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A NOTE ON THE INTEGRAL POINTS ON SOME HYPERBOLAS

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Abstract. In this paper, we study the Lie-generalized Fibonacci sequence and the root system of rank 2 symmetric hyperbolic Kac-Moody algebras. We derive several interesting properties of the Lie-Fibonacci sequence and relationship between them. We also give a couple of sufficient conditions for the existence of the integral points on the hyperbola $\mathfrak{h}^a : x^2 - axy + y^2 = 1$ and $\mathfrak{h}_k : x^2 - axy + y^2 = -k$ $(k \in \mathbb{Z}_{>0})$. To list all the integral points on that hyperbola, we find the number of elements of Ω_k.

1. Introduction

Let A be a symmetric Cartan matrix $A = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ with $a \geq 3$ and $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated symmetric rank 2 hyperbolic Kac-Moody Lie algebra over the field of complex numbers. Let $\Pi = \{\alpha_0, \alpha_1\}$ denote the set of simple roots with Δ its root system. A root $\alpha \in \Delta$ is called a *real root* if there exists $w \in W$ such that $w(\alpha)$ is a *simple root*, and a root which is not real is called an *imaginary root*. We denote by Δ^{re}, Δ^{re}_+, Δ^{im}, and Δ^{im}_+ the set of all real, positive real, imaginary and positive imaginary roots, respectively. We also denote by $\Delta^{im}_{+,k}$ the set of all positive imaginary roots of the algebra $\mathfrak{g}(A)$ with square length $-2k$. In [2], A.J.Feingold show that the Fibonacci numbers are intimately related to the rank 2 hyperbolic GCM Lie algebras. In [5], S.J.Kang and D.J.Melville show that all the roots of a given length are Weyl conjugate to roots in a small region. These information help in determining the sufficient conditions for the existence of integral points on the hyperbola $\mathfrak{h}_k : x^2 - axy + y^2 = -k$ $(k \in \mathbb{Z}_{>0})$.

In this paper, we give some results on the Lie-Fibonacci sequence and symmetric hyperbolic Kac-Moody algebra of rank 2.

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In section 2, we derive several interesting properties of the Lie-Fibonacci sequence. And then we give the following results:
1. If n increases, then the ratio of two successive Lie-Fibonacci number approaches

$$\frac{a-2+\sqrt{a^2-4}}{2}, \text{ or } \left(\frac{1}{a-2}\right)\left(\frac{a-2+\sqrt{a^2-4}}{2}\right)$$

(which is the golden ratio if $a=3$).
2. Two successive Lie-Fibonacci numbers $F_n^{(a)}$ and $F_{n+1}^{(a)}$ are relatively prime.

In section 3, we give some definitions and known results on the Kac-Moody algebras and the study of their elementary properties. We derive the relations among the Lie-Fibonacci numbers. We also give some sufficient conditions for the existence of integral points. We find the number of elements of Ω_k for some k. Lastly, we give the following theorem:

Theorem. *Let $x^2-axy+y^2=-(a-2)\gamma^2$ for $a\geq 3$ and $\gamma\in\mathbb{Z}_{>0}$ be the hyperbola. If $a+2=\gamma^2$, and $a-2$ is a square free integer, then $|\Omega_{(a-2)\gamma^2}|=2$.*

This procedure finds all the integral points on these hyperbolas far more easily than the traditional number-theoretic algorithm.

2. Lie-Fibonacci Sequence

In this section, we introduce the Lie generalized Fibonacci sequence $\{F_n^{(a)}\}$, and generalize the several interesting properties of the Fibonacci sequence $\{F_n\}$.

Define a new sequence $\{F_n^{(a)}\}$ by the recurrence relations

$$\begin{aligned} F_0^{(a)} &= F_1^{(a)}=1, \\ F_{2n+2}^{(a)} &= aF_{2n}^{(a)}-F_{2n-2}^{(a)} \\ F_{2n+1}^{(a)} &= F_{2n+2}^{(a)}-F_{2n}^{(a)} \quad (n>0). \end{aligned} \tag{1}$$

Clearly $\{F_n^{(3)}\}=\{F_n\}$, the Fibonacci sequence defined by:

$$F_0=0,\ F_1=1,\ F_{n+2}=F_n+F_{n+1}. \tag{2}$$

We call this sequence $\{F_n^{(a)}\}$, the Lie-Fibonacci sequence, and $F_n^{(a)}$ the Lie-Fibonacci number.

It is well known that there are many interesting identities for the Fibonacci sequence. In this section, we derive several similar identities for the Lie-Fibonacci sequence. Among the several known results concerning Fibonacci numbers, we quote below some interesting ones:

Proposition 2.1 ([7]). *Let $\{F_n\}$ be the Fibonacci sequence. Then we have the followings.*
(a) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.
(b) $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$.
(c) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.
(d) $F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1$.
(e) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$.

To prove several identities for the Lie-Fibonacci sequence , we need the following Proposition.

Lemma 2.2 ([8]). *For any positive integer n, we have*
(a) $F_{2n+3}^{(a)} = aF_{2n+1}^{(a)} - F_{2n-1}^{(a)}$.
(b) $F_{2n}^{(a)} + F_{2n+1}^{(a)} = F_{2n+2}^{(a)}$.
(c) $F_{2n-1}^{(a)} + (a-2)F_{2n}^{(a)} = F_{2n+1}^{(a)}$.

We deduce from Proposition 2.2 the following theorem.

Theorem 2.3. *Let $\{F_n^{(a)}\}$ be the Lie-Fibonacci sequence. Then we have the following.*
(a) $F_1^{(a)} + F_3^{(a)} + F_5^{(a)} + \cdots + F_{2n-1}^{(a)} = F_{2n}^{(a)}$.
(b) $F_2^{(a)} + F_4^{(a)} + \cdots + F_{2n}^{(a)} = \frac{1}{a-2}(F_{2n+1}^{(a)} - 1)$.
(c) $F_1^{(a)} + F_2^{(a)} + \cdots + F_{2n-1}^{(a)} = \frac{1}{a-2}(F_{2n+1}^{(a)} - 1)$.
(d) $F_1^{(a)} + (a-2)F_2^{(a)} + F_3^{(a)} + (a-2)F_4^{(a)} + \cdots + F_{2n-1}^{(a)} + (a-2)F_{2n}^{(a)} = F_{2n+2}^{(a)} - 1$.
(e) $F_1^{(a)} - F_2^{(a)} + F_3^{(a)} - F_4^{(a)} + \cdots + F_{2n-1}^{(a)} - F_{2n}^{(a)} = \frac{1}{a-2}(1 - F_{2n-1}^{(a)})$.
(f) $(F_1^{(a)})^2 + (a-2)(F_2^{(a)})^2 + F_3^{(a)} + \cdots + (a-2)^{l_n}(F_n^{(a)})^2 = F_{2n}^{(a)}F_{2n+1}^{(a)}$, *where*

$$l_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd .} \end{cases}$$

(g) $(F_{2n}^{(a)})^2 = F_{2n}^{(a)}F_{2n+2}^{(a)} - F_{2n}^{(a)}F_{2n+1}^{(a)}$.

Proof. Since $F_{2n+1}^{(a)} = F_{2n+2}^{(a)} - F_{2n}^{(a)}$, we have

$$\begin{aligned} & F_1^{(a)} + F_3^{(a)} + F_5 + \cdots + F_{2n-1}^{(a)} \\ & = F_1^{(a)} + (F_4^{(a)} - F_2^{(a)}) + (F_6^{(a)} - F_4^{(a)}) \cdots + (F_{2n}^{(a)} - F_{2n-2}^{(a)}) \\ & = F_1^{(a)} - F_2^{(a)} + F_{2n}^{(a)} \\ & = F_{2n}^{(a)}, \end{aligned} \tag{3}$$

which proves part (a). For part (b), using Proposition 2.2(e), we have:

$$
\begin{aligned}
&F_2^{(a)} + F_4^{(a)} + F_6^{(a)} + \cdots + F_{2n}^{(a)} \\
&= \frac{1}{a-2}\{ (F_3^{(a)} - F_1^{(a)}) + (F_5^{(a)} - F_3^{(a)}) + \cdots + (F_{2n+1}^{(a)} - F_{2n-1}^{(a)}) \} \\
&= \frac{1}{a-2}(F_{2n+1}^{(a)} - 1),
\end{aligned}
\tag{4}
$$

the desired result. Using parts (a) and (b), we have

$$
\begin{aligned}
&F_1^{(a)} + F_2^{(a)} + \cdots + F_{2n-1}^{(a)} \\
&= (F_1^{(a)} + F_3^{(a)} + F_5^{(a)} + \cdots + F_{2n-1}^{(a)}) + (F_2^{(a)} + F_4^{(a)} + \cdots + F_{2n-2}^{(a)}) \\
&= F_{2n}^{(a)} + F_2^{(a)} + F_4^{(a)} + \cdots + F_{2n-2}^{(a)} \\
&= \frac{1}{a-2}(F_{2n+1}^{(a)} - 1),
\end{aligned}
\tag{5}
$$

which proves part (c). In a similar manner, we can derive parts (d),(e) and (f). □

It is well known that two successive Fibonacci numbers F_n and F_{n+1} are disjoint. The following Theorem shows that the Lie-Fibonacci numbers have the same property.

Theorem 2.4. *Let $\{F_n^{(a)}\}$ be the Lie-Fibonacci sequence. Then two successive Lie-Fibonacci numbers $F_{2n}^{(a)}$ and $F_{2n+1}^{(a)}$ are relatively prime.*

Proof. Clearly, $F_1^{(a)}$ and $F_2^{(a)}$ are relatively prime. Let d be a gcd of $F_{2n}^{(a)}$ and $F_{2n+1}^{(a)}$ $(n \geq 1)$. Since $F_{2n+2}^{(a)} = F_{2n}^{(a)} + F_{2n+1}^{(a)}$, d divides $F_{2n+2}^{(a)}$. On the other hands, $F_{2n+2}^{(a)} = aF_{2n}^{(a)} + F_{2n-2}^{(a)}$. Thus d also divides $F_{2n-2}^{(a)}$ and $F_{2n-1}^{(a)}$. Continuing this process, we arrive at $d = 1$. □

Let $\{F_n\}$ be the Fibonacci sequence. Robert Simson stated that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n,$$

for every positive integer n, as it is to see, by induction on n.

To generalize the Simson's identity concerning the Fibonacci sequence, we need the following Proposition.

Proposition 2.5 ([7]). (*Generalization of Binet formula*) *Let $\{F_n^{(a)}\}$ be the Lie-Fibonacci sequence, and let $\alpha = \dfrac{a+\sqrt{a^2-4}}{2}$ be a zero of $1-(a^2-2)x^2+x^4$. Then we have the following:*

$$\text{(a) } F_{2n}^{(a)} = \frac{1}{\sqrt{a^2-4}} \left(\left(\frac{(\alpha-1)^2}{a-2} \right)^n - \left(\frac{(\frac{1}{\alpha}-1)^2}{a-2} \right)^n \right)$$
$$= \frac{1}{(a-2)^n\sqrt{a^2-4}} \left(\left(\frac{a-2+\sqrt{a^2-4}}{2} \right)^{2n} - \left(\frac{a-2-\sqrt{a^2-4}}{2} \right)^{2n} \right),$$

$$\text{(b) } F_{2n+1}^{(a)} = \frac{1}{\sqrt{a^2-4}} \left(\left(\frac{(\alpha-1)^2}{a-2} \right)^n - \left(\frac{(\frac{1}{\alpha}-1)^2}{a-2} \right)^n \right)$$
$$= \frac{1}{(a-2)^n\sqrt{a^2-4}} \left(\left(\frac{a-2+\sqrt{a^2-4}}{2} \right)^{2n+1} - \left(\frac{a-2-\sqrt{a^2-4}}{2} \right)^{2n+1} \right).$$

Theorem 2.6. *Let $\{F_n^{(a)}\}$ be the Lie-Fibonacci sequence and $n \in \mathbb{Z}_{>0}$. Then we have:*

(a) $F_{2n-1}^{(a)}F_{2n+1}^{(a)} - (a-2)(F_{2n}^{(a)})^2 = 1$.
(b) $(a-2)F_{2n}^{(a)}F_{2n+2}^{(a)} - (F_{2n+1}^{(a)})^2 = -1$.
(c) $F_n^{(a)}F_{n+1}^{(a)} - F_{n-1}^{(a)}F_{n+2}^{(a)} = (-1)^{n+1}$.

Proof. Let $\beta = \frac{1}{\alpha}$, $\alpha' = \alpha - 1$ and $\beta' = \beta - 1$. Then we have

$$F_{2n-1}^{(a)}F_{2n+1}^{(a)} - (a-2)(F_{2n}^{(a)})^2$$
$$= \frac{1}{(a-2)^{n-1}\sqrt{a^2-4}} \left(\left(\frac{a-2+\sqrt{a^2-4}}{2} \right)^{2n-1} - \left(\frac{a-2-\sqrt{a^2-4}}{2} \right)^{2n-1} \right)$$
$$\frac{1}{(a-2)^n\sqrt{a^2-4}} \left(\left(\frac{a-2+\sqrt{a^2-4}}{2} \right)^{2n+1} - \left(\frac{a-2-\sqrt{a^2-4}}{2} \right)^{2n+1} \right)$$
$$-(a-2)\left(\frac{1}{(a-2)^n\sqrt{a^2-4}} \left(\left(\frac{a-2+\sqrt{a^2-4}}{2} \right)^{2n} - \left(\frac{a-2-\sqrt{a^2-4}}{2} \right)^{2n} \right) \right)^2$$
$$= \frac{1}{(a-2)^{2n-1}(a^2-4)} \left(\alpha'^{4n} - (\alpha'\beta')^{2n-1}\left(\beta'^2+\alpha'^2\right) + \beta'^{4n} - \alpha'^{4n} + 2(\alpha'\beta')^{2n} - \beta'^{4n} \right)$$
$$= \frac{-(\alpha'\beta')^{2n-1}}{(a-2)^{2n-1}(a^2-4)} \left(\alpha'^2 + \beta'^2 - 2\alpha'\beta' \right)$$
$$= \frac{-(2-a)^{2n-1}}{(a-2)^{2n-1}(a^2-4)} \left(\alpha' - \beta' \right)^2$$
$$= 1,$$

which proves part (a). In a similar manner, we can derive parts (b) and (c). □

It is well known that as n increases the ratio $\frac{F_{n+1}}{F_n}$ approaches $\frac{1+\sqrt{5}}{2}$, the golden ratio. The following Theorem shows that the Lie-Fibonacci sequence $\{F_n^{(a)}\}$ has similar properties.

Theorem 2.7. *Let $\{F_n^{(a)}\}$ be the Lie-Fibonacci sequence, and let $\alpha = \frac{a+\sqrt{a^2-4}}{2}$. Then we have*
(a) $\lim_{n\to\infty} \frac{F_{2n+1}^{(a)}}{F_{2n}^{(a)}} = \frac{a-2+\sqrt{a^2-4}}{2} = \alpha - 1.$
(b) $\lim_{n\to\infty} \frac{F_{2n+2}^{(a)}}{F_{2n+1}^{(a)}} = \left(\frac{1}{a-2}\right)\left(\frac{a-2+\sqrt{a^-4}}{2}\right) = \frac{1}{a-2}(\alpha-1).$

In particular, $\lim_{n\to\infty} \frac{F_{2n+1}^{(3)}}{F_{2n}^{(3)}} = \frac{1+\sqrt{5}}{2}$, *the golden ratio.*

Proof. Let $p_n = \frac{F_{2n+2}^{(a)}}{F_{2n}^{(a)}}$. Then we have

$$
\begin{aligned}
p_n &= \frac{aF_{2n}^{(a)} - F_{2n-2}^{(a)}}{F_{2n}^{(a)}} \\
&= a - \frac{1}{p_{n-1}} \qquad (6) \\
&= a - \cfrac{1}{a - \cfrac{1}{p_{n-2}} \cdots}
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty} P_n \text{ is a zero of } \ x = a - \frac{1}{x},$$

and hence,

$$\lim_{n\to\infty} P_n = \frac{a-\sqrt{a^2-4}}{2}.$$

Let

$$q_n = \frac{F_{2n+1}^{(a)}}{F_{2n}^{(a)}}.$$

Then we have

$$
\begin{aligned}
q_n &= \frac{F_{2n+2}^{(a)} - F_{2n}^{(a)}}{F_{2n}^{(a)}} \\
&= p_n - 1.
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty} q_n = \frac{a-\sqrt{a^2-4}}{2} - 1 = \frac{a-2+\sqrt{a^2-4}}{2},$$

which proves for part (a). In a similar manner, we can derive part (b). □

3. Existence of Integral Points on the Hyperbolas

In this section, we study the root system of the rank 2 hyperbolic Kac-Moody algebras $\mathfrak{g}(A)$ with symmetric generalized Cartan matrix $A = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ with $a \geq 3$. Let W be the Weyl group of $g(A)$, generated by simple reflections r_1 and r_2.

We identify an element

$$\alpha = x\alpha_1 + y\alpha_2 \in Q \text{ with an ordered pair } (x, y) \in \mathbb{Z} \times \mathbb{Z}. \tag{7}$$

We call a root $\alpha \in \mathbb{Z}\times\mathbb{Z}$ the *positive integral point* if $x, y \in \mathbb{Z}_{\geq 0}$. Define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ by the following equation:

$$(\alpha_1 \,|\, \alpha_1) = (\alpha_2 \,|\, \alpha_2) = 2, \qquad (\alpha_1|\alpha_2) = -a. \tag{8}$$

Then for $\alpha = x\alpha_1 + y\alpha_2$, we have $(\alpha \,|\, \alpha) = 2(x^2 - axy + y^2)$.

It is well known that there is a one-to-one correspondence between the set of real roots of $\mathfrak{g}(A)$ and the set of integral points on the hyperbola $x^2 - axy + y^2 = 1$. Since there is no root α such that $(\alpha \,|\, \alpha) = 0$, the imaginary roots of $\mathfrak{g}(A)$ correspondence to the set of integral points on the hyperbolas $\mathfrak{h}_k : x^2 - axy + y^2 = -k$ for $k \geq 1$. In other words, for each $k \geq 1$, there is a one-to-one correspondence between the set of all imaginary roots α with square length $(\alpha \,|\, \alpha) = -2k$ and the set of all integral points on the hyperbola $\mathfrak{h}_k$.

We introduce the sequences of integers $\{B_n\}$ for $n \geq 0$ by the recurrence relations

$$B_0 = 0, \qquad B_1 = 1, \quad \text{and} \quad B_{n+2} = aB_{n+1} - B_n \quad \text{for} \quad n \geq 1. \tag{9}$$

Clearly, we have

$$F^{(a)}_{2n} = B_n, \quad \text{and} \quad F^{(a)}_{2n-2} = B_n - B_{n-1}. \tag{10}$$

The following Proposition is well known.

Proposition 3.1 ([3]). $\Delta^{re}_+ = \{(B_n, B_{n+1}), \ (B_{n+1}, B_n) \mid n \geq 0\}$. *Furthermore,*

$$\Delta^{re}_+ = \{(F_{2j}, F_{2j+2}), \ (F_{2j+2}, F_{2j}) \,|\, j \in \mathbb{Z}_{\geq 0}\}$$

for $a = 3$.

For a positive integer k, let $\Delta^{im}_{+,k}$ be the set of all positive imaginary roots α of $\mathfrak{g}(A)$ with square length $(\alpha\,|\,\alpha) = -2k$. That is, $\Delta^{im}_{+,k}$ is the set of all positive integral points on the hyperbola $\mathfrak{h}_k$. The following Proposition gives a nice description of the set of positive imaginary roots of length $-2k$.

Proposition 3.2 ([5]).

$$\begin{aligned}\Delta^{im}_{+,k} =\{&(m,n),\ (n,m),\ (mB_{j+1}-nB_j, mB_{j+2}-nB_{j+1}),\\ &(mB_{j+2}-nB_{j+1}, mB_{j+1}-nB_j),\ (nB_{j+1}-mB_j, nB_{j+2}-mB_{j+1}),\\ &(nB_{j+2}-mB_{j+1}, nB_{j+1}-mB_j)\,|\,(m,n)\in\Omega_k\},\end{aligned}$$

where

$$\Omega_k=\left\{(m,n)\in\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 0}\Big|\frac{2\sqrt{k}}{\sqrt{a^2-4}}\le m\le\sqrt{\frac{k}{a-2}}, n=\frac{am-\sqrt{(a^2-4)m^2-4k}}{2}\right\}.$$

Since $F^{(a)}_{2n} = B_n$, and $F^{(a)}_{2n-2} = B_n - B_{n-1}$. Proposition 3.1 and Proposition 3.2 can be rewritten as follows:

Proposition 3.3. *Let $\{F^{(a)}_n\}$ be the Lie-Fibonacci sequence. Then*
(a) *The set of all nonnegative integral points on the hyperbola*

$$x^2 - axy + y^2 = 1$$

is $\{(F^{(a)}_{2n}, F^{(a)}_{2n+2}), (F^{(a)}_{2n+2}, F^{(a)}_{2n})\,|\,n\in\mathbb{Z}_{\ge 0}\}$.
(b) *The set of all nonnegative integral points on the hyperbola*

$$x^2 - axy + y^2 = -(a-2)$$

is $\{(1,1), (F^{(a)}_{2n-1}, F^{(a)}_{2n+1}), (F^{(a)}_{2n+1}, F^{(a)}_{2n-1})\,|\,n\in\mathbb{Z}_{\ge 0}\}$.

Proof. (a) is immediate consequence of Proposition 3.1 and definition of the Lie-Fibonacci sequence. For (b), after simple calculation, we have $\Omega_k = \{(1,1)\}$ and hence

$$\Delta^{im}_{+} = \{(1,1), (F^{(a)}_{2n-1}, F^{(a)}_{2n+1}), (F^{(a)}_{2n+1}, F^{(a)}_{2n-1})\,|\,n\in\mathbb{Z}_{\ge 0}\}.$$

□

To list all the integral points on those hyperbolas, we also find the number of elements of Ω_k.

Proposition 3.4. ([7]) *Let $x^2 - axy + ay^2 = -k$ be the hyperbola and let $k = t\gamma^2$ be any positive integer where t is a square free integer and $\gamma\in\mathbb{Z}_{>0}$. If $(\gamma,\delta)\in\Omega_k$ for some positive integer δ, then*

$$a-2 \le t \le \frac{a^2-4}{4} \quad \textit{for} \quad a \ge 3,$$

where

$$\Omega_k = \left\{ (m,n) \in \Delta^{im}_{+,k} \Big| \frac{2\sqrt{k}}{\sqrt{a^2-4}} \le m \le \sqrt{\frac{k}{a-2}}, n = \frac{am - \sqrt{(a^2-4)m^2-4k}}{2} \right\}.$$

Since W is infinite, $\Omega_k \neq \emptyset$ implies that there are infinitely many integral points on the hyperbola $x^2 - axy + y^2 = -k$. Proposition 3.3 tells us that Ω_k have crucial information for the set of integral points on the hyperbola $x^2 - axy + y^2 = -k$. We have the following Lemma.

Lemma 3.5. *Let $x^2 - axy + ay^2 = -k$ be the hyperbola. If $k < a-2$, then there is no integral point on that hyperbola.*

Proof. Since there is no integer m with $\frac{2\sqrt{k}}{\sqrt{a^2-4}} \le m \le \sqrt{\frac{k}{a-2}}$, we have $\Omega_k = \emptyset$, and hence we get the desired result. □

The following Proposition is obtained by the definition of Ω_k.

Proposition 3.6. *Let $x^2-axy+y^2 = -(a-2)\gamma^2$ be the hyperbola for $a \ge 3$ and $\gamma \in \mathbb{Z}_{>0}$. If $\gamma < \frac{n\sqrt{a+2}}{\sqrt{a+2}-2}$, then $1 \le |\,\Omega_{(a-2)\gamma^2}| \le n$. Furthermore, $\gamma < \frac{\sqrt{a+2}}{\sqrt{a+2}-2}$, then $|\,\Omega_{(a-2)\gamma^2}| = 1$.*

Proof. Clearly, we have $(\gamma,\gamma) \in \Omega_{(a-2)\gamma^2}$, and hence,

$$|\,\Omega_{(a-2)\gamma^2}| \ge 1.$$

Consider the set

$$\Omega_{(a-2)\gamma^2} = \left\{ (m,n) \in \Delta^{im}_{+,k} \Big| \frac{2\gamma}{\sqrt{a+2}} \le m \le \gamma, n = \frac{am - \sqrt{(a^2-4)m^2 - 4(a-2)\gamma^2}}{2} \right\}.$$

Since

$$\gamma - \frac{2\gamma}{\sqrt{a+2}} < n \ \text{ implies } \ \gamma < \frac{n\sqrt{a+2}}{\sqrt{a+2}-2},$$

at most n positive integers exist between $\frac{2\gamma}{\sqrt{a+2}}$ and γ, we get the desired result. □

Example 3.7. Let $x^2-7xy+y^2 = -5$ be the hyperbola. Then we have $\Omega_5 = \{(1,1)\}$. Therefore,

$$\begin{aligned} \Delta^{im}_{+} &= \{(1,1),\, (F^{(7)}_{2n-1}, F^{(7)}_{2n+1}),\, (F^{(7)}_{2n+1}, F^{(7)}_{2n-1}) \,|\, n \ge 1\} \\ &= \{(1,1),\, (1,6),\, (6,1)(6,41),\, (41,6), \cdots\cdots\}. \end{aligned}$$

Example 3.8. Let $x^2 - 3xy + y^2 = -\gamma^2$ be the hyperbola. Since $a = 3$, $\gamma < \dfrac{\sqrt{5}}{\sqrt{5}-2}$ implies $|\,\Omega_{(a-2)\gamma^2}| = 1$, thus we have $|\Omega_{(a-2)\gamma^2}| = 1$ for $1 \le \gamma \le 9$. Therefore,

$$\Omega_{(a-2)\gamma^2} = \{(\gamma,\gamma)\} \text{ for } 1 \le \gamma \le 9,$$

and hence

$$\Delta^{re} = \{\sigma(\gamma,\gamma) \,|\, \sigma \in W\}.$$

For the case of $a = 4$, similarly we have, $|\Omega_{(a-2)\gamma^2}\,| = 1$ for $1 \le \gamma \le 5$.

Lemma 3.9. *Let $x^2 - axy + y^2 = -(a-2)\gamma^2$ for $a \ge 3$ and $\gamma \in \mathbb{Z}_{>0}$ be the hyperbola. If $a + 2 = \gamma^2$, then $|\,\Omega_{(a-2)\gamma^2}| \ge 2$.*

Proof. Clearly, $(\gamma,\gamma) \in \Omega_{(a-2)\gamma^2}$. If we substitute γ^2 for $a+2$, then we have $\gamma \ge 3$ and

$$\Omega_{a^2-4} = \left\{ (m,n) \in \Delta^{im}_{+,k} \,\middle|\, 2 \le m \le \gamma, n = \frac{am - \sqrt{(a-2)(m^2-4)}\gamma}{2} \right\}.$$

Thus we have $\{(2,a),\ (\gamma,\gamma)\} \subseteq \Omega_{a^2-4}$, and hence we get the desired result. □

Theorem 3.10. *Let $x^2 - axy + y^2 = -(a-2)\gamma^2$ for $a \ge 3$ and $\gamma \in \mathbb{Z}_{>0}$ be the hyperbola. If $a + 2 = \gamma^2$, and $a - 2$ is a square free integer, then $|\,\Omega_{(a-2)\gamma^2}| = 2$.*

Proof. If $(m,n) \in \Omega_{(a-2)\gamma^2}$ for some $n \in \mathbb{Z}_{>0}$, then we have $m^2 - 4 = (a-2)l^2$ for some $l \in \mathbb{Z}_{\ge}$. Since $a - 2 = \gamma^2 - 4$, and $m \le \gamma$, we have $\gamma^2 - 4 \ge m^2 - 4 = (\gamma^2-4)l^2$, and hence either $l = 0$ or $l = 1$. This implies that either $m = 2$ or $m = \gamma$, and hence $\Omega_{a^2-4} = \{(2,a),(\gamma,\gamma)\}$. □

Example 3.11. Let $x^2 - 7xy + y^2 = -5 \cdot 3^2$ be the hyperbola. Then we have

$$\Omega_{7\cdot 3^2} = \{(2,7),(3,3)\},$$

and hence

$$\begin{aligned}\Delta^{im}_{+} \ &= \{3(F^{(7)}_{2n-1},\ F^{(7)}_{2n+1}),\ 3(F^{(7)}_{2n+1}, F^{(7)}_{2n-1}),\ (2F^{(7)}_{2n+2} - 7F^{(7)}_{2n}, 2F^{(7)}_{2n+4} - 7F^{(7)}_{2n+2}),\\ &\quad (2F^{(7)}_{2n+4} - 7F^{(7)}_{2n+2}, 2F^{(7)}_{2n+2} - 7F^{(7)}_{2n})(7F^{(7)}_{2n+2} - 2F^{(7)}_{2n}, 7F^{(7)}_{2n+4} - 2F^{(7)}_{2n+2}),\\ &\quad (7F^{(7)}_{2n+4} - 2F^{(7)}_{2n+2}, 7F^{(7)}_{2n+2} - 2F^{(7)}_{2n}) \,|\, n \ge 1\,\}.\end{aligned}$$

Corollary 3.12. *There are many integral solutions $x^2 - axy + y^2 = 4 - a^2$ for $a \ge 2$.*

Theorem 3.13. *If $a \ne 2 \pmod 4$, then there is a one-to-one correspondence between the set of integral points on the hyperbolas $x^2 - axy + y^2 = 1$ and $(a+2)x^2 - (a-2)y^2 = 4$.*

Proof. $(a+2)x'^2-(a-2)y'^2=4$ is obtained from $x^2-axy+y^2=1$ by substituting $(x,y)=\frac{1}{2}(x'+y',\ -x'+y')$, that is $(x',\ y')=(x-y,\ x+y)$.

If x, y are integers, then clearly x' and y' are also integers. On the other hand, we need to show that $(x',y')\in\mathbb{Z}\times\mathbb{Z}$ implies that $(x,y)\in 2\mathbb{Z}\times 2\mathbb{Z}$ or $(x,y)\in(2n+1)\mathbb{Z}\times(2n+1)\mathbb{Z}$. If $a=4k$, then $(4k+2)x'^2-(4k-2)y'^2=4$. That is $(2k+1)x'^2=(2k-1)y'^2+2$. This implies x' and y' are both even or both odd.

Similarly, we can show in the other cases: $a\equiv 1 \pmod 4$ and $a\equiv 3 \pmod 4$. □

Example 3.14. Since the set of all nonnegative integral points on the hyperbola

$$x^2-5xy+y^2=1$$

is $\{(0,1),\ (1,0),\ (1,5),\ (5,1),\ (5,24),\ (24,5),\ (24,115),\ (115,24),\cdots\cdots\}$, and $\{(-1,1),\ (1,1),\ (-4,6),\ (4,6),\ (-19,29),\ (19,29),\ (-91,139),\ (91,139),\cdots\cdots\}$ is the set of integral points on the hyperbola

$$7x^2-3y^2=4.$$

Corollary 3.15. *There are infinitely many integral points on the hyperbola*

$$(a+2)x^2-(a-2)y^2=4\ (a\geq 3,\ a\not\equiv 2 \pmod 4).$$

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