5n} + T_{5(n-1)} + T_{5(n-2)}$, $T_{6(n+1)} = 39T_{6n} - 11T_{6(n-1)} + T_{6(n-2)}$,
$T_{7(n+1)} = 71T_{7n} + 15T_{7(n-1)} + T_{7(n-2)}$, $T_{8(n+1)} = 131T_{8n} - 3T_{8(n-1)} + T_{8(n-2)}$,
$T_{9(n+1)} = 241T_{9n} - 23T_{9(n-1)} + T_{9(n-2)}$, $T_{10(n+1)} = 443T_{10n} + 41T_{10(n-1)} + T_{10(n-2)}$.

**Example 2.** Consider $T_{50}$. By taking $k = 7$ for instance, we have

$$T_{50} = T_{7(7)+1} = \mu_1 T_{7(6)+1} + \mu_2 T_{7(5)+1} + \mu_3 T_{7(4)+1}$$

with $(\mu_1, \mu_2, \mu_3) = (3T_7 - 1,\ 15,\ 1) = (71, 15, 1)$. So it follows immediately

$$\begin{aligned} T_{50} &= 71T_{7(6)+1} + 15T_{7(5)+1} + T_{7(4)+1} \\ &= (71 \cdot 71 + 15)T_{7(5)+1} + (15 \cdot 71 + 1)T_{7(4)+1} + 71T_{7(3)+1} \\ &= 5056\ T_{7(5)+1} + 1066\ T_{7(4)+1} + 71\ T_{7(3)+1} \\ &= 360042\ T_{7(4)+1} + 75911\ T_{7(3)+1} + 5056\ T_{7(2)+1} \\ &= 25638893\ T_{7(3)+1} + 5405686\ T_{7(2)+1} + 360042\ T_{7(1)+1} \\ &= 1825767089\ T_{7(2)+1} + 384943437\ T_{7+1} + 25638893\ T_1 = 5,742,568,741,225 \end{aligned}$$

by plugging $T_{7(2)+1} = 3136$, $T_{7+1} = 44$ and $T_1 = 1$.

Now taking tribonacci number $T_{kt+r}$ by modular tribonacci number $T_k$, the next theorem follows from Theorem 3.1.

**Theorem 3.3.** *Let $n = kt + r$ $(1 \le r \le k)$ and $4 \le k \le 10$. By mod $T_k$,*

$$T_{kt+r} \equiv \nu_1 T_{k(t-1)+r} + \nu_2 T_{k(t-2)+r} + \nu_3 T_{k(t-3)+r} \pmod{T_k}$$

*where the coefficients $(\nu_1, \nu_2, \nu_3)$ are*

| $k$ | $(\nu_1, \nu_2, \nu_3)$ | $k$ | $(\nu_1, \nu_2, \nu_3)$ | $k$ | $(\nu_1, \nu_2, \nu_3)$ | $k$ | $(\nu_1, \nu_2, \nu_3)$ |
|---|---|---|---|---|---|---|---|
| 4 | $(-1, 1, 1)$ | 5 | $(0, 1, 1)$ | 6 | $(0, 2, 1)$ | 7 | $(-1, 15, 1)$ |
| 8 | $(-1, -3, 1)$ | 9 | $(-2, -23, 1)$ | 10 | $(-4, 41, 1)$ | | |

**Example 3.** Take $k = 5$ for instance. Then $T_{50} \pmod{T_5 = 7}$ is

$$\begin{aligned} T_{50} = T_{5\cdot9+5} &\equiv T_{5\cdot7+5} + T_{5\cdot6+5} \equiv (T_{5\cdot5+5} + T_{5\cdot4+5}) + T_{5\cdot6+5} \\ &\equiv T_{5\cdot6+5} + T_{5\cdot5+5} + T_{5\cdot4+5} \equiv (T_{5\cdot4+5} + T_{5\cdot3+5}) + T_{5\cdot5+5} + T_{5\cdot4+5} \\ &\equiv T_{5\cdot5+5} + 2T_{5\cdot4+5} + T_{5\cdot3+5} \equiv 2T_{5\cdot4+5} + 2T_{5\cdot3+5} + T_{5\cdot2+5} \end{aligned}$$

$$\equiv 2T_{5\cdot 3+5} + 3T_{5\cdot 2+5} + 2T_{5+5} \equiv 3T_{5\cdot 2+5} + 4T_{5+5} + 2T_5 \equiv 1.$$

If we regard the $k$ columns tribonacci table as a matrix with $k$ columns, we may treat $T_{kt+r}$ $(1 \le r \le k)$ as the entry $e_{(t+1,r)}$ at the place of $(t+1)$th row and $r$th column in the matrix. Thus due to Theorem 3.1, it can be written as

$$T_{kt+r} = e_{(t+1,r)} = \mu_1 e_{(t,r)} + \mu_2 e_{(t-1,r)} + \mu_3 e_{(t-2,r)},$$

i.e., $T_{kt+r}$ is a linear sum of three successive entries in the same $r$th column. The coefficients $(\mu_1, \mu_2, \mu_3)$ are strongly dependent on the number $k$.

**Theorem 3.4.** *Let $n = 4t + r$ $(1 \le r \le 4)$ and $u \in \mathbb{Z}$. Let $A_{(4)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $X_{(4)} = [1\ 1\ -1]$. Then in the 4 columns tribonacci matrix (mod $T_4 = 4$)*

$$T_{4t+r} \equiv X_{(4)} \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(4)} A_{(4)}^u \begin{bmatrix} e_{(t-u-2,r)} \\ e_{(t-u-1,r)} \\ e_{(t-u,r)} \end{bmatrix} \equiv X_{(4)} A_{(4)}^{t-3} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}.$$

*Proof.* In the 4 columns tribonacci matrix, Theorem 3.3 shows that

$$\begin{aligned} T_{4t+r} &= e_{(t+1,r)} \equiv -e_{(t,r)} + e_{(t-1,r)} + e_{(t-2,r)} \equiv 2e_{(t-1,r)} - e_{(t-3,r)} \\ &\equiv -2e_{(t-2,r)} + e_{(t-3,r)} + 2e_{(t-4,r)} \equiv 3e_{(t-3,r)} - 2e_{(t-5,r)} \equiv \cdots \end{aligned}$$

by mod $T_4$. By making use of $X_{(4)}$ and $A_{(4)}$, it can be written by

$$\begin{aligned} T_{4t+r} &\equiv X_{(4)} \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(4)} A_{(4)} \begin{bmatrix} e_{(t-3,r)} \\ e_{(t-2,r)} \\ e_{(t-1,r)} \end{bmatrix} \equiv X_{(4)} A_{(4)}^2 \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} \\ &\equiv \cdots \equiv X_{(4)} A_{(4)}^u \begin{bmatrix} e_{(t-u-2,r)} \\ e_{(t-u-1,r)} \\ e_{(t-u,r)} \end{bmatrix} \equiv X_{(4)} A_{(4)}^{t-3} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}. \end{aligned}$$

□

**Theorem 3.5.** *Let $A_{(5)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $A_{(6)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $X_{(5)} = [1\ 1\ 1]$ and $X_{(6)} = [2\ 2^2\ 1]$. If $n = 5t + r$ $(1 \le r \le 5)$, then in the 5 columns matrix (mod $T_5$),*

$$T_{5t+r} = [1\ 1\ 0] \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(5)} A_{(5)}^u \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix} \equiv X_{(5)} A_{(5)}^{t-5} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}.$$

*Similarly, if $n = 6t + r$ $(1 \le r \le 6)$, then in the 6 columns matrix (mod $T_6$),*

$$T_{6t+r} \equiv [1\ 2\ 0] \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(6)} A_{(6)}^u \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix} \equiv X_{(6)} A_{(6)}^{t-5} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}.$$

*Proof.* In the 5 columns matrix, Theorem 3.3 shows by mod $T_5 = 7$ that

$$\begin{aligned} T_{5t+r} &= e_{(t+1,r)} \equiv e_{(t-1,r)} + e_{(t-2,r)} \equiv e_{(t-2,r)} + e_{(t-3,r)} + e_{(t-4,r)} \\ &\equiv e_{(t-3,r)} + 2e_{(t-4,r)} + e_{(t-5,r)} \equiv 2e_{(t-4,r)} + 2e_{(t-5,r)} + e_{(t-6,r)} \\ &\equiv 2e_{(t-5,r)} + 3e_{(t-6,r)} + 2e_{(t-7,r)} \equiv 3e_{(t-6,r)} + 4e_{(t-7,r)} + 2e_{(t-8,r)}. \end{aligned}$$

By means of $X_{(5)}$ and $A_{(5)}$, these identities can be expressed by

$$\begin{aligned} T_{5t+r} &\equiv [1\ 1\ 0] \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv [1\ 1\ 1] \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} = X_{(5)} \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} \\ &\equiv X_{(5)} A_{(5)} \begin{bmatrix} e_{(t-5,r)} \\ e_{(t-4,r)} \\ e_{(t-3,r)} \end{bmatrix} \equiv X_{(5)} A_{(5)}^2 \begin{bmatrix} e_{(t-6,r)} \\ e_{(t-5,r)} \\ e_{(t-4,r)} \end{bmatrix} \equiv \cdots \\ &\equiv X_{(5)} A_{(5)}^u \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix} \equiv \cdots \equiv X_{(5)} A_{(5)}^{t-5} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}. \end{aligned}$$

Similarly $T_{6t+r} \equiv 2T_{6(t-2)+r} + T_{6(t-3)+r} \pmod{T_6 = 13}$ shows

$$\begin{aligned} T_{6t+r} &\equiv e_{(t+1,r)} \equiv 2e_{(t-1,r)} + e_{(t-2,r)} \\ &\equiv e_{(t-2,r)} + 2^2 e_{(t-3,r)} + 2e_{(t-4,r)} \equiv 2^2 e_{(t-3,r)} + 2^2 e_{(t-4,r)} + e_{(t-5,r)} \\ &\equiv 2^2 e_{(t-4,r)} + (2^3+1) e_{(t-5,r)} + 2^2 e_{(t-6,r)} \\ &\equiv (2^3+1) e_{(t-5,r)} + (2^3+2^2) e_{(t-6,r)} + 2^2 e_{(t-7,r)} \\ &\equiv (2^3+2^2) e_{(t-6,r)} + (2(2^3+1)+2^2) e_{(t-7,r)} + (2^3+1) e_{(t-8,r)}. \end{aligned}$$

Hence in terms of $X_{(6)}$ and $A_{(6)}$, this is equivalent to write

$$\begin{aligned} T_{6t+r} &\equiv [1\ \ 2\ \ 0] \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(6)} \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} \\ &\equiv X_{(6)} A_{(6)} \begin{bmatrix} e_{(t-5,r)} \\ e_{(t-4,r)} \\ e_{(t-3,r)} \end{bmatrix} \equiv X_{(6)} A_{(6)}^u \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix} \equiv X_{(6)} A_{(6)}^{t-5} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}. \qquad \square \end{aligned}$$

**Theorem 3.6.** *Let* $X_{(k)} = [\nu_3\ \nu_2\ \nu_1]$ *and* $A_{(k)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \nu_3 & \nu_2 & \nu_1 \end{bmatrix}$ *where* $4 \le k \le 10$ *and* $\nu_i$ *are in Theorem 3.3. Then in the* $k$ *columns tribonacci matrix,*

$$T_{kt+r} \equiv X_{(k)} \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(k)} A_{(k)}^{t-3} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}.$$

*Proof.* In the $k$ columns tribonacci matrix, it follows that

$$\begin{aligned} T_{kt+r} &= e_{(t+1,r)} \equiv \nu_1 e_{(t,r)} + \nu_2 e_{(t-1,r)} + \nu_3 e_{(t-2,r)} \\ &\equiv (\nu_1^2 + \nu_2) e_{(t-1,r)} + (\nu_1\nu_2 + \nu_3) e_{(t-2,r)} + \nu_1\nu_3 e_{(t-3,r)} \\ &\equiv (\nu_1^3 + 2\nu_1\nu_2 + \nu_3) e_{(t-2,r)} + (\nu_2(\nu_1^2+\nu_2) + \nu_1\nu_3) e_{(t-3,r)} + \nu_3(\nu_1^2+\nu_2) e_{(t-4,r)}. \end{aligned}$$

These identities can be expressed as

$$T_{kt+r} \equiv X_{(k)} \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv X_{(k)} A_{(k)} \begin{bmatrix} e_{(t-3,r)} \\ e_{(t-2,r)} \\ e_{(t-1,r)} \end{bmatrix} \equiv X_{(k)} A_{(k)}^2 \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix}$$

$$\equiv X_{(k)} A_{(k)}^u \begin{bmatrix} e_{(t-u-2,r)} \\ e_{(t-u-1,r)} \\ e_{(t-u,r)} \end{bmatrix} \equiv \cdots \equiv X_{(k)} A_{(k)}^{t-3} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}. \qquad \square$$

**Example 4.** $T_{50} = T_{4(12)+2} \pmod{T_4 = 4}$ in the 4 columns matrix is

$$T_{50} \equiv [1\ 1\ -1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^9 \begin{bmatrix} e_{(1,2)} \\ e_{(2,2)} \\ e_{(3,2)} \end{bmatrix} \equiv [1\ 1\ -1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \equiv 1.$$

Also in the 5 columns matrix, $T_{50} = T_{5(9)+5} \pmod{T_5 = 7}$ is

$$T_{50} \equiv [1\ 1\ 1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^4 \begin{bmatrix} e_{(1,5)} \\ e_{(2,5)} \\ e_{(3,5)} \end{bmatrix} \equiv [1\ 1\ 1] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \equiv 1.$$

Similarly in the 6 columns tribonacci matrix, we are able to show

$$T_{50} = T_{6(8)+2} = e_{(9,2)} \equiv 4 \pmod{T_6 = 13}.$$

Comparing to Example 3, this makes it easier to have modular tribonacci.

## 4. Period of the Tribonacci Sequences

The smallest number $h$ is called the period of the tribonacci sequence by mod $n$ denoting by $h = \mathrm{per}_T(n)$ such that $T_{h-1} \equiv T_h \equiv 0$ and $T_{h+1} \equiv 1 \pmod{n}$. Refer to [3] and [8] for the period of Fibonacci sequence. By the order of matrix $M$ by mod $n$, we mean the smallest number $u$ to be $M^u \equiv I \pmod{n}$ ($I$ the identity matrix). We denote it by $u = o(M \mod n)$. And the smallest number $s$ satisfying $v^s \equiv 1 \pmod{n}$ is called the order of $v \in \mathbb{Z}$ by mod $n$, and is denoted by $s = o(v \mod n)$. Let us consider the period of tribonacci sequences by mod tribonacci $T_k$.

**Lemma 4.1.** *The matrices $A_{(4)}$, $A_{(5)}$ and $A_{(6)}$ in Theorem 3.4 and 3.5 are of order $o(A_{(4)}$ mod $T_4) = 8$, $o(A_{(5)}$ mod $T_5) = 48$ and $o(A_{(6)}$ mod $T_6) = 28$.*

*Proof.* By some matrix calculation, it is easy to see that, by mod $T_4 = 4$,

$$A_{(4)}^2 \equiv \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \ A_{(4)}^4 \equiv \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \text{ and } A_{(4)}^8 \equiv I,$$

hence $o(A_{(4)}$ *mod* $T_4) = 8$. Moreover by mod $T_5 = 7$,

$$A_{(5)}^4 \equiv \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \ A_{(5)}^8 \equiv \begin{bmatrix} 2 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 5 & 4 \end{bmatrix}, \text{ and } A_{(5)}^{16} \equiv \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I,$$

thus $A_{(5)}^{32} \equiv 2^2 I$ and $A_{(5)}^{48} \equiv 2^3 I \equiv I$, so $o(A_{(5)}\ mod\ T_5) = 48$.

On the other hand, by mod $T_6 = 13$,

$$A_{(6)}^4 \equiv \begin{bmatrix} 0 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 4 & 4 \end{bmatrix},\ A_{(6)}^7 \equiv \begin{bmatrix} 4 & 9 & 4 \\ 4 & 12 & 9 \\ 9 & 9 & 12 \end{bmatrix}, \text{ and } A_{(6)}^{14} \equiv \begin{bmatrix} -3 & -2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix},$$

so $A_{(6)}^{28} \equiv I$ and $o(A_{(6)}\ mod\ T_6) = 28$. □

**Theorem 4.2.** *For any* $n \in \mathbb{Z}$,

$T_n \equiv T_{n+4\cdot 8} \pmod{T_4}$, $T_n \equiv T_{n+5\cdot 48} \pmod{T_5}$ *and* $T_n \equiv T_{n+6\cdot 28} \pmod{T_6}$.

*Proof.* For $n = 4t + r$ $(1 \le r \le 4)$ and $u \in \mathbb{Z}$, Theorem 3.5 says

$$T_n = T_{4t+r} = e_{(t+1,r)} \equiv X_{(4)} A_{(4)}^u \begin{bmatrix} e_{(t-u-2,r)} \\ e_{(t-u-1,r)} \\ e_{(t-u,r)} \end{bmatrix} \pmod{T_4}$$

with $X_{(4)} = [1\ 1\ -1]$. Since $A_{(4)}^8 \equiv I$ in Lemma 4.1, we have

$$T_{4t+r} \equiv X_{(4)} A_{(4)}^8 \begin{bmatrix} e_{(t-10,r)} \\ e_{(t-9,r)} \\ e_{(t-8,r)} \end{bmatrix} \equiv X_{(4)} \begin{bmatrix} e_{(t-10,r)} \\ e_{(t-9,r)} \\ e_{(t-8,r)} \end{bmatrix} = T_{4(t-8)+r},$$

so $T_n = T_{4t+r} \equiv T_{4(t\pm 8)+r} = T_{n+4\cdot 8} \pmod{T_4}$. Similarly in Theorem 3.6,

$$T_{5t+r} = e_{(t+1,r)} \equiv X_{(5)} A_{(5)}^u \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix} \pmod{T_5}$$

with $1 \le r \le 5$ and $X_{(5)} = [1\ 1\ 0]$. But since $A_{(5)}^{48} = I$, it follows that

$$T_{5t+r} \equiv X_{(5)} A_{(5)}^{48} \begin{bmatrix} e_{(t-52,r)} \\ e_{(t-51,r)} \\ e_{(t-50,r)} \end{bmatrix} \equiv X_{(5)} \begin{bmatrix} e_{(t-52,r)} \\ e_{(t-51,r)} \\ e_{(t-50,r)} \end{bmatrix} = T_{5(t-48)+r},$$

so $T_n = T_{5t+r} \equiv T_{5(t\pm 48)+r} \equiv T_{n+5\cdot 48} \pmod{T_5}$. Analogously since

$$T_{6t+r} = e_{(t+1,r)} \equiv X_{(6)} A_{(6)}^u \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix}$$

with $1 \le r \le 6$ and $X_{(6)} = [2\ 4\ 1]$, and $A_{(6)}^{28} = I$, we have

$$T_{6t+r} \equiv X_{(6)} A_{(6)}^{28} \begin{bmatrix} e_{(t-32,r)} \\ e_{(t-31,r)} \\ e_{(t-30,r)} \end{bmatrix} \equiv X_{(6)} \begin{bmatrix} e_{(t-32,r)} \\ e_{(t-31,r)} \\ e_{(t-30,r)} \end{bmatrix} = T_{6(t-28)+r},$$

thus $T_n = T_{6t+r} \equiv T_{6(t\pm 28)+r} \equiv T_{n+6\cdot 28} \pmod{T_6}$. □

**Theorem 4.3.** $per_T(T_4)\ |4 \cdot 8$, $per_T(T_5)\ |5 \cdot 48$ *and* $per_T(T_6)\ |6 \cdot 28$. *In fact,* $per_T(T_4) = 8 = o(A_{(4)})$, $per_T(T_5) = 48 = o(A_{(5)})$ *and* $per_T(T_6) = 168 = 6 \cdot o(A_{(6)})$.

*Proof.* Since $T_n \equiv T_{n+4\cdot 8} \pmod{T_4}$ and $8 = o(A_{(4)})$ by Theorem 4.2, the tribonacci sequence by mod $T_4$ is periodic with $\mathrm{per}_T(T_4)\ |4 \cdot 8$. Similarly since $T_n \equiv T_{n+5\cdot 48}$ $\pmod{T_5}$ and $o(A_{(5)}) = 48$, the sequence by mod $T_5$ is periodic with $\mathrm{per}_T(T_5)\ |5 \cdot 48$. Again since $T_n \equiv T_{n+6\cdot 28} \pmod{T_6}$ and $o(A_{(6)}) = 28$, the sequence by mod $T_6$ is periodic with $\mathrm{per}_T(T_6)\ |6 \cdot 28$. □

Furthermore, for matrices $A_{(k)}$ $(7 \le k \le 10)$, we have the followings that

· $o(A_{(7)}\ mod\ T_7) = 208$ and $T_n \equiv T_{n+7\cdot 208} \pmod{T_7}$. So $\mathrm{per}_T(T_7)\ |7 \cdot 208$.

· $o(A_{(8)}\ mod\ T_8) = 440$ and $T_n \equiv T_{n+8\cdot 440} \pmod{T_8}$. So $\mathrm{per}_T(T_8)\ |8 \cdot 440$.

· $o(A_{(9)}\ mod\ T_9) = 39$ and $T_n \equiv T_{n+9\cdot 39} \pmod{T_9}$. So $\mathrm{per}_T(T_9)\ |9 \cdot 39$.

· $o(A_{(10)}\ mod\ T_{10}) = 740$ and $T_n \equiv T_{n+10\cdot 740} \pmod{T_{10}}$. So $\mathrm{per}_T(T_{10})\ |10 \cdot 740$.

This consideration provides a lower and upper bound of the period that

$$o(A_{(k)}) \mid \mathrm{per}_T(T_k) \ \text{ and } \ \mathrm{per}_T(T_k) \mid (k \cdot o(A_{(k)})) \text{ for } \ 4 \le k \le 10.$$

The length of period of tribonacci sequence is usually long, but the periods of tribonacci by tribonacci modules are as follows.

| $k$ | $T_k$ | $\mathrm{per}_T(T_k)$ |
|---|---|---|
| 3 | 2 | $\mathrm{per}_T(2) = 4 = 2^2$ |
| 4 | 4 | $\mathrm{per}_T(4) = 8 = 2^3$ |
| 5 | 7 | $\mathrm{per}_T(7) = 48 = 2^4(3)$ |
| 6 | 13 | $\mathrm{per}_T(13) = 168 = 2^3(3)(7)$ |
| 7 | $24 = 2^3(3)$ | $\mathrm{per}_T(24) = 208 = 2^4(13)$ |
| 8 | $44 = 2^2(11)$ | $\mathrm{per}_T(44) = 440 = 2^3(5)(11)$ |
| 9 | $81 = 3^4$ | $\mathrm{per}_T(81) = 351 = 3^3(13)$ |
| 10 | 149 | $\mathrm{per}_T(149) = 7400 = 2^3\ 5^2(37)$ |
| 11 | $274 = 2(137)$ | $\mathrm{per}_T(274) = 75628 = 2^2(7)(37)(73)$ |
| 12 | $504 = 2^3\ 3^2(7)$ | $\mathrm{per}_T(504) = 624 = 2^4(3)(13)$ |
| 13 | $927 = 3^2(103)$ | $\mathrm{per}_T(927) = 662 = 2(331)$ |
| 14 | $1705 = 5(11)(31)$ | $\mathrm{per}_T(1705) > 120,000$ |

Lemma 4.1 shows that the smallest $u$ and $v$ such that $A^u_{(k)} \equiv vI \pmod{T_k}$ are $(u, v) = (8, 1)$ if $k = 4$, while $(16, 2)$ if $k = 5$. The following is useful to determine the period of tribonacci.

**Theorem 4.4.** *Let $(k, u, v)$ be the triple such that $u, v > 0$ are the smallest satisfying $A^u_{(k)} \equiv vI \pmod{T_k}$ for $4 \le k \le 10$. Then $(k, u, v)$ are*

$$(4, 8, 1),\ (5, 16, 2),\ (6, 28, 1),\ (7, 208, 1),\ (8, 440, 1),\ (9, 13, 28),\ (10, 740, 1).$$

*Proof.* From $A^u_{(k)} \equiv vI \pmod{T_k}$, the determinants of both sides yield $v^3 \equiv 1$ $\pmod{T_k}$. By mod $T_k$ $(k = 4, 7, 8, 10)$, it is easy to see that the congruence equation

$v^3 \equiv 1 \pmod{T_k}$ has unique solution $v \equiv 1$. And
$A_{(4)}^{8} \pmod{T_4} \equiv I$, $A_{(7)}^{208} \pmod{T_7} \equiv I$, $A_{(8)}^{440} \pmod{T_8} \equiv I$, $A_{(10)}^{740} \pmod{T_{10}} \equiv I$ yield the triples of integers

$$(k,u,v) = (4,8,1),\ (7,208,1),\ (8,440,1),\ (10,740,1).$$

On the other hand, if $k = 5$ then the equation $v^3 \equiv 1 \pmod{T_5}$ has solutions $\{1,2,4\} \equiv \{2,2^2,2^3\} \pmod{T_5}$, while if $k = 9$ then $v^3 \equiv 1 \pmod{T_9}$ has solutions $\{1,28,55\} \equiv \{28,28^2,28^3\} \pmod{T_9}$ respectively. Thus, since $A_{(5)}^{16} \equiv 2I \pmod{T_5}$ and $A_{(9)}^{13} \equiv 28I \pmod{T_9}$, we have triples

$$(k,u,v) = (5,16,2),\ (9,13,28).$$

In particular when $k = 6$, $v^3 \equiv 1 \pmod{T_6}$ has solution $\{1,3,9\} \equiv \{3,3^2,3^3\}$ (mod $T_6$). But $A_{(6)}^{28} \equiv I$, so we have $(k,u,v) = (6,28,1)$. □

The periods of the tribonacci sequence by mod either $T_5$ or $T_9$ are as follows.

**Corollary 4.5.** *Let $t \in \mathbb{Z}$. In the 5 columns tribonacci matrix, $e_{(t+16,r)} \equiv 2e_{(t,r)}$ and $e_{(t+48,r)} \equiv e_{(t,r)} \pmod{T_5}$ for $1 \le r \le 5$. Similarly in the 9 columns matrix, $e_{(t+13,r)} \equiv 28e_{(t,r)}$ and $e_{(t+39,r)} \equiv e_{(t,r)} \pmod{T_9}$ for $1 \le r \le 9$.*

*Proof.* With respect to $X_{(5)} = [1\ 1\ 1]$, Theorem 3.5 shows

$$e_{(t+1,r)} \equiv X_{(5)} \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} \equiv X_{(5)} A_{(5)}^{u} \begin{bmatrix} e_{(t-u-4,r)} \\ e_{(t-u-3,r)} \\ e_{(t-u-2,r)} \end{bmatrix} \equiv X_{(5)} A_{(5)}^{t-5} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}$$

by mod $T_5$ for any $t,u \in \mathbb{Z}$. Since $A_{(5)}^{16} = 2I$, by plugging $u = 16$, we have

$$e_{(t+1,r)} \equiv X_{(5)} A_{(5)}^{16} \begin{bmatrix} e_{(t-20,r)} \\ e_{(t-19,r)} \\ e_{(t-18,r)} \end{bmatrix} \equiv 2X_{(5)} \begin{bmatrix} e_{(t-20,r)} \\ e_{(t-19,r)} \\ e_{(t-18,r)} \end{bmatrix}.$$

This means that

$$e_{(t+16,r)} \equiv 2X_{(5)} \begin{bmatrix} e_{(t-5,r)} \\ e_{(t-4,r)} \\ e_{(t-3,r)} \end{bmatrix} \equiv 2e_{(t,r)},$$

hence $\dfrac{e_{(t+48,r)}}{e_{(t,r)}} \equiv \dfrac{e_{(t+48,r)}}{e_{(t+32,r)}} \dfrac{e_{(t+32,r)}}{e_{(t+16,r)}} \dfrac{e_{(t+16,r)}}{e_{(t,r)}} \equiv 2^3 \equiv 1 \pmod{T_5 = 7}$.

Similarly in the 9 columns matrix, with $X_{(9)} = [1\ \ -23\ \ -2]$, we have

$$e_{(t+1,r)} \equiv X_{(9)} \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} \equiv X_{(9)} A_{(9)}^{u} \begin{bmatrix} e_{(t-u-2,r)} \\ e_{(t-u-1,r)} \\ e_{(t-u,r)} \end{bmatrix} \equiv X_{(9)} A_{(9)}^{t-3} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \\ e_{(3,r)} \end{bmatrix}.$$

Since $A_{(9)}^{13} \equiv 28I$, it follows

$$e_{(t+1,r)} \equiv X_{(9)} A_{(9)}^{13} \begin{bmatrix} e_{(t-15,r)} \\ e_{(t-14,r)} \\ e_{(t-13,r)} \end{bmatrix} \equiv 28X_{(9)} \begin{bmatrix} e_{(t-15,r)} \\ e_{(t-14,r)} \\ e_{(t-13,r)} \end{bmatrix},$$

so

$$e_{(t+13,r)} \equiv 28X_{(9)} \begin{bmatrix} e_{(t-3,r)} \\ e_{(t-2,r)} \\ e_{(t-1,r)} \end{bmatrix} \equiv 28e_{(t,r)}.$$

Thus, $\dfrac{e_{(t+39,r)}}{e_{(t,r)}} = \dfrac{e_{(t+39,r)}}{e_{(t+26,r)}} \dfrac{e_{(t+26,r)}}{e_{(t+13,r)}} \dfrac{e_{(t+13,r)}}{e_{(t,r)}} \equiv 28^3 \equiv 1 \pmod{T_9 = 81}$. □

**Example 5.** By Theorem 3.1, in the 5 columns tribonacci matrix,

$$T_{99} = T_{5(19)+4} = e_{(20,4)} \equiv [1\ 1\ 0] \begin{bmatrix} e_{(17,4)} \\ e_{(18,4)} \\ e_{(19,4)} \end{bmatrix} \pmod{T_5 = 7}.$$

Since $e_{(1,4)} = 4$, $e_{(2,4)} = 4$ and $e_{(3,4)} = 4$, we have
$e_{(4,4)} = e_{(1,4)} + e_{(2,4)} = 1,\ \ e_{(5,4)} = e_{(2,4)} + e_{(3,4)} = 1,\ \ e_{(6,4)} = e_{(3,4)} + e_{(4,4)} = 5.$
Continuing, we have $e_{(17,4)} = 1$, $e_{(18,4)} = 1$ and $e_{(19,4)} = 6$, so $T_{99} = [1\ 1\ 0] \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \equiv 2$.
However by making use of Corollary 4.5, it follows immediately that

$$\begin{aligned} T_{99} &\equiv [1\ 1\ 1]\ A_{(5)}^{14} \begin{bmatrix} e_{(1,4)} \\ e_{(2,4)} \\ e_{(3,4)} \end{bmatrix} \equiv [1\ 1\ 1]\ A_{(6)}^{16} A_{(6)}^{-2} \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \\ &\equiv 2 \cdot 4\ [1\ 1\ 1]\ A_{(6)}^{-2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \equiv [1\ 1\ 1] \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \equiv 2 \pmod{T_5}. \end{aligned}$$

Note $T_{99} = 53324762928098149064722658$ is 2 (mod 4), 2 (mod 7) and 6 (mod 13).

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## References

1. D.K. Chang: Higher order Fibonacci sequence modulo $M$. *Fibonacci Quarterly* **24** (1986), no. 2, 138-139.
2. D.J. DeCarli: A Generalized Fibonacci sequence over an arbitrary ring. *Fibonacci Quarterly* **8** (1970), no. 2, 182-184.

3. A. Ehrlich: On the periods of the Fibonacci sequence modulo $M$. *Fibonacci Quarterly* **27** (1989), no. 1, 11-13.
4. D.L. Johnson: Fibonacci sequences in groups. *Irish Math. Soc. Bulletin* **56** (2005), 81-85.
5. D. Kalman: Generalized Fibonacci numbers by matrix method. *Fibonacci Quarterly* **20** (1982), no. 1, 73-76.
6. E. Kilic: Tribonacci sequences with certain indices and their sums. *Ars. Combinatorics* **86** (2008), 13-22.
7. E. Kilic & H. Prodinger: Some double binomial sums related to Fibonacci, Pell and generalized ordered $k$-Fibonacci numbers. *Rocky Mountain J. Math.* **43** (2013), no. 3, 975-987.
8. D.D. Wall: Fibonacci series modulo $m$. *Amer. Math. Monthly* **67** (1960), 525-532.
9. H.J. Wilcox: Fibonacci sequences of period $n$ in groups. *Fibonacci Quarterly* **24** (1986), no. 4, 356-361.



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