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Integrability of Distributions in GCR-lightlike Submanifolds of Indefinite Sasakian Manifolds

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Abstract. In this paper, we study GCR-lightlike submanifolds of indefinite Sasakian manifold. We give some necessary and sufficient conditions on integrability of various distributions of GCR-lightlike submanifold of an indefinite Sasakian manifold. We also find the conditions for each leaf of holomorphic distribution and radical distribution is totally geodesic.

1. Introduction

The geometry of CR-submanifolds of Kaehler manifolds were introduced by Bejancu [2] in 1978 and further developed by Bejancu [3, 4], Chen [5, 6], Duggal [8, 9], Yano and Kon [16, 18] and others. Contact CR-submanifolds of Sasakian manifolds with definite metric were introduced by Yano and Kon [17] in 1982. For indefinite metric, Duggal and Sahin [11] introduced the notion of contact CR-lightlike submanifolds of indefinite Sasakian manifolds. In continuation, recently Duggal and Sahin [12] introduced a new class of submanifolds namely generalized Cauchy-Riemannian (GCR)-lightlike submanifolds of indefinite Sasakian manifolds, which is an umbrella of invariant, screen real, contact CR-lightlike submanifolds. Since significant applications of contact geometry (Arnol'd [1], Nazaikinskii et. al. [14], Maclane [15]) with definite and indefinite metric and limited information available on its GCR-lightlike case motivated us to extend the work of them. After the

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brief information of GCR-lightlike submanifolds of indefinite Sasakian manifolds, we study the integrability of various distributions.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [10] by Duggal and Bejancu. A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a r-lightlike submanifold if the metric g induced from $\bar{g}$ is degenerate and the radical distribution $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$. Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $(S(TM))^{\perp}$ such that $\bar{g}(\xi_i, N_j) = ?delta_{ij}$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}$. Then,

$$T\bar{M} \mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.1}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.1), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall \quad X, Y \in \Gamma(TM), \tag{2.2}$$

$$\bar{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \quad \forall \quad X \in \Gamma(TM), \quad U \in \Gamma(tr(TM)), \tag{2.3}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^{\perp} U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called a second fundamental form, A_U is linear a operator on M and known as a shape operator. Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (2.2) and (2.3) give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N), \tag{2.5}$$

$$\bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W), \tag{2.6}$$

where $X \in \Gamma(TM)$, $\quad N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (2.4)-(2.5), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.7}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.8}$$

for any $\xi \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Let P be the projection morphism of TM on $S(TM)$. Then we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \tag{2.9}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively.

Next, we recall some basic definitions and results of indefinite Sasakian manifolds from [13]. An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an ϵ-contact metric manifold, if there is a $(1,1)$ tensor field ϕ, a vector field V and a 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \epsilon\eta(X)\eta(Y), \quad \bar{g}(V,V) = \epsilon, \tag{2.11}$$

$$\phi^2(X) = -X + \eta(X)V, \quad \bar{g}(X,V) = \epsilon\eta(X), \tag{2.12}$$

$$d\eta(X,Y) = \bar{g}(X, \phi Y), \forall X, Y \in \Gamma(TM), where \quad \epsilon = \pm 1.$$

It follows that

$$\phi V = 0, \quad \eta o \phi = 0, \quad \eta(V) = 1. \tag{2.13}$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ -contact metric structure of $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is Nijenhuis tensor field of ϕ. A normal ϵ -contact metric manifold is called an ϵ-Sasakian manifold. For which we have

$$\bar{\nabla}_X V = \phi X, \quad (\bar{\nabla}_X \phi)Y = -\bar{g}(X,Y)V + \epsilon\eta(Y)X \tag{2.14}$$

3. Generalized Cauchy-Riemann (GCR)-lightlike submanifolds

Calin[7] proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. We assume that the characteristic vector V is tangent to M throughout this paper.

Definition 2.2. ([12]). Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S.$$

where D_0 is an invariant non degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then tangent bundle TM of M is decomposed as

$$TM = \{D \oplus \bar{D} \oplus \{V\}\}, \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2).$$

An r-lightlike submanifold M is called a proper GCR-lightlike submanifold if $D_1 \neq 0$, $D_2 \neq 0$, $D_0 \neq 0$ and $S \neq 0$. Let Q, P_1, P_2 be the projection morphism on D, $\phi S \subset \bar{D}$, $\phi L \subset \bar{D}$ respectively, therefore

$$X = QX + V + P_1 X + P_2 X, \tag{3.1}$$

for $X \in \Gamma(TM)$. Applying ϕ to (3.1), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{3.2}$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(S)$ and $\omega P_2 X \in \Gamma(L)$ or we can write (3.2) as

$$\phi X = fX + \omega X,$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.
Similarly

$$\phi U = BU + CU, \quad U \in \Gamma(tr(TM)), \tag{3.3}$$

where BU and CU are the sections of TM and $tr(TM)$, respectively. Differentiating (3.2) and using (2.5)-(2.7), (2.8) and (3.3), we have

$$D^l(X, \omega P_1 Y) = -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y).$$

$$D^s(X, \omega P_2 Y) = -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y),$$

for all $X, Y \in \Gamma(TM)$.

4. Integrability of Distributions

Let $\bar{M}$ be a real $m+n$-dimensional smooth manifold then a distribution of rank t on $\bar{M}$ is a mapping D defined on $\bar{M}$, which assign to each point x of $\bar{M}$ a t-dimensional linear subspace D_x of $T_x(M)$. A vector field X on $\bar{M}$ is said to belong to D if $X(x) \in D_x$ for each x of $\bar{M}$. The distribution D is said to be involutive if $[X, Y] \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. Then from page no. 34 of [10], a distribution D is integrable if and only if it is involutive.

Theorem 4.1. *Let M be a proper GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $\bar{D}$ is integrable if and only if $\nabla_X \phi Y = \nabla_Y \phi X$ for any $X, Y \in \Gamma(\bar{D})$.*

Proof. For any $X, Y \in \Gamma(\bar{D})$ we have

$$h(X, \phi Y) = \bar{\nabla}_X \phi Y - \nabla_X \phi Y.$$

Replacing X by Y and subtracting the resulting equation from the above equation, we get

$$\begin{aligned} h(X, \phi Y) - h(Y, \phi X) &= \bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X - \nabla_X \phi Y + \nabla_Y \phi X \\ &= \phi[X, Y] - \nabla_X \phi Y + \nabla_Y \phi X \\ &= f[X, Y] + \omega[X, Y] - \nabla_X \phi Y + \nabla_Y \phi X. \end{aligned}$$

Taking tangential parts of this equation , we have

$$f[X, Y] = \nabla_X \phi Y - \nabla_Y \phi X. \tag{4.1}$$

Hence from (4.1), the result follows. □

Theorem 4.2. *Let proper M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then D_0 is integrable if and only if*

(i) $\bar{g}(h^*(X, Y), N) = \bar{g}(h^*(Y, X), N)$

(ii) $g(\nabla^*_X Y, V) = g(\nabla^*_Y X, V)$

(iii) $\bar{g}(h^*(X, \phi Y), N_1) = \bar{g}(h^*(Y, \phi X), N_1)$

(iv) $h^s(X, \phi Y) = h^s(Y, \phi X)$

(v) $g(\nabla^*_X Y, \phi\xi) = g(\nabla^*_Y X, \phi\xi)$,

for any $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), N_1 \in \Gamma(L), W \in \Gamma(S)$, and $\xi \in \Gamma(D_2)$.

Proof. Using the definition of GCR-lightlike submanifold D_0 is integrable if and only if

$$\bar{g}([X, Y], \phi W) = \bar{g}([X, Y], \phi\xi) = \bar{g}([X, Y], V) = \bar{g}([X, Y], \phi N_1) = \bar{g}([X, Y], N) = 0,$$

for any $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), N_1 \in \Gamma(L), W \in \Gamma(S)$, and $\xi \in \Gamma(D_2)$. Using (2.9) we have

$$\bar{g}([X, Y], N) = \bar{g}(h^*(X, Y), N) - \bar{g}(h^*(Y, X), N), \tag{4.2}$$

and

$$\bar{g}([X,Y],V) = g(\nabla^*_X Y, V) - g(\nabla^*_Y X, V) \tag{4.3}$$

On the other hand using (2.4) and (2.14) we get

$$\begin{aligned} \bar{g}([X,Y],\phi N_1) &= \bar{g}(\bar{\nabla}_X Y, \phi N_1) - \bar{g}(\bar{\nabla}_Y X, \phi N_1) \\ &= -\bar{g}(\nabla_X \phi Y, N_1) + \bar{g}(\nabla_Y \phi X, N_1) \\ &= -\bar{g}(h^*(X,\phi Y), N_1) + \bar{g}(h^*(Y,\phi X), N_1). \end{aligned} \tag{4.4}$$

From (2.4) and (2.14) and using the indefinite Sasakian character of $\bar{M}$, we derive

$$\begin{aligned} \bar{g}([X,Y],\phi W) &= \bar{g}(\bar{\nabla}_X Y, \phi W) - \bar{g}(\bar{\nabla}_Y X, \phi W) \\ &= -\bar{g}(\bar{\nabla}_X \phi Y, W) + \bar{g}(\bar{\nabla}_Y \phi X, W) \\ &= -\bar{g}(h^s(X,\phi Y), W) + \bar{g}(h^s(Y,\phi X), W). \end{aligned} \tag{4.5}$$

Finally from equation (2.4) and (2.9) we obtain

$$\begin{aligned} \bar{g}([X,Y],\phi\xi) &= g(\nabla_X Y, \phi\xi) - g(\nabla_Y X, \phi\xi) \\ &= g(\nabla^*_X Y, \phi\xi) - g(\nabla^*_Y X, \phi\xi). \end{aligned} \tag{4.6}$$

Thus from (4.2)-(4.6) the proof is complete. □

Theorem 4.3. *Let M be a proper GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $RadTM$ is integrable if and only if*

(i) $g(A^*_{\xi''}\xi', V) = g(A^*_{\xi'}\xi'', V)$

(ii) $\bar{g}(h^l(\xi', Z), \xi'') = \bar{g}(h^l(\xi'', Z), \xi')$

(iii) $\bar{g}(h^l(\xi', \phi\xi), \xi'') = \bar{g}(h^l(\xi'', \phi\xi), \xi')$

(iv) $h^s(\xi', \phi\xi'') = h^s(\xi'', \phi\xi')$

(v) $g(h^*(\xi', \phi\xi''), N_1) = g(h^*(\xi'', \phi\xi'), N_1)$.

for any $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L)$, $W \in \Gamma(S)$, $Z \in \Gamma(D_0)$.

Proof. Using the definition of GCR-lightlike submanifold of an indefinite Sasakian manifold, $Rad(TM)$ is integrable if and only if

$$\bar{g}([\xi',\xi''],V) = \bar{g}([\xi',\xi''],Z) = \bar{g}([\xi',\xi''],\phi\xi) = \bar{g}([\xi',\xi''],\phi W) = \bar{g}([\xi',\xi''],\phi N_1) = 0,$$

for any $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L), W \in \Gamma(S)$, and $Z \in \Gamma(D_0)$. Using (2.10), we obtain

$$\bar{g}([\xi',\xi''],V) = -g(A^*_{\xi''}\xi', V) + g(A^*_{\xi'}\xi'', V), \tag{4.7}$$

and

$$\bar{g}([\xi', \xi''], Z) = -g(A^*_{\xi''}\xi', Z) + g(A^*_{\xi'}\xi'', Z) = -\bar{g}(h^l(\xi', Z), \xi'') + \bar{g}(h^l(\xi'', Z), \xi'). \tag{4.8}$$

Using (2.4) we get

$$\begin{aligned}\bar{g}([\xi', \xi''], \phi\xi) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi\xi) - \bar{g}(\bar{\nabla}_{\xi''}\xi', \phi\xi)\\ &= -\bar{g}(\xi'', \bar{\nabla}_{\xi'}\phi\xi) + \bar{g}(\xi', \bar{\nabla}_{\xi''}\phi\xi)\\ &= -\bar{g}(h^l(\xi', \phi\xi), \xi'') + \bar{g}(h^l(\xi'', \phi\xi), \xi').\end{aligned} \tag{4.9}$$

On the other hand using (2.4), (2.9), (2.11) and (2.14) we get

$$\begin{aligned}\bar{g}([\xi', \xi''], \phi W) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi W) - \bar{g}(\bar{\nabla}_{\xi''}\xi', \phi W)\\ &= -\bar{g}(\phi\bar{\nabla}_{\xi'}\xi'', W) + \bar{g}(\phi\bar{\nabla}_{\xi''}\xi', W)\\ &= -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'', W) + \bar{g}(\bar{\nabla}_{\xi''}\phi\xi', W)\\ &= -\bar{g}(h^s(\xi', \phi\xi''), W) + \bar{g}(h^s(\xi'', \phi\xi'), W),\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\bar{g}([\xi', \xi''], \phi N_1) &= -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - (\bar{\nabla}_{\xi'}\phi)\xi'', N_1) + \bar{g}(\bar{\nabla}_{\xi''}\phi\xi' - (\bar{\nabla}_{\xi''}\phi)\xi', N_1)\\ &= -\bar{g}(\nabla_{\xi'}\phi\xi'', N_1) + \bar{g}(\nabla_{\xi''}\phi\xi', N_1)\\ &= -\bar{g}(h^*(\xi', \phi\xi''), N_1) + \bar{g}(h^*(\xi'', \phi\xi'), N_1).\end{aligned} \tag{4.11}$$

Thus from (4.7) - (4.11) the proof is complete. □

Theorem 4.4. *Let M be a proper GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then D_1 is integrable if and only if*

(i) $\nabla^{*t}_X\phi Y - \nabla^{*t}_Y\phi X \in \Gamma(D_1)$.

(ii) $Bh(X, \phi Y) = Bh(Y, \phi X)$

(iii) $A^*_{\phi Y}X = A^*_{\phi X}Y$ *and belongs to* $\Gamma[(\bar{D} \oplus \phi D_2) \perp D_0]$

for any $X, Y \in \Gamma(D_1)$.

Proof. For any $X, Y \in \Gamma Rad(TM)$ using (2.14), we have

$$\bar{\nabla}_X\phi Y = \phi\bar{\nabla}_X Y$$

$$\bar{\nabla}_X Y = -\phi\bar{\nabla}_X\phi Y - \epsilon g(A^*_Y X, V)V$$

$$\nabla_X Y + h(X, Y) = -\phi(\nabla_X\phi Y + h(X, \phi Y)) - \epsilon g(A^*_Y X, V)V. \tag{4.12}$$

Let $X, Y \in \Gamma(D_1)$ and using (2.10) we have

$$\nabla_X Y + h(X, Y) = -\phi(-A^*_{\phi Y}X + \nabla^{*t}_X\phi Y + h(X, \phi Y)) - \epsilon g(A^*_Y X, V)V, \tag{4.13}$$

equating the tangential components of above equation both sides, we get

$$\nabla_X Y = fA^*_{\phi Y}X - f\nabla^{*t}_X \phi Y - Bh(X, \phi Y)) - \epsilon g(A^*_Y X, V)V, \tag{4.14}$$

replacing X by Y and subtracting resulting equation from this equation, we get

$$\begin{aligned}[X, Y] &= f(A^*_{\phi Y}X - A^*_{\phi X}Y) - f(\nabla^{*t}_X \phi Y - \nabla^{*t}_Y \phi X) + Bh(X, \phi Y) - Bh(Y, \phi X) \\ &\quad -\epsilon g(A^*_Y X, V)V + \epsilon g(A^*_X Y, V)V,\end{aligned}$$

thus $[X, Y] \in \Gamma(D_1)$ if and only if $\nabla^{*t}_X \phi Y - \nabla^{*t}_Y \phi X \in \Gamma(D_1)$, $Bh(X, \phi Y) = Bh(Y, \phi X)$, $A^*_{\phi Y}X = A^*_{\phi X}Y$ and belong to $\Gamma[(\bar{D} \oplus \phi D_2)\perp D_0]$, this completes the proof. □

Theorem 4.5. *Let M be a proper GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then D_2 is integrable if and only if*

(i) $\nabla^*_X \phi Y - \nabla^*_Y \phi X \in \Gamma(\phi D_2)$.

(ii) $Bh(X, \phi Y) = Bh(Y, \phi X)$.

(iii) $h^*(X, \phi Y) - h^*(Y, \phi X)$

(iv) $A^*_X Y$ *and belongs to* $\Gamma[(\bar{D} \oplus \phi D_2)\perp D_0]$,

for any $X, Y \in \Gamma(D_2)$.

Proof. Let $X, Y \in \Gamma(D_2)$ and using (2.10) in (4.12)we have

$$\nabla_X Y + h(X, Y) = -\phi(\nabla^*_X \phi Y + h^*(X, \phi Y) + h(X, \phi Y)) - \epsilon g(A^*_Y X, V)V, \tag{4.15}$$

equating the tangential components of above equation, we get

$$\nabla_X Y = -f\nabla^*_X \phi Y - fh^*(X, \phi Y) - Bh(X, \phi Y)) - \epsilon g(A^*_Y X, V)V, \tag{4.16}$$

replacing X by Y and subtracting resulting equation from this equation, we get

$$\begin{aligned}[X, Y] &= -f(\nabla^*_X \phi Y - \nabla^*_Y \phi X) - f(h^*(X, \phi Y) - h^*(Y, \phi X)) - Bh(X, \phi Y) \\ &\quad +Bh(Y, \phi X) - \epsilon g(A^*_Y X, V)V + \epsilon g(A^*_X Y, V)V,\end{aligned}$$

thus $[X, Y] \in \Gamma(D_2)$ if and only if $\nabla^*_X \phi Y - \nabla^*_Y \phi X \in \Gamma(\phi D_2)$, $h^*(X, \phi Y) = h^*(Y, \phi X)$, $Bh(X, \phi Y) = Bh(Y, \phi X)$, $A^*_X Y$ and belongs to $\Gamma[(\bar{D} \oplus \phi D_2)\perp D_0]$, which proves the theorem. □

Theorem 4.6. *Let M be a proper GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then ϕD_2 is integrable if and only if*

(i) $g(A^*_Y \phi X, \phi Z) = g(A^*_X \phi Y, \phi Z)$

(ii) $h^s(\phi X, Y) = h^s(\phi Y, X)$

(iii) $\bar{g}(h^l(\phi X, Y), \xi) = \bar{g}(h^l(\phi Y, X)$

(iv) $g(\phi Y, A_N\phi X) = g(\phi X, A_N\phi Y)$

for any $X, Y \in \Gamma(D_2), Z \in \Gamma(D_0), W \in \Gamma(S), N \in \Gamma(ltr(TM))$, *and* $\xi \in \Gamma(D_2)$.

Proof. Using the definition of GCR-lightlike submanifolds, it is clear that ϕD_2 is integrable if and only if

$$\begin{aligned}\bar{g}([\phi X, \phi Y], Z) &= \bar{g}([\phi X, \phi Y], V) = \bar{g}([\phi X, \phi Y], \phi W)\\ &= \bar{g}([\phi X, \phi Y], \phi\xi) = \bar{g}([\phi X, \phi Y], N) = 0,\end{aligned}$$

for any $X, Y \in \Gamma(D_2), Z \in \Gamma(D_0), W \in \Gamma(S), \xi \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$. From (2.2), (2.10)-(2.12), (2.13) and (2.14), we obtain

$$\begin{aligned}\bar{g}([\phi X, \phi Y], Z) &= \bar{g}(\bar{\nabla}_{\phi X}\phi Y, Z) - \bar{g}(\bar{\nabla}_{\phi Y}\phi X, Z)\\ &= \bar{g}(\phi\bar{\nabla}_{\phi X}Y, Z) - \bar{g}(\phi\bar{\nabla}_{\phi Y}X, Z)\\ &= -\bar{g}(\bar{\nabla}_{\phi X}Y, \phi Z) + \bar{g}(\bar{\nabla}_{\phi Y}X, \phi Z)\\ &= -g(\nabla_{\phi X}Y, \phi Z) + g(\nabla_{\phi Y}X, \phi Z)\\ &= g(A^*_Y\phi X, \phi Z) - g(A^*_X\phi Y, \phi Z),\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\bar{g}([\phi X, \phi Y], V) &= \bar{g}(\bar{\nabla}_{\phi X}\phi Y, V) - \bar{g}(\bar{\nabla}_{\phi Y}\phi X, V)\\ &= -\bar{g}(\phi Y, \bar{\nabla}_{\phi X}V) + \bar{g}(\phi X, \bar{\nabla}_{\phi Y}V)\\ &= -g(\phi Y, \phi^2 X) + g(\phi X, \phi^2 Y)\\ &= 0.\end{aligned} \tag{4.18}$$

Using (2.11), we obtain

$$\begin{aligned}\bar{g}([\phi X, \phi Y], \phi W) &= \bar{g}(\bar{\nabla}_{\phi X}\phi Y, \phi W) - \bar{g}(\bar{\nabla}_{\phi Y}\phi X, \phi W)\\ &= \bar{g}(\phi\bar{\nabla}_{\phi X}Y, \phi W) - \bar{g}(\phi\bar{\nabla}_{\phi Y}X, \phi W)\\ &= \bar{g}(\bar{\nabla}_{\phi X}Y, W) + \bar{g}(\bar{\nabla}_{\phi Y}X, W)\\ &= \bar{g}(h^s(\phi X, Y), W) - \bar{g}(h^s(\phi Y, X), W),\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\bar{g}([\phi X, \phi Y], \phi\xi) &= \bar{g}(\bar{\nabla}_{\phi X}\phi Y, \phi\xi) - \bar{g}(\bar{\nabla}_{\phi Y}\phi X, \phi\xi)\\ &= \bar{g}(\phi\bar{\nabla}_{\phi X}Y, \phi\xi) - \bar{g}(\phi\bar{\nabla}_{\phi Y}X, \phi\xi)\\ &= \bar{g}(\bar{\nabla}_{\phi X}Y, \xi) + \bar{g}(\bar{\nabla}_{\phi Y}X, \xi)\\ &= \bar{g}(h^l(\phi X, Y), \xi) - \bar{g}(h^l(\phi Y, X), \xi),\end{aligned} \tag{4.20}$$

finally

$$\begin{aligned}\bar{g}([\phi X, \phi Y], N) &= \bar{g}(\bar{\nabla}_{\phi X}\phi Y, N) - \bar{g}(\bar{\nabla}_{\phi Y}\phi X, N)\\ &= -\bar{g}(\phi Y, \bar{\nabla}_{\phi X}N) + \bar{g}(\phi X, \bar{\nabla}_{\phi Y}N)\\ &= \bar{g}(\phi Y, A_N\phi X) - \bar{g}(\phi X, A_N\phi Y).\end{aligned} \tag{4.21}$$

Thus from (4.12)-(4.21) the proof is complete. □

Theorem 4.7. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then each leaf of radical distribution is totally geodesic in M if and only if*

(i) $A^*_\xi \xi' \notin \Gamma(D_0 \perp M_1)$

(ii) $\bar{g}(h^s(\xi', \phi\xi''), W) = 0$

(iii) $\bar{g}(h^*(\xi', \phi\xi''), N_1) = 0,$

where $M_1 = \phi(L)$ *for any* $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L)$, $W \in \Gamma(S)$ *and* $Z \in \Gamma(D_0)$.

Proof. Using the definition of GCR-lightlike submanifold of an indefinite Sasakian manifold, each leaf of $Rad(TM)$ defines totally geodesic foliation in M if and only if

$$\bar{g}(\nabla_{\xi'}\xi'', V) = \bar{g}(\nabla_{\xi'}\xi'', Z) = \bar{g}(\nabla_{\xi'}\xi'', \phi\xi) = \bar{g}(\nabla_{\xi'}\xi'', \phi W) = \bar{g}(\nabla_{\xi'}\xi'', \phi N_1) = 0,$$

for any $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L), W \in \Gamma(S)$, and $Z \in \Gamma(D_0)$. From (2.4), (2.10) we obtain

$$\begin{aligned} g(\nabla_{\xi'}\xi'', V) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', V) \\ &= -\bar{g}(\xi'', \bar{\nabla}_{\xi'}V) \\ &= -g(\xi'', -\phi\xi') \\ &= 0, \end{aligned} \tag{4.22}$$

and

$$g(\nabla_{\xi'}\xi'', Z) = -g(A^*_{\xi''}\xi', Z), \tag{4.23}$$

and

$$g(\nabla_{\xi'}\xi'', \phi\xi) = -g(A^*_{\xi''}\xi', \phi\xi). \tag{4.24}$$

On the other hand using (2.4), (2.9), (2.11) and (2.14) we get

$$\begin{aligned} g(\nabla_{\xi'}\xi'', \phi W) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi W) \\ &= -\bar{g}(\phi\bar{\nabla}_{\xi'}\xi'', W) \\ &= -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'', W) \\ &= -\bar{g}(h^s(\xi', \phi\xi''), W), \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} g(\nabla_{\xi'}\xi'', \phi N_1) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi N_1) \\ &= -\bar{g}(\phi\bar{\nabla}_{\xi'}\xi'', N_1) \\ &= -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - (\bar{\nabla}_{\xi'}\phi)\xi'', N_1) \\ &= -\bar{g}(\nabla_{\xi'}\phi\xi'', N_1) \\ &= -\bar{g}(h^*(\xi', \phi\xi''), N_1). \end{aligned} \tag{4.26}$$

Hence form (4.22) - (4.26), the assertion follows. □

Theorem 4.8. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If M is D-geodesic then each leaf of holomorphic distribution is totally geodesic foliation in M.*

Proof. For $X, Y \in \Gamma(D)$ we have

$$\begin{aligned} h(X,Y) &= \bar{\nabla}_X Y - \nabla_X Y \\ &= \bar{\nabla}_X(-\phi^2 Y + \eta(Y)V) - \nabla_X Y \\ &= -(\bar{\nabla}_X \phi)(\phi Y) - \phi \bar{\nabla}_X \phi Y - \nabla_X Y \\ &= \bar{g}(X, \phi Y)V - \phi \bar{\nabla}_X \phi Y - \nabla_X Y \\ &= \bar{g}(X, \phi Y)V - \phi \nabla_X \phi Y - \phi h(X, \phi Y) - \nabla_X Y \\ &= \bar{g}(X, \phi Y)V - f\nabla_X \phi Y - \omega \nabla_X \phi Y - Bh(X, \phi Y) - Ch(X, \phi Y) - \nabla_X Y. \end{aligned}$$

Equating the transversal parts both sides we get

$$h(X,Y) = -\omega \nabla_X \phi Y - Ch(X, \phi Y),$$

M is D-geodesic implies $-\omega \nabla_X \phi Y = 0$. Thus the proof is complete. □

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