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COMMON COUPLED FIXED FOINT THEOREMS FOR NONLINEAR CONTRACTIVE CONDITION ON INTUITIONISTIC FUZZY METRIC SPACES WITH APPLICATION TO INTEGRAL EQUATIONS

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Abstract. We establish a common fixed point theorem for mappings under ϕ-contractive conditions on intuitionistic fuzzy metric spaces. As an application of our result we study the existence and uniqueness of the solution to a nonlinear Fredholm integral equation. We also give an example to validate our result.

1. Introduction

Many authors have extensively developed the theory of fuzzy sets and applications. George and Veeramani [6] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [8] and defined the Hausdorff topology of fuzzy metric spaces which has very important applications in quantum particle physics.

Atanassov [2, 3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Alaca et al. [1] using the idea of intuitionistic fuzzy sets defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [8]. In [11] Park generalized the notion of fuzzy metric space given by George and Veeramani [6] with the help of continuous t-norms and continuous t-conorms and introduced the notion of intuitionistic fuzzy metric space.

In [4] Bhaskar and Lakshmikantham introduced the notion of coupled fixed point and mixed monotone mappings and gave some coupled fixed point theorems.

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Bhaskar and Lakshmikantham [4] apply these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [9] introduced the concept of coupled coincidence point and proved some common coupled fixed point theorems. Sedghi et al. [13] gave a coupled fixed point theorem for contractions in fuzzy metric spaces. On the other hand, integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. The potential theory contributed more than any field to give rise to integral equations. Mathematical physics models such as diffraction problems, scattering in quantum mechanics, conformal mapping and water waves also contributed to the creation of integral equations. Many other applications in science and engineering are described by integral equations or integro-differential equations. The Volterra's population growth model, biological species living together, propagation of stocked fish in a new lake, the heat radiation are among many areas that are described by integral equations. Many scientific problems give rise to integral equations with logarithm kernels. Integral equations often arise in electrostatics, low frequency electromagnetic problems, electromagnetic scattering problems and propagation of acoustical and elastical waves.

In this paper, we prove a common fixed point theorem for mappings under ϕ-contractive conditions on intuitionistic fuzzy metric spaces. As an application of our result we study the existence and uniqueness of the solution to a nonlinear Fredholm integral equation, which arise naturally in the theory of signal processing, linear forward modeling and inverse problems. First time effort has been made by us to give application of coupled fixed point theorems in the settings of intuitionistic fuzzy metric spaces to integral equations. We also give an example to validate our result. We extend the result of Sedghi, Altun and Shobe [13]. Our result intuitionistically fuzzifies the result of Hu [16].

2. Preliminaries

Definition 2.1 ([12]). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

A few examples of continuous t-norm are

$$a * b = ab,\ \ a * b = \min\{a, b\} \text{ and } a * b = \max\{a + b - 1, 0\}.$$

Definition 2.2 ([12])**.** A binary operation $\Diamond : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ is continuous t-conorm if it satisfies the following conditions:

(1) $\Diamond$ is commutative and associative,

(2) $\Diamond$ is continuous,

(3) $a\ \Diamond\ 0 = a$ for all $a \in [0,\ 1]$,

(4) $a\ \Diamond\ b \leq c\ \Diamond\ d$ whenever $a \leq c$ and $b \leq d$ for all $a,\ b,\ c,\ d \in [0,\ 1]$.

A few examples of continuous t-conorm are

$$a\Diamond b = a + b - ab,\ \ a\Diamond b = \max\{a, b\} \text{ and } a\Diamond b = \min\{a + b, 1\}.$$

Remark 2.1. The concept of triangular norms (t-norm) and triangular conorms (t-conorm) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger [10] in his study of statistical metric spaces.

Definition 2.3 ([7])**.** Let $\sup_{0<t<1} \Delta(t,\ t) = 1$. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t\Delta t, \Delta^{m+1}(t) = t\Delta(\Delta^m(t)),\ m = 1, 2, ..., t \in [0, 1]. \tag{2.1}$$

The t-norm $\Delta_M = \min$ is an example of t-norm of H-type, but there are some other t-norms Δ of H-type [7].

Obviously, Δ is a H-type t-norm if and only if for any $\lambda \in (0,\ 1)$, there exists $\delta(\lambda) \in (0,\ 1)$ such that $\Delta^m(t) > 1 - \lambda$ for all $m \in N$, when $t > 1 - \delta$.

Definition 2.4 ([15])**.** Let $\inf_{0<t<1} \bigtriangledown(t,\ t) = 0$. A t-conorm $\bigtriangledown$ is said to be of H-type if the family of functions $\{\bigtriangledown^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 0$, where

$$\bigtriangledown^1(t) = t \bigtriangledown t, \bigtriangledown^{m+1}(t) = t \bigtriangledown (\bigtriangledown^m(t)),\ m = 1, 2, ..., t \in [0, 1]. \tag{2.2}$$

The t-conorm $\bigtriangledown_M = \max$ is an example of t-conorm of H-type.

Obviously, $\bigtriangledown$ is a H-type t-conorm if and only if for any $\lambda \in (0,\ 1)$, there exists $\delta(\lambda) \in (0,\ 1)$ such that $\bigtriangledown^m(t) < \lambda$ for all $m \in N$, when $t < \delta$.

Definition 2.5 ([11])**.** A 5-tuple $(X,\ M,\ N,\ *,\ \Diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M and N are fuzzy sets on $X^2 \times [0,\ \infty)$ satisfying the following conditions: for all $x,\ y,\ z \in X$ and $s,\ t > 0$,

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$,
(IFM-2) $M(x, y, t) > 0$,
(IFM-3) $M(x, y, t) = 1$ iff $x = y$,
(IFM-4) $M(x, y, t) = M(y, x, t)$,
(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(IFM-6) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous,
(IFM-7) $N(x, y, t) > 0$,
(IFM-8) $N(x, y, t) = 0$ iff $x = y$,
(IFM-9) $N(x, y, t) = N(y, x, t)$,
(IFM-10) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$,
(IFM-11) $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

Then (M, N) is called an *intuitionistic fuzzy metric* on X. The function $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r, N(x,y,t) < r\}. \tag{2.3}$$

A subset $A \subset X$ is called *open* if, for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let $\Im$ denote the family of all open subsets of X. Then $\Im$ is called the *topology* on X induced by the intuitionistic fuzzy metric (M, N). This topology is Hausdorff and first countable. In [11] Park proved that the family of sets of the form $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$ is a base for the topology $\Im$ on X.

Example 2.1 ([11])**.** Let (X, d) be a metric space. Define t-norm and t-conorm by $a * b = \min\{a, b\}$ and $a \lozenge b = \max\{a, b\}$ respectively. Let

$$M(x,y,t) = \frac{t}{t + d(x,y)} \text{ and } N(x,y,t) = \frac{d(x,y)}{t + d(x,y)}, \text{ for all } x,y \in X \text{ and } t > 0.$$

Then $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric (M, N) induced by the metric d the standard intuitionistic fuzzy metric. It is obvious that $N(x, y, t) = 1 - M(x, y, t)$.

Definition 2.6 ([11])**.** Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space, then

(1) A sequence $\{x_n\}$ in X is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x, t) = 0, \text{ for all } t > 0. \tag{2.4}$$

(2) A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$M(x_n, x_m, t) > 1 - \varepsilon \text{ and } N(x_n, x_m, t) < \varepsilon, \text{ for all } t > 0 \text{ and } n, m \geq n_0. \tag{2.5}$$

(3) An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent.

Remark 2.2 ([11])**.** In intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all x, $y \in X$.

Remark 2.3. It is easy to prove that if $x_n \to x$, $y_n \to y$, $t_n \to t$, then

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t) \text{ and } \lim_{n\to\infty} N(x_n, y_n, t_n) = N(x, y, t).$$

Remark 2.4. In an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, whenever $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$ for x, $y \in X$, $t > 0$ and $0 < r < 1$, we can find a t_0, $0 < t_0 < t$ such that $M(x, y, t_0) > 1 - r$ and $N(x, y, t_0) < r$.

Remark 2.5 ([11])**.** It is clear that if $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space then $(X, M, *)$ is a fuzzy metric space. Conversely if $(X, M, *)$ is a fuzzy metric space then $(X, M, 1 - M, *, \Diamond)$ is an intuitionistic fuzzy metric space where $a \Diamond b = 1 - [(1 - a) * (1 - b)]$ for all a, $b \in [0, 1]$.

Remark 2.6. For convenience, we denote

$$\begin{aligned} &[M(x,y,t)]^n = \underbrace{M(x,y,t) * M(x,y,t) * ... * M(x,y,t)}_{n} \\ &\text{and} \\ &[N(x,y,t)]^n = \underbrace{N(x,y,t) \Diamond N(x,y,t) \Diamond ... \Diamond N(x,y,t)}_{n} \end{aligned} \tag{2.6}$$

for all $n \in N$.

Definition 2.7. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. (M, N) is said to satisfy the p-property on $X^2 \times (0, \infty)$ if

$$\lim_{p\to\infty} [M(x, y, k^p t)]^p = 1 \text{ and } \lim_{p\to\infty} [N(x, y, k^p t)]^p = 0, \tag{2.7}$$

whenever x, $y \in X$, $k > 1$ and $p \in N$.

Lemma 2.1. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and (M, N) satisfy the p-property, then*

$$\lim_{t\to\infty} M(x, y, t) = 1 \textit{ and } \lim_{t\to\infty} N(x, y, t) = 0, \textit{ for all } x, y \in X. \tag{2.8}$$

Proof. If not, since $0 \leq M(x,\ y,\ \cdot) \leq 1$ and $0 \leq N(x,\ y,\ \cdot) \leq 1$, $M(x,\ y,\ \cdot)$ is non-decreasing and $N(x,\ y,\ \cdot)$ is non-increasing, there exist x_0, $y_0 \in X$ such that

$$\lim_{t\to\infty} M(x_0, y_0, t) = \lambda < 1 \text{ and } \lim_{t\to\infty} N(x_0, y_0, t) = 1 - \lambda > 0,$$

then for $k > 1$, $k^p t \to \infty$ when $p \to \infty$ as $t > 0$ and we get

$$\lim_{p\to\infty} [M(x_0, y_0, k^p t)]^p = 0 \text{ and } \lim_{p\to\infty} [N(x_0, y_0, k^p t)]^p = 1,$$

which is a contradiction. □

Remark 2.7. Condition (2.8) cannot guarantee the p-property. See the following example.

Example 2.2. Let $(X,\ d)$ be an ordinary metric space, $a * b \leq ab \leq a \diamondsuit b$ for all a, $b \in [0,\ 1]$ and ψ be defined as following:

$$\psi(t) = \begin{cases} \alpha\sqrt{t}, & 0 \leq t \leq 4 \\ 1 - \frac{1}{\ln t}, & t > 4, \end{cases} \tag{2.9}$$

where $\alpha = \frac{1}{2}\left(1 - \frac{1}{\ln 4}\right)$. Then $\psi(t)$ is continuous and increasing in $(0,\ \infty)$, $\psi(t) \in (0,\ 1)$ and $\lim_{t\to\infty} \psi(t) = 1$. Let

$$M(x, y, t) = [\psi\,(t)]^{d(x,y)} \text{ and } N(x, y, t) = 1 - [\psi\,(t)]^{d(x,y)},\ \forall x, y \in X \text{ and } t > 0.$$

Then $(X,\ M,\ N,\ *,\ \diamondsuit)$ is an intuitionistic fuzzy metric space with

$$\lim_{t\to\infty} M(x, y, t) = \lim_{t\to\infty} [\psi\,(t)]^{d(x,y)} = 1$$

and

$$\lim_{t\to\infty} N(x, y, t) = \lim_{t\to\infty} \left(1 - [\psi\,(t)]^{d(x,y)}\right) = 0,\ \forall x, y \in X.$$

But for any $x \neq y$, $k > 1$ and $t > 0$,

$$\lim_{p\to\infty} [M(x, y, k^p t)]^p = \lim_{p\to\infty} [\psi(k^p t)]^{d(x,y)\cdot p} = \lim_{p\to\infty} \left[1 - \frac{1}{\ln(k^p t)}\right]^{d(x,y)\cdot p} = e^{-\frac{d(x,y)}{\ln k}} \neq 1$$

and

$$\begin{aligned} \lim_{p\to\infty} [N(x, y, k^p t)]^p &= \lim_{p\to\infty} \left[1 - [\psi(k^p t)]^{d(x,y)}\right]^p \\ &= \lim_{p\to\infty} \left[1 - [\psi(k^p t)]^{d(x,y)\cdot p}\right] \\ &= \lim_{p\to\infty} \left[1 - \left[1 - \frac{1}{\ln(k^p t)}\right]^{p\cdot d(x,y)}\right] \\ &= 1 - e^{-\frac{d(x,y)}{\ln k}} \neq 0. \end{aligned}$$

Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

$(\phi - 1)$ ϕ is non-decreasing,

$(\phi - 2)$ ϕ is continuous,

$(\phi - 3)$ $\Sigma_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi^n(\phi(t))$, $n \in N$.

It is easy to prove that, if $\phi \in \Phi$ then $\phi(t) < t$ for all $t > 0$.

Lemma 2.2 ([14])**.** *Let $\phi \in \Phi$. Then for all $t > 0$ it holds that* $\lim_{n\to\infty} \phi^n(t) = 0$, *where $\phi^n(t)$ denote the n-th iteration of ϕ.*

Lemma 2.3 ([14])**.** *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space, where $*$ is a continuous t-norm of H-type and $\Diamond$ is a continuous t-conorm of H-type. If there exists $\phi \in \Phi$ such that if*

$$M(x, y, \phi(t)) \geq M(x, y, t) \text{ and } N(x, y, \phi(t)) \leq N(x, y, t), \forall t > 0, \tag{2.10}$$

then $x = y$.

Definition 2.8 ([4])**.** An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 2.9 ([9])**.** An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x) \text{ and } F(y, x) = g(y).$$

Definition 2.10 ([9])**.** An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = F(x, y) = g(x) \text{ and } y = F(y, x) = g(y).$$

Definition 2.11 ([9])**.** An element $x \in X$ is called a *common fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = F(x, x) = g(x).$$

Definition 2.12 ([9])**.** The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *commutative* if

$$gF(x, y) = F(gx, gy), \text{ for all } (x, y) \in X^2.$$

Definition 2.13 ([5])**.** The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(gx_n, gy_n), t) = 1, \lim_{n\to\infty} N(gF(x_n, y_n), F(gx_n, gy_n), t) = 0$$

and

$$\lim_{n\to\infty} M(gF(y_n, x_n), F(gy_n, gx_n), t) = 1, \lim_{n\to\infty} N(gF(y_n, x_n), F(gy_n, gx_n), t) = 0,$$

for all $t > 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y,$$

for all x, $y \in X$.

Remark 2.8. It is easy to prove that if F and g are commutative, then they are compatible.

3. Main Results

Theorem 3.1. *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space, where $*$ and $\Diamond$ respectively are continuous t-norm and continuous t-conorm of H-type satisfying* (2.8). *Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

$$\begin{aligned} & M(F(x,y), F(u,v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t) \\ & \text{and} \\ & N(F(x,y), F(u,v), \phi(t)) \leq N(gx, gu, t) \Diamond N(gy, gv, t), \end{aligned} \tag{3.1}$$

for all x, y, u, $v \in X$ and $t > 0$. Suppose that $F(X \times X) \subseteq g(X)$, F and g are compatible and g is continuous. Then there exists $x \in X$ such that

$$x = g(x) = F(x, x),$$

that is, F and g have a unique common fixed point in X.

Proof. Let x_0, $y_0 \in X$ be two arbitrary points in X. Since $F(X \times X) \subseteq g(X)$, we can choose x_1, $y_1 \in X$ such that

$$gx_1 = F(x_0, y_0) \text{ and } gy_1 = F(y_0, x_0).$$

Continuing in this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n), \ gy_{n+1} = F(y_n, x_n), \ \forall n \geq 0. \tag{3.2}$$

The proof is divided into 4 steps.

Step 1. Prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since $*$ and $\Diamond$ are continuous t-norm and continuous t-conorm of H-type respectively. Therefore for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu) * (1-\mu) * ... * (1-\mu)}_{k} \geq 1-\lambda \text{ and } \underbrace{\mu\Diamond\mu\Diamond...\Diamond\, \mu}_{k} \leq \lambda, \tag{3.3}$$

for all $k \in N$.

Since $M(x, y, \cdot), N(x, y, \cdot)$ are continuous, $\lim_{t\to\infty} M(x, y, t) = 1$ and $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$, there exists $t_0 > 0$ such that

$$\begin{gathered} M(gx_0, gx_1, t_0) \geq 1-\mu,\ N(gx_0, gx_1, t_0) \leq \mu \\ \text{and} \\ M(gy_0, gy_1, t_0) \geq 1-\mu,\ N(gy_0, gy_1, t_0) \leq \mu. \end{gathered} \tag{3.4}$$

On the other hand, since $\phi \in \Phi$, by condition $(\phi - 3)$ we have $\Sigma_{n=1}^{\infty}\, \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.5}$$

From condition (3.1), we have

$$\begin{aligned} M(gx_1,\, gx_2,\, \phi(t_0)) &= M(F(x_0,\, y_0),\, F(x_1,\, y_1),\, \phi(t_0)) \\ &\geq M(gx_0,\, gx_1,\, t_0) * M(gy_0,\, gy_1,\, t_0), \\ M(gy_1,\, gy_2,\, \phi(t_0)) &= M(F(y_0,\, x_0),\, F(y_1,\, x_1),\, \phi(t_0)) \\ &\geq M(gy_0,\, gy_1,\, t_0) * M(gx_0,\, gx_1,\, t_0) \end{aligned}$$

and

$$\begin{aligned} N(gx_1,\, gx_2,\, \phi(t_0)) &= N(F(x_0,\, y_0),\, F(x_1,\, y_1),\, \phi(t_0)) \\ &\leq N(gx_0,\, gx_1,\, t_0) \Diamond N(gy_0,\, gy_1,\, t_0) \\ N(gy_1,\, gy_2,\, \phi(t_0)) &= N(F(y_0,\, x_0),\, F(y_1,\, x_1),\, \phi(t_0)) \\ &\leq N(gy_0,\, gy_1,\, t_0) \Diamond N(gx_0,\, gx_1,\, t_0). \end{aligned}$$

Similarly, we can also get

$$\begin{aligned} M(gx_2,\, gx_3,\, \phi^2(t_0)) &= M(F(x_1,\, y_1),\, F(x_2,\, y_2),\, \phi^2(t_0)) \\ &\geq M(gx_1,\, gx_2,\, \phi(t_0)) * M(gy_1,\, gy_2,\, \phi(t_0)) \\ &\geq [M(gx_0,\, gx_1,\, t_0)]^2 * [M(gy_0,\, gy_1,\, t_0)]^2, \\ M(gy_2,\, gy_3,\, \phi^2(t_0)) &= M(F(y_1,\, x_1),\, F(y_2,\, x_2),\, \phi^2(t_0)) \\ &\geq M(gy_1,\, gy_2,\, \phi(t_0)) * M(gx_1,\, gx_2,\, \phi(t_0)) \\ &\geq [M(gy_0,\, gy_1,\, t_0)]^2 * [M(gx_0,\, gx_1,\, t_0)]^2 \end{aligned}$$

and

$$\begin{aligned} N(gx_2,\, gx_3,\, \phi^2(t_0)) &= N(F(x_1,\, y_1),\, F(x_2,\, y_2),\, \phi^2(t_0)) \\ &\le N(gx_1,\, gx_2,\, \phi(t_0)) \lozenge N(gy_1,\, gy_2,\, \phi(t_0)) \\ &\le [N(gx_0,\, gx_1,\, t_0)]^2 \lozenge [N(gy_0,\, gy_1,\, t_0)]^2, \\ N(gy_2,\, gy_3,\, \phi^2(t_0)) &= N(F(y_1,\, x_1),\, F(y_2,\, x_2),\, \phi^2(t_0)) \\ &\le N(gy_1,\, gy_2,\, \phi(t_0)) \lozenge N(gx_1,\, gx_2,\, \phi(t_0)) \\ &\le [N(gy_0,\, gy_1,\, t_0)]^2 \lozenge [N(gx_0,\, gx_1,\, t_0)]^2. \end{aligned}$$

Continuing in the same way, we can get

$$\begin{aligned} M(gx_n,\, gx_{n+1},\, \phi^n(t_0)) &\ge [M(gx_0,\, gx_1,\, t_0)]^{2^{n-1}} * [M(gy_0,\, gy_1,\, t_0)]^{2^{n-1}}, \\ M(gy_n,\, gy_{n+1},\, \phi^n(t_0)) &\ge [M(gy_0,\, gy_1,\, t_0)]^{2^{n-1}} * [M(gx_0,\, gx_1,\, t_0)]^{2^{n-1}} \end{aligned}$$

and

$$\begin{aligned} N(gx_n,\, gx_{n+1},\, \phi^n(t_0)) &\le [N(gx_0,\, gx_1,\, t_0)]^{2^{n-1}} \lozenge [N(gy_0,\, gy_1,\, t_0)]^{2^{n-1}}, \\ N(gy_n,\, gy_{n+1},\, \phi^n(t_0)) &\le [N(gy_0,\, gy_1,\, t_0)]^{2^{n-1}} \lozenge [N(gx_0,\, gx_1,\, t_0)]^{2^{n-1}}. \end{aligned}$$

Now from (3.3), (3.4) and (3.5), for $m > n \ge n_0$, we have

$$\begin{aligned} & M(gx_n,\, gx_m,\, t) \\ & \ge M\left(gx_n, gx_m, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ & \ge M\left(gx_n, gx_m, \Sigma_{k=n}^{m-1} \phi^k(t_0)\right) \\ & \ge M(gx_n,\, gx_{n+1},\, \phi^n(t_0)) * M(gx_{n+1},\, gx_{n+2},\, \phi^{n+1}(t_0)) \\ & \quad *\ldots * M(gx_{m-1},\, gx_m,\, \phi^{m-1}(t_0)) \\ & \ge [M(gx_0,\, gx_1,\, t_0)]^{2^{n-1}} * [M(gy_0,\, gy_1,\, t_0)]^{2^{n-1}} \\ & \quad * [M(gx_0,\, gx_1,\, t_0)]^{2^n} * [M(gy_0,\, gy_1,\, t_0)]^{2^n} \\ & \quad *\ldots* [M(gx_0,\, gx_1,\, t_0)]^{2^{m-2}} * [M(gy_0,\, gy_1,\, t_0)]^{2^{m-2}} \\ & \ge [M(gx_0,\, gx_1,\, t_0)]^{2^{(m-n)(m+n-3)}} * [M(gy_0,\, gy_1,\, t_0)]^{2^{(m-n)(m+n-3)}} \\ & \ge \underbrace{(1-\mu) * \ (1-\mu) * \ldots * \ (1-\mu)}_{2^{2(m-n)(m+n-3)}} \ge 1 - \lambda \end{aligned}$$

and

$$\begin{aligned} & N(gx_n,\, gx_m,\, t) \\ & \le N\left(gx_n, gx_m, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ & \le N\left(gx_n, gx_m, \Sigma_{k=n}^{m-1} \phi^k(t_0)\right) \\ & \le N(gx_n,\, gx_{n+1},\, \phi^n(t_0)) \lozenge N(gx_{n+1},\, gx_{n+2},\, \phi^{n+1}(t_0)) \\ & \quad \lozenge \ldots \lozenge N(gx_{m-1},\, gx_m,\, \phi^{m-1}(t_0)) \\ & \le [N(gx_0,\, gx_1,\, t_0)]^{2^{n-1}} \lozenge [N(gy_0,\, gy_1,\, t_0)]^{2^{n-1}} \\ & \quad \lozenge [N(gx_0,\, gx_1,\, t_0)]^{2^n} \lozenge [N(gy_0,\, gy_1,\, t_0)]^{2^n} \\ & \quad \lozenge \ldots \lozenge N[(gx_0,\, gx_1,\, t_0)]^{2^{m-2}} \lozenge [N(gy_0,\, gy_1,\, t_0)]^{2^{m-2}} \end{aligned}$$

$$\leq [N(gx_0,\, gx_1,\, t_0)]^{2^{(m-n)(m+n-3)}} \lozenge [N(gy_0,\, gy_1,\, t_0)]^{2^{(m-n)(m+n-3)}}$$
$$\leq \underbrace{\mu \lozenge \mu \lozenge ... \lozenge\, \mu}_{2^{2(m-n)(m+n-3)}} \leq \lambda,$$

which implies that

$$M(gx_n, gx_m, t) > 1 - \lambda \text{ and } N(gx_n, gx_m, t) < \lambda,$$

for all m, $n \in N$ with $m > n \geq n_0$ and $t > 0$. So $\{gx_n\}$ is a Cauchy sequence. Similarly, we can get that $\{gy_n\}$ is also a Cauchy sequence.

Step 2. Prove that g and F have a coupled coincidence point.
Since X is complete, there exist x, $y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$$
$$\text{and} \tag{3.6}$$
$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y.$$

Since g and F are compatible, we have by (3.6),

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(gx_n, gy_n), t) = 1,$$
$$\lim_{n\to\infty} N(gF(x_n, y_n), F(gx_n, gy_n), t) = 0$$
$$\text{and} \tag{3.7}$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(gy_n, gx_n), t) = 1,$$
$$\lim_{n\to\infty} N(gF(y_n, x_n), F(gy_n, gx_n), t) = 0,$$

for all $t > 0$. Next we prove that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

For all $t > 0$, by condition (3.1), we have

$$\begin{aligned} &M(gx,\, F(x,\, y),\, \phi(t)) \\ &\geq M(ggx_{n+1},\, F(x,\, y),\, \phi(k_1t)) * M(gx,\, ggx_{n+1},\, \phi(t) - \phi(k_1t)) \\ &\geq M(gF(x_n,\, y_n),\, F(x,\, y),\, \phi(k_1t)) * M(gx,\, ggx_{n+1},\, \phi(t) - \phi(k_1t)) \\ &\geq M(gF(x_n,\, y_n),\, F(gx_n,\, gy_n),\, \phi(k_1t) - \phi(k_2t)) \\ &\quad * M(F(gx_n,\, gy_n),\, F(x,\, y),\, \phi(k_2t)) * M(gx,\, ggx_{n+1},\, \phi(t) - \phi(k_1t)) \\ &\geq M(gF(x_n,\, y_n),\, F(gx_n,\, gy_n),\, \phi(k_1t) - \phi(k_2t)) \\ &\quad * M(ggx_n,\, gx,\, k_2t) * M(ggy_n,\, gy,\, k_2t) * M(gx,\, ggx_{n+1},\, \phi(t) - \phi(k_1t)) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &N(gx,\, F(x,\, y),\, \phi(t)) \\ &\leq N(ggx_{n+1},\, F(x,\, y),\, \phi(k_1t)) \lozenge N(gx,\, ggx_{n+1},\, \phi(t) - \phi(k_1t)) \\ &\leq N(gF(x_n,\, y_n),\, F(x,\, y),\, \phi(k_1t)) \lozenge N(gx,\, ggx_{n+1},\, \phi(t) - \phi(k_1t)) \\ &\leq N(gF(x_n,\, y_n),\, F(gx_n,\, gy_n),\, \phi(k_1t) - \phi(k_2t)) \end{aligned} \tag{3.9}$$

$$\begin{aligned}&\lozenge\ N(F(gx_n,\, gy_n),\, F(x,\, y),\, \phi(k_2t))\ \lozenge\ N(gx,\, ggx_{n+1},\, \phi(t)-\phi(k_1t))\\ &\le N(gF(x_n,\, y_n),\, F(gx_n,\, gy_n),\, \phi(k_1t)-\phi(k_2t))\\ &\lozenge\ N(ggx_n,\, gx,\, k_2t)\ \lozenge\ N(ggy_n,\, gy,\, k_2t)\lozenge\ N(gx,\, ggx_{n+1},\, \phi(t)-\phi(k_1t)),\end{aligned}$$

for all $0 < k_2 < k_1 < 1$. Letting $n \to \infty$ in (3.8) and (3.9), by using (3.6), (3.7) and with the continuity of g, we get

$$M(gx, F(x,y), \phi(t)) \ge 1 \text{ and } N(gx, F(x,y), \phi(t)) \le 0,$$

which implies that $gx = F(x,\, y)$. Similarly, we can get $gy = F(y,\, x)$.

Step 3. Prove that $gx = y$ and $gy = x$.

Since $*$ and $\lozenge$ are continuous t-norm and continuous t-conorm of H-type respectively. Therefore for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu) *\ (1-\mu) * ... *\ (1-\mu)}_{k} \ge 1-\lambda \ \text{ and }\ \underbrace{\mu\lozenge\mu\lozenge ... \lozenge\ \mu}_{k} \le \lambda, \tag{3.10}$$

for all $k \in N$.

Since $M(x, y, \cdot)$, $N(x, y, \cdot)$ are continuous, $\lim_{t\to\infty} M(x, y, t) = 1$ and $\lim_{t\to\infty} N(x,\, y,\, t) = 0$ for all $x,\, y \in X$, there exists $t_0 > 0$ such that

$$\begin{aligned}&M(gx, y, t_0) \ge 1-\mu,\ \ N(gx, y, t_0) \le \mu\\ &\text{and}\\ &M(gy, x, t_0) \ge 1-\mu,\ \ N(gy, x, t_0) \le \mu.\end{aligned} \tag{3.11}$$

On the other hand, since $\phi \in \Phi$, by condition $(\phi - 3)$ we have $\Sigma_{n=1}^{\infty}\ \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.12}$$

Since

$$\begin{aligned}M(gx, gy_{n+1}, \phi(t_0)) &= M(F(x,y), F(y_n, x_n), \phi(t_0))\\ &\ge M(gx, gy_n, t_0) * M(gy, gx_n, t_0)\\ \text{and}&\\ N(gx, gy_{n+1}, \phi(t_0)) &= N(F(x,y), F(y_n, x_n), \phi(t_0))\\ &\le N(gx, gy_n, t_0)\lozenge N(gy, gx_n, t_0),\end{aligned} \tag{3.13}$$

letting $n \to \infty$ in (3.13), by using (3.6), we get

$$\begin{aligned}&M(gx, y, \phi(t_0)) \ge M(gx, y, t_0) * M(gy, x, t_0)\\ &\text{and}\\ &N(gx, y, \phi(t_0)) \le N(gx, y, t_0)\lozenge N(gy, x, t_0).\end{aligned} \tag{3.14}$$

Similarly, we can get

$$M(gy,x,\phi(t_0)) \geq M(gx,y,t_0) * M(gy,x,t_0)$$

(3.15) and

$$N(gy,x,\phi(t_0)) \leq N(gx,y,t_0) \Diamond N(gy,x,t_0).$$

From (3.14) and (3.15), we have

$$M(gx,\, y,\, \phi(t_0)) * M(gy,\, x,\, \phi(t_0)) \geq [M(gx,\, y,\, t_0)]^2 * [M(gy,\, x,\, t_0)]^2$$

and

$$N(gx,\, y,\, \phi(t_0)) \Diamond N(gy,\, x,\, \phi(t_0)) \leq [N(gx,\, y,\, t_0)]^2 \Diamond [N(gy,\, x,\, t_0)]^2.$$

By this way, we can get for all $n \in N$,

$$\begin{aligned} & M(gx,y,\phi^n(t_0)) * M(gy,x,\phi^n(t_0)) \\ & \geq [M(gx,y,\phi^{n-1}(t_0))]^2 * [M(gy,x,\phi^{n-1}(t_0))]^2 \\ & \geq [M(gx,y,t_0)]^{2^n} * [M(gy,x,t_0)]^{2^n} \end{aligned}$$

(3.16) and

$$\begin{aligned} & N(gx,y,\phi^n(t_0)) \Diamond N(gy,x,\phi^n(t_0)) \\ & \leq [N(gx,y,\phi^{n-1}(t_0))]^2 \Diamond [N(gy,x,\phi^{n-1}(t_0))]^2 \\ & \leq [N(gx,y,t_0)]^{2^n} \Diamond [N(gy,x,t_0)]^{2^n}. \end{aligned}$$

Then, by (3.10), (3.11), (3.12) and (3.16), we have

$$\begin{aligned} M(gx,\, y,\, t) * M(gy,\, x,\, t) & \geq M\left(gx, y, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) * M\left(gy, x, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ & \geq M(gx,\, y,\, \phi^{n_0}(t_0)) * M(gy,\, x,\, \phi^{n_0}(t_0)) \\ & \geq [M(gx,y,t_0)]^{2^{n_0}} * [M(gy,x,t_0)]^{2^{n_0}} \\ & \geq \underbrace{(1-\mu) * \ (1-\mu) * \ldots * \ (1-\mu)}_{2^{2n_0}} \geq 1-\lambda \end{aligned}$$

and

$$\begin{aligned} N(gx,\, y,\, t) \Diamond N(gy,\, x,\, t) & \leq N\left(gx, y, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) \Diamond N\left(gy, x, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ & \leq N(gx,\, y,\, \phi^{n_0}(t_0)) \Diamond N(gy,\, x,\, \phi^{n_0}(t_0)) \\ & \leq [N(gx,y,t_0)]^{2^{n_0}} \Diamond [N(gy,x,t_0)]^{2^{n_0}} \\ & \leq \underbrace{\mu \Diamond \ \mu \Diamond \ldots \Diamond \ \mu}_{2^{2n_0}} \leq \lambda. \end{aligned}$$

So for any $\lambda > 0$ we have

$M(gx,\, y,\, t) * M(gy,\, x,\, t) \geq 1-\lambda$ and $N(gx,\, y,\, t) \Diamond N(gy,\, x,\, t) \leq \lambda$, for all $t > 0$. We can get that $gx = y$ and $gy = x$.

Step 4. Prove that $x = y$.

Since $*$ and $\Diamond$ are continuous t-norm and continuous t-conorm of H-type respectively. Therefore for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu) * \ (1-\mu) * ... * \ (1-\mu)}_{k} \geq 1-\lambda \ \text{ and } \ \underbrace{\mu \Diamond \mu \Diamond ... \Diamond \mu}_{k} \leq \lambda \tag{3.17}$$

for all $k \in N$.

Since $M(x, y, \cdot)$, $N(x, y, \cdot)$ are continuous, $\lim_{t\to\infty} M(x, y, t) = 1$ and $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$, there exists $t_0 > 0$:

$$M(x, y, t_0) \geq 1-\mu \ \text{ and } \ N(x, y, t_0) \leq \mu. \tag{3.18}$$

On the other hand, since $\phi \in \Phi$, by condition $(\phi - 3)$ we have $\Sigma_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.19}$$

Since for $t_0 > 0$,

$$\begin{aligned} M(gx_{n+1}, gy_{n+1}, \phi(t_0)) &= M(F(x_n, y_n), F(y_n, x_n), \phi(t_0)) \\ &\geq M(gx_n, gy_n, t_0) * M(gy_n, gx_n, t_0) \\ \text{and} & \\ N(gx_{n+1}, gy_{n+1}, \phi(t_0)) &= N(F(x_n, y_n), F(y_n, x_n), \phi(t_0)) \\ &\leq N(gx_n, gy_n, t_0) \Diamond N(gy_n, gx_n, t_0). \end{aligned} \tag{3.20}$$

Letting $n \to \infty$ in (3.20) yields

$$\begin{aligned} &M(x, y, \phi(t_0)) \geq M(x, y, t_0) * M(y, x, t_0) \\ &\text{and} \\ &N(x, y, \phi(t_0)) \leq N(x, y, t_0) \Diamond N(y, x, t_0). \end{aligned} \tag{3.21}$$

Thus, by (3.17), (3.18), (3.19) and (3.21), we have

$$\begin{aligned} M(x, y, t) &\geq M\left(x, y, \Sigma_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M(x, y, \phi^{n_0}(t_0)) \\ &\geq [M(x, y, t_0)]^{2^{n_0}} * [M(y, x, t_0)]^{2^{n_0}} \\ &\geq \underbrace{(1-\mu) * \ (1-\mu) * ... * \ (1-\mu)}_{2^{2n_0}} \geq 1-\lambda \end{aligned}$$

and

$$
\begin{aligned}
N(x,\, y,\, t) &\le N\left(x, y, \Sigma_{k=n_0}^{\infty}\phi^k(t_0)\right) \\
&\le N(x,\, y,\, \phi^{n_0}(t_0)) \\
&\le [N(x,\, y,\, t_0)]^{2^{n_0}} \ \Diamond\ [N(x,\, y,\, t_0)]^{2^{n_0}} \\
&\le \underbrace{\mu\Diamond\mu\Diamond ... \Diamond\mu}_{2^{2n_0}} \le \lambda,
\end{aligned}
$$

which implies that $x = y$. Thus we have proved that F and g have a unique common fixed point in X. This completes the proof of the Theorem 3.1. □

Taking $g = I$ (the identity mapping) in Theorem 3.1, we get the following consequence.

Corollary 3.2. *Let* $(X,\, M,\, N,\, *,\, \Diamond)$ *be a complete intuitionistic fuzzy metric space, where* $*$ *and* $\Diamond$ *respectively are continuous t-norm and continuous t-conorm of H-type satisfying* (2.8). *Let* $F : X \times X \to X$ *and there exist* $\phi \in \Phi$ *such that*

$$
\begin{aligned}
&M(F(x,y), F(u,v), \phi(t)) \ge M(x,u,t) * M(y,v,t) \\
&\quad \textit{and} \\
&N(F(x,y), F(u,v), \phi(t)) \le N(x,u,t) \Diamond N(y,v,t),
\end{aligned}
\tag{3.22}
$$

for all $x,\, y,\, u,\, v \in X$ *and* $t > 0$. *Then there exists* $x \in X$ *such that* $x = F(x,\, x)$, *that is,* F *admits a unique fixed point in* X.

Put $\phi(t) = kt$, where $0 < k < 1$. By Lemma 2.1, we get the following.

Corollary 3.3. *Let* $a \,\Diamond\, b \le ab \le a * b$ *for all* $a,\, b \in [0,\, 1]$ *and* $(X,\, M,\, N,\, *,\, \Diamond)$ *be a complete intuitionistic fuzzy metric space such that* $(M,\, N)$ *has p-property. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two functions such that*

$$
\begin{aligned}
&M(F(x,y), F(u,v), kt) \ge M(gx, gu, t) * M(gy, gv, t) \\
&\quad \textit{and} \\
&N(F(x,y), F(u,v), kt) \le N(gx, gu, t) \Diamond N(gy, gv, t),
\end{aligned}
\tag{3.23}
$$

for all $x,\, y,\, u,\, v \in X$ *and* $t > 0$ *where* $0 < k < 1$. *Suppose* $F(X \times X) \subseteq g(X)$, g *is continuous and commutes with* F. *Then there exists a unique* $x \in X$ *such that* $x = g(x) = F(x,\, x)$.

Remark 3.1. The Corollary 3.3 is the intuitionistic fuzzy version of theorem of Sedghi, Altun and Shobe [13].

Next we give an example to demonstrate Theorem 3.1.

Example 3.1. Let $X = [-2, 2]$, $a * b = ab$, $a \diamondsuit b = 1 - [(1-a)*(1-b)]$ for all a, $b \in [0, 1]$ and ψ is defined as (2.9). Let

$$M(x,y,t) = [\psi(t)]^{|x-y|} \text{ and } N(x,y,t) = 1 - [\psi(t)]^{|x-y|}, \forall x, y \in X \text{ and } t > 0. \tag{3.24}$$

Then $(X, M, N, *, \diamondsuit)$ is a complete intuitionistic fuzzy metric space. Let $\phi(t) = \frac{t}{2}$, $g(x) = x$ and $F : X \times X \to X$ be defined as

$$F(x,y) = \frac{x^2}{8} + \frac{y^2}{8} - 2, \forall x, y \in X. \tag{3.25}$$

Then F satisfies all conditions of Theorem 3.1, and there exists a point $x = 2 - 2\sqrt{3}$ which is the unique common fixed point of g and F. In fact, it is easy to see that

$$F(X \times X) = [-2, -1] \subset [-2, 2] = g(X).$$

Now,

$$M(F(x,y), F(u,v), \phi(t)) = [\psi(\phi(t))]^{\frac{|x^2-u^2+y^2-v^2|}{8}} \text{ and } N(F(x,y), F(u,v), \phi(t)) = 1 - [\psi(\phi(t))]^{\frac{|x^2-u^2+y^2-v^2|}{8}}, \tag{3.26}$$

for all x, $y \in X$ and $t > 0$. (3.22) is equivalent to

$$\left[\psi\left(\frac{t}{2}\right)\right]^{\frac{|x^2-u^2+y^2-v^2|}{8}} \geq [\psi(t)]^{|x-u|} \cdot [\psi(t)]^{|y-v|}. \tag{3.27}$$

Since $\psi(t) \in (0, 1)$, we can get

$$\left[\psi\left(\frac{t}{2}\right)\right]^{\frac{|x^2-u^2+y^2-v^2|}{8}} \geq \left[\psi\left(\frac{t}{2}\right)\right]^{\frac{|x-u|}{2}} \cdot \left[\psi\left(\frac{t}{2}\right)\right]^{\frac{|y-v|}{2}}. \tag{3.28}$$

From (3.27), we only need to verify the following:

$$\left[\psi\left(\frac{t}{2}\right)\right]^{\frac{|x-u|}{2}} \geq [\psi(t)]^{|x-u|}, \tag{3.29}$$

that is,

$$\psi\left(\frac{t}{2}\right) \geq [\psi(t)]^2, \forall t > 0. \tag{3.30}$$

We consider the following cases.

Case 1. $(0 < t \leq 4)$. Then (3.30) is equivalent to

$$\alpha\sqrt{\frac{t}{2}} \geq [\alpha\sqrt{t}]^2,$$

it is easy to verified.

Case 2. $(t \geq 8)$. Then (3.30) is equivalent to

$$1 - \frac{1}{\ln \frac{t}{2}} \geq \left(1 - \frac{1}{\ln t}\right)^2,$$

which is

$$2 \ln t \cdot \ln \frac{t}{2} \geq \ln^2 t + \ln \frac{t}{2},$$

since

$$\ln^2 t + \ln^2 \frac{t}{2} - 2 \ln t \cdot \ln \frac{t}{2} + \ln \frac{t}{2} - \ln^2 \frac{t}{2} \leq 0,$$

that is,

$$\ln^2 2 + \ln \frac{t}{2} - \ln^2 \frac{t}{2} \leq 0,$$

holds for all $t \geq 8$. So (3.30) holds for $t \geq 8$.

Case 3. $(4 < t < 8)$. Then (3.30) is equivalent to

$$\alpha\sqrt{\frac{t}{2}} \geq \left(1 - \frac{1}{\ln t}\right)^2.$$

Let $t = e^x$, we only need to verify

$$\frac{\alpha}{\sqrt{2}} e^{\frac{x}{2}} - \left(1 - \frac{1}{x}\right)^2 \geq 0,$$

for all x that $2 \ln 2 < x < 3 \ln 2$. We can verify it holds.

Thus it is verified that the functions F, g, ϕ satisfy all the conditions of Theorem 3.1, $x = 2 - 2\sqrt{3}$ which is the common fixed point of g and F in X.

4. Application to Integral Equations

As an application of the coupled fixed point theorems established in section 3 of our paper, we study the existence and uniqueness of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation,

$$x(p) = \int_a^b (K_1(p,q) + K_2(p,q)) \left[f(q, x(q)) + g(q, x(q))\right] dq + h(p), \tag{4.1}$$

for all $p \in I = [a, b]$.

Let Θ denote the set of all functions $\theta : [0,\, 1] \to [0,\, 1]$ satisfying

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) \leq p$.

We assume that the functions K_1, K_2, f, g fulfill the following conditions:

Assumption 4.1. (i) $K_1(p, q) \geq 0$ and $K_2(p, q) \leq 0$ for all $p, q \in I$,

(ii) There exists $\theta \in \Theta$ such that for all x, $y \in R$ with $x \geq y$, the following conditions hold:

$$0 \leq f(q,x) - f(q,y) \leq \lambda\theta(x-y) \tag{4.2}$$

and

$$-\mu\theta(x-y) \leq g(q,x) - g(q,y) \leq 0, \tag{4.3}$$

(iii)

$$\max\{\lambda,\mu\} \sup_{p\in I} \int_a^b [K_1(p,q) - K_2(p,q)]dq \leq \frac{1}{4}. \tag{4.4}$$

Theorem 4.1. *Consider the integral equation* (4.1) *with* K_1, $K_2 \in C(I \times I, R)$ *and* $h \in C(I, R)$. *Suppose that Assumption 4.1 is satisfied. Then the integral equation* (4.1) *has a unique solution in* $C(I, R)$.

Proof. Consider $X = C(I, R)$. It is easy to check that $(X, M, N, *, \Diamond)$ is a complete intuitionistic fuzzy metric space with respect to the intuitionistic fuzzy metric

$$M(x,y,t) = [\psi(t)]^{|x-y|} \text{ and } N(x,y,t) = 1 - [\psi(t)]^{|x-y|},$$

for all $x, y \in X$ and $t > 0$ with $x*y = xy$, $x \Diamond y = 1-[(1-x)*(1-y)]$ for all x, $y \in I$ and ψ is defined as (2.9). Obviously $\psi(t)$ is an increasing function and $\psi(t) \in (0, 1)$. Define now the mapping $F : X \times X \to X$ by

$$\begin{aligned} F(x,y)(p) = &\int_a^b K_1(p,q)\,[f(q,x(q)) + g(q,y(q))]\,dq \\ &+ \int_a^b K_2(p,q)\,[f(q,y(q)) + g(q,x(q))]\,dq + h(p), \end{aligned}$$

for all $p \in I$ and $\phi(t) = \frac{t}{2}$ for all $t > 0$. Now for all x, y, u, $v \in X$, using (4.2) and (4.3), we have

$$
\begin{aligned}
& F(x,y)(p) - F(u,v)(p) \\
& = \int_a^b K_1(p,q)\left[f(q,x(q)) + g(q,y(q))\right] dq \\
& \quad + \int_a^b K_2(p,q)\left[f(q,y(q)) + g(q,x(q))\right] dq \\
& \quad - \int_a^b K_1(p,q)\left[f(q,u(q)) + g(q,v(q))\right] dq \\
& \quad - \int_a^b K_2(p,q)\left[f(q,v(q)) + g(q,u(q))\right] dq \\
& = \int_a^b K_1(p,q)\left[f(q,x(q)) - f(q,u(q)) + g(q,y(q)) - g(q,v(q))\right] dq \\
& \quad + \int_a^b K_2(p,q)\left[f(q,y(q)) - f(q,v(q)) + g(q,x(q)) - g(q,u(q))\right] dq \\
& = \int_a^b K_1(p,q)\left[(f(q,x(q)) - f(q,u(q))) - (g(q,v(q)) - g(q,y(q)))\right] dq \\
& \quad - \int_a^b K_2(p,q)\left[(f(q,v(q)) - f(q,y(q))) - (g(q,x(q)) - g(q,u(q)))\right] dq \\
& \leq \int_a^b K_1(p,q)\left[\lambda\theta\left(x(q) - u(q)\right) + \mu\theta\left(v(q) - y(q)\right)\right] dq \\
& \quad - \int_a^b K_2(p,q)\left[\lambda\theta\left(v(q) - y(q)\right) + \mu\theta\left(x(q) - u(q)\right)\right] dq.
\end{aligned}
\tag{4.5}
$$

Since the function θ is non-decreasing and so we have

$$\theta\left(x(q) - u(q)\right) \leq \theta\left(|x(q) - u(q)|\right)$$

and

$$\theta\left(v(q) - y(q)\right) \leq \theta\left(|v(q) - y(q)|\right),$$

hence by (4.5), in view of the fact $K_2(p,\, q) \leq 0$, we get

(4.6)

$$\begin{aligned}
&|F(x,y)(p) - F(u,v)(p)| \\
&\le \int_a^b K_1(p,q)\left[\lambda\theta\left(|x(q)-u(q)|\right) + \mu\theta\left(|v(q)-y(q)|\right)\right] dq \\
&\quad - \int_a^b K_2(p,q)\left[\lambda\theta\left(|v(q)-y(q)|\right) + \mu\theta\left(|x(q)-u(q)|\right)\right] dq \\
&\le \int_a^b K_1(p,q)\left[\max\{\lambda,\mu\}\theta\left(|x(q)-u(q)|\right) + \max\{\lambda,\mu\}\theta\left(|v(q)-y(q)|\right)\right] dq \\
&\quad - \int_a^b K_2(p,q)\left[\max\{\lambda,\mu\}\theta\left(|v(q)-y(q)|\right) + \max\{\lambda,\mu\}\theta\left(|x(q)-u(q)|\right)\right] dq,
\end{aligned}$$

as all the quantities on the right hand side of (4.5) are non-negative. Now by using (4.4), we get

$$\begin{aligned}
&|F(x,y) - F(u,v)| \\
&\le \max\{\lambda,\mu\} \int_a^b \left[K_1(p,q) - K_2(p,q)\right] dq \cdot \left[\theta\left(|x(q)-u(q)|\right) + \theta\left(|v(q)-y(q)|\right)\right] \\
&\le \max\{\lambda,\mu\} \sup_{p\in I} \int_a^b \left[K_1(p,q) - K_2(p,q)\right] dq \cdot \left[\theta\left(|x(q)-u(q)|\right) + \theta\left(|v(q)-y(q)|\right)\right] \\
&\le \frac{\theta\left(|x-u|\right) + \theta\left(|v-y|\right)}{4},
\end{aligned}$$

which implies, by $\psi(t) \in (0, 1)$, that

$$\left[\psi\left(\frac{t}{2}\right)\right]^{|F(x,y)-F(u,v)|} \ge \left[\psi\left(\frac{t}{2}\right)\right]^{\frac{\theta(|x-u|)+\theta(|v-y|)}{4}},$$

it follows, by (3.30), that

$$\left[\psi\left(\frac{t}{2}\right)\right]^{|F(x,y)-F(u,v)|} \ge \left[\psi(t)\right]^{\frac{\theta(|x-u|)+\theta(|v-y|)}{2}}. \tag{4.7}$$

Now, since θ is non-decreasing, we have

$$\theta(|x-u|) \le \theta\left(|x-u| + |y-v|\right) \text{ and } \theta(|y-v|) \le \theta\left(|x-u| + |y-v|\right),$$

and so

$$\frac{\theta\left(|x-u|\right) + \theta\left(|y-v|\right)}{2} \le \theta\left(|x-u| + |y-v|\right),$$

which implies, by $\psi(t) \in (0,\, 1)$, that

$$[\psi(t)]^{\frac{\theta(|x-u|)+\theta(|y-v|)}{2}} \geq [\psi(t)]^{\theta(|x-u|+|y-v|)}. \tag{4.8}$$

Thus by (4.7), (4.8) and (ii_θ), we have

$$\left[\psi\left(\frac{t}{2}\right)\right]^{|F(x,y)-F(u,v)|} \geq [\psi(t)]^{\theta(|x-u|+|y-v|)} \geq [\psi(t)]^{|x-u|+|y-v|}. \tag{4.9}$$

Now by (4.9), it follows that

$$\begin{aligned} M(F(x,y), F(u,v), \phi(t)) &= M\left(F(x,y), F(u,v), \frac{t}{2}\right) \\ &= \left[\psi\left(\frac{t}{2}\right)\right]^{|F(x,y)-F(u,v)|} \\ &\geq [\psi(t)]^{|x-u|+|y-v|} \\ &\geq M(x,u,t) * M(y,v,t) \end{aligned}$$

and

$$\begin{aligned} N(F(x,y), F(u,v), \phi(t)) &= N\left(F(x,y), F(u,v), \frac{t}{2}\right) \\ &= \ 1 - \left[\psi\left(\frac{t}{2}\right)\right]^{|F(x,y)-F(u,v)|} \\ &\leq 1 - [\psi(t)]^{|x-u|+|y-v|} \\ &\leq N(x,u,t) \Diamond N(y,v,t). \end{aligned}$$

Thus

$$M(F(x,y), F(u,v), \phi(t)) \geq M(x,u,t) * M(y,v,t)$$

and

$$N(F(x,y), F(u,v), \phi(t)) \leq N(x,u,t) \Diamond N(y,v,t),$$

which are the conditions in (3.22), show that all hypotheses of Corollary 3.2 are satisfied. This proves that F has a unique fixed point $\widetilde{x} \in X$, that is, $\widetilde{x} = F(\widetilde{x},\, \widetilde{x})$ and therefore $\widetilde{x} \in C(I,\, R)$ is the unique solution of the integral equation (4.1). □

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