Journal of the Korean Data & Information Science Society
2013, **24**(4), 949–957

http://dx.doi.org/10.7465/jkdi.2013.24.4.949
한국데이터정보과학회지

Default Bayesian testing for the scale parameters in two parameter exponential distributions

Sang Gil Kang[1] · Dal Ho Kim[2] · Woo Dong Lee[3]

[1]Department of Computer and Data Information, Sangji University
[2]Department of Statistics, Kyungpook National University
[3]Department of Asset Management, Daegu Haany University

Received 8 June 2013, revised 9 July 2013, accepted 14 July 2013

Abstract

In this paper, we consider the problem of testing the equality of the scale parameters in two parameter exponential distributions. We propose Bayesian testing procedures for the equality of the scale parameters under the noninformative priors. The noninformative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. Thus, we propose the default Bayesian testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors. Simulation study and an example are provided.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, reference prior, scale parameter, two parameter exponential distribution.

1. Introduction

The exponential distribution plays an important role in the field of life testing and reliability. One is referred to Zelen (1966), Johnson and Kotz (1970), Bain (1978) and Lawless and Singhal (1980). The probability density function of two parameter exponential distribution $\mathcal{E}(\mu, \sigma)$ with the location parameter μ and the scale parameter σ is given by

$$f(x|\mu,\sigma) = \frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma}\right\}, x \geq \mu > 0, \sigma > 0. \tag{1.1}$$

The decision theoretic estimation of the scale parameter was firstly studied by Arnold (1970). Zidek (1973), Brewster (1974), Kubokawa (1994) and Petropoulos and Kourouklis (2002) considered Bayesian estimation of scale parameter based on decision theory. Also, the estimator of the ratio of the scale parameters from the decision theoretic point of view was studied by Madi and Tsui (1990), Madi (2008) and Bobotas and Kourouklis (2011). All the papers mentioned above were focused on Bayesian point estimation.

[1] Professor, Department of Computer and Data Information, Sangji University, Wonju 220-702, Korea.
[2] Professor, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea.
[3] Corresponding author : Professor, Department of Asset Management, Daegu Haany University, Kyungsan 712-715, Korea. E -mail: wdlee@dhu.ac.kr

The problem of comparison for two scale parameters is not considered by Bayesian viewpoint, yet. In this paper, we focus on Bayesian testing procedures for the equality of two scale parameters.

The equality problem of two scale parameters arises when one wants to know the equality of the hazard rate of these distributions. For example, experiment with two different conditions is performed. Usually, an experiment condition changes the hazard rate of a test item. After performing experiment, we want to know whether the condition changes the hazard rate of an item or not. At this moment, the equality problem of two scale parameters is of interest.

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training sample in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang *et al.*, 2011, 2012). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference priors, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors. In Section 4, simulation study and an example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses $H_1, H_2, \cdots, H_q$ are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i. The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis $H_i, i = 1, 2, \cdots, q$. Then, the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \frac{p(H_i)p(\mathbf{x}|H_i)}{\sum_{j=1}^q p(H_j)p(\mathbf{x}|H_j)} = \frac{p_i \int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i}{\sum_{j=1}^q p_j \int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j} = \left(\sum_{j=1}^q \frac{p_j}{p_i} \cdot B_{ji}\right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Often in Bayesian analysis, one can use noninformative prior π_i^N. Common choices are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence, the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996, 1998) proposed the intrinsic Bayes factor and the median Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the training sample and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \cdots, q. \tag{2.3}$$

In view (2.3), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Then the idea is to compute the Bayes factors with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors. The result is

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and the training sample $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus, the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L}\sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also, the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all the training sample Bayes factors. Therefore, we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.5) and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction b of each likelihood function $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$ with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then, the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \times \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \times \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.7}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. See O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995).

3. Bayesian hypothesis testing

Let $x_i, i = 1, \cdots, n_1$ denote observations from $\mathcal{E}(\mu_1, \sigma_1)$, and $y_i, i = 1, \cdots, n_2$ denote observations from $\mathcal{E}(\mu_2, \sigma_2)$. Then likelihood function is given by

$$f(\mathbf{x}, \mathbf{y}|\mu_1, \mu_2, \sigma_1, \sigma_2) = \sigma_1^{-n_1}\sigma_2^{-n_2} \exp\left\{-\frac{\sum_{i=1}^{n_1}(x_i - \mu_1)}{\sigma_1} - \frac{\sum_{i=1}^{n_2}(y_i - \mu_2)}{\sigma_2}\right\}, \tag{3.1}$$

where $\mathbf{x} = (x_1, \cdots, x_{n_1})$, $\mathbf{y} = (y_1, \cdots, y_{n_2})$, $\mu_1 > 0$, $\mu_2 > 0$, $\sigma_1 > 0$ and $\sigma_2 > 0$. We are interested in testing the hypotheses $H_1 : \sigma_1 = \sigma_2$ versus $H_2 : \sigma_1 \neq \sigma_2$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \sigma_1 = \sigma_2 \equiv \sigma$ is

$$L_1(\sigma, \mu_1, \mu_2|\mathbf{x}, \mathbf{y}) = \sigma^{-n_1-n_2} \exp\left\{-\frac{1}{\sigma}\left[\sum_{i=1}^{n_1}(x_i - \mu_1) + \sum_{i=1}^{n_2}(y_i - \mu_2)\right]\right\}. \tag{3.2}$$

And under H_1, the reference prior for (σ, μ_1, μ_2) developed by Ghosal (1997) and Kang *et al.* (2008) is

$$\pi_1^N(\sigma, \mu_1, \mu_2) \propto \sigma^{-1}. \tag{3.3}$$

Then, from the likelihood (3.2) and the reference prior (3.3), the element $m_1^b(\mathbf{x}, \mathbf{y})$ of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \int_0^{y_{(1)}} \int_0^{x_{(1)}} L_1^b(\sigma, \mu_1, \mu_2|\mathbf{x}, \mathbf{y})\pi_1^N(\sigma, \mu_1, \mu_2)d\mu_1 d\mu_2 d\sigma \\ &= \frac{b^{-(n_1+n_2)b}}{n_1 n_2}\Gamma\left[b(n_1+n_2) - 2\right] \\ &\times \Big\{[n_1\bar{x} + n_2\bar{y} - n_1 x_{(1)} - n_2 y_{(1)}]^{-(n_1+n_2)b+2} + [n_1\bar{x} + n_2\bar{y}]^{-(n_1+n_2)b+2} \\ &\quad -[n_1\bar{x} + n_2\bar{y} - n_1 x_{(1)}]^{-(n_1+n_2)b+2} - [n_1\bar{x} + n_2\bar{y} - n_2 y_{(1)}]^{-(n_1+n_2)b+2}\Big\}, \end{aligned} \tag{3.4}$$

where $x_{(1)} = \min\{x_1, \cdots, x_{n_1}\}$, $y_{(1)} = \min\{y_1, \cdots, y_{n_2}\}$, $\bar{x} = \sum_{i=1}^{n_1} x_i/n_1$, $\bar{y} = \sum_{i=1}^{n_2} y_i/n_2$.

For the hypothesis $H_2 : \sigma_1 \neq \sigma_2$, the reference prior for $(\sigma_1, \sigma_2, \mu_1, \mu_2)$ is

$$\pi^N(\sigma_1, \sigma_2, \mu_1, \mu_2) \propto \sigma_1^{-1}\sigma_2^{-1} \quad (3.5)$$

which is derived by Ghosal (1997). The likelihood function under H_2 is

$$L_2(\sigma_1, \sigma_2, \mu_1, \mu_2|\mathbf{x}, \mathbf{y}) = \sigma_1^{-n_1}\sigma_2^{-n_2} \exp\left\{-\frac{\sum_{i=1}^{n_1}(x_i - \mu_1)}{\sigma_1} - \frac{\sum_{i=1}^{n_2}(y_i - \mu_2)^2}{\sigma_2}\right\}. \quad (3.6)$$

Thus, from the reference prior (3.5) and the likelihood (3.6), the element $m_2^b(\mathbf{x}, \mathbf{y})$ of FBF under H_2 is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \int_0^\infty \int_0^{y_{(1)}} \int_0^{x_{(1)}} L_2^b(\sigma_1, \sigma_2, \mu_1, \mu_2|\mathbf{x}, \mathbf{y})\pi_2^N(\sigma_1, \sigma_2, \mu_1, \mu_2)d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 \\ &= \frac{b^{-(n_1+n_2)b}}{n_1 n_2}\Gamma[bn_1 - 1]\Gamma[bn_2 - 1]\left\{[n_1(\bar{x} - x_{(1)})]^{-(bn_1-1)} - [n_1\bar{x}]^{-(bn_1-1)}\right\} \\ &\times \left\{[n_2(\bar{y} - y_{(1)})]^{-(bn_2-1)} - [n_2\bar{y}]^{-(bn_2-1)}\right\}. \end{aligned} \quad (3.7)$$

Therefore, the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{\Gamma[n_1 + n_2 - 2]}{\Gamma[n_1 - 1]\Gamma[n_2 - 1]} \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})}, \quad (3.8)$$

where

$$\begin{aligned} S_1(\mathbf{x}, \mathbf{y}) &= \Big\{[n_1\bar{x} + n_2\bar{y} - n_1 x_{(1)} - n_2 y_{(1)}]^{-(n_1+n_2)+2} + [n_1\bar{x} + n_2\bar{y}]^{-(n_1+n_2)+2} \\ &\quad -[n_1\bar{x} + n_2\bar{y} - n_1 x_{(1)}]^{-(n_1+n_2)+2} - [n_1\bar{x} + n_2\bar{y} - n_2 y_{(1)}]^{-(n_1+n_2)+2}\Big\} \end{aligned}$$

and

$$S_2(\mathbf{x}, \mathbf{y}) = \left\{[n_1(\bar{x} - x_{(1)})]^{-(n_1-1)} - [n_1\bar{x}]^{-(n_1-1)}\right\}\left\{[n_2(\bar{y} - y_{(1)})]^{-(n_2-1)} - [n_2\bar{y}]^{-(n_2-1)}\right\}.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x}, \mathbf{y})}{m_2^b(\mathbf{x}, \mathbf{y})} = \frac{\Gamma[bn_1 - 1]\Gamma[bn_2 - 1]}{\Gamma[b(n_1 + n_2) - 2]} \frac{S_1(\mathbf{x}, \mathbf{y}; b)}{S_2(\mathbf{x}, \mathbf{y}; b)}, \quad (3.9)$$

where

$$\begin{aligned} S_1(\mathbf{x}, \mathbf{y}; b) &= \Big\{[n_1\bar{x} + n_2\bar{y} - n_1 x_{(1)} - n_2 y_{(1)}]^{-b(n_1+n_2)+2} + [n_1\bar{x} + n_2\bar{y}]^{-b(n_1+n_2)+2} \\ &\quad -[n_1\bar{x} + n_2\bar{y} - n_1 x_{(1)}]^{-b(n_1+n_2)+2} - [n_1\bar{x} + n_2\bar{y} - n_2 y_{(1)}]^{-b(n_1+n_2)+2}\Big\} \end{aligned}$$

and

$$\begin{aligned} S_2(\mathbf{x}, \mathbf{y}; b) &= \left\{[n_1(\bar{x} - x_{(1)})]^{-(bn_1-1)} - [n_1\bar{x}]^{-(bn_1-1)}\right\} \\ &\times \left\{[n_2(\bar{y} - y_{(1)})]^{-(bn_2-1)} - [n_2\bar{y}]^{-(bn_2-1)}\right\}. \end{aligned}$$

Thus, the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{\Gamma[bn_1 - 1]\Gamma[bn_2 - 1]\Gamma[n_1 + n_2 - 2]}{\Gamma[n_1 - 1]\Gamma[n_2 - 1]\Gamma[b(n_1 + n_2) - 2]} \frac{S_2(\mathbf{x}, \mathbf{y})S_1(\mathbf{x}, \mathbf{y}; b)}{S_1(\mathbf{x}, \mathbf{y})S_2(\mathbf{x}, \mathbf{y}; b)}. \quad (3.10)$$

3.2. Bayesian hypothesis testing based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor (IBF) is given in (3.8). To obtain IBF, we only calculate the marginal densities of a minimal training sample under the hypotheses H_1 and H_2. The marginal density of (X_{j_1}, X_{j_2}) and (Y_{k_1}, Y_{k_2}) is finite for all $1 \le j_1 < j_2 \le n_1$ and $1 \le k_1 < k_2 \le n_2$ under hypotheses H_1 and H_2. Thus we conclude that any training sample of size 4, which is 2 sample from each population, is a minimal training sample.

The marginal density $m_1^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})$ under H_1 is given by

$$\begin{aligned}
& m_1^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) \\
&= \int_0^\infty \int_0^{z_{(k_1)}} \int_0^{z_{(j_1)}} f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2} | \sigma, \mu_1, \mu_2) \pi_1^N(\sigma, \mu_1, \mu_2) d\mu_1 d\mu_2 d\sigma \\
&= \frac{1}{16} \left\{ \left[\frac{x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2}}{2} - z_{(j_1)} - z_{(k_1)} \right]^{-2} + \left[\frac{x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2}}{2} \right]^{-2} \right. \\
& \left. - \left[\frac{x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2}}{2} - z_{(j_1)} \right]^{-2} - \left[\frac{x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2}}{2} - z_{(k_1)} \right]^{-2} \right\}, \qquad (3.11)
\end{aligned}$$

where $z_{(j_1)} = \min\{x_{j_1}, x_{j_2}\}$ and $z_{(k_1)} = \min\{y_{k_1}, y_{k_2}\}$. And under H_2 the marginal density $m_2^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})$ is given by

$$\begin{aligned}
m_2^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) &= \int_0^\infty \int_0^\infty \int_0^{z_{(k_1)}} \int_0^{z_{(j_1)}} f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2} | \sigma_1, \sigma_2, \mu_1, \mu_2) \\
&\times \pi_2^N(\sigma_1, \sigma_2, \mu_1, \mu_2) d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 \\
&= \frac{1}{16} \left\{ \left[(x_{j_1} + x_{j_2})/2 - z_{(j_1)} \right]^{-1} - \left[(x_{j_1} + x_{j_2})/2 \right]^{-1} \right\} \\
&\times \left\{ \left[(y_{k_1} + y_{k_2})/2 - z_{(k_1)} \right]^{-1} - \left[(y_{k_1} + y_{k_2})/2 \right]^{-1} \right\}. \qquad (3.12)
\end{aligned}$$

Therefore, the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{\Gamma[n_1 + n_2 - 2]}{\Gamma[n_1 - 1]\Gamma[n_2 - 1]} \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} \left[\frac{1}{L} \sum_{j_1 < j_2}^{n_1} \sum_{k_1 < k_2}^{n_2} \frac{T_1(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})}{T_2(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})} \right], \qquad (3.13)$$

where $L = [n_1 n_2 (n_1 - 1)(n_2 - 1)]/4$,

$$\begin{aligned}
& T_1(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) \\
&= \left\{ \left[(x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2})/2 - z_{(j_1)} - z_{(k_1)} \right]^{-2} + \left[(x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2})/2 \right]^{-2} \right. \\
& \left. - \left[(x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2})/2 - z_{(j_1)} \right]^{-2} - \left[(x_{j_1} + x_{j_2} + y_{k_1} + y_{k_2})/2 - z_{(k_1)} \right]^{-2} \right\}
\end{aligned}$$

and

$$\begin{aligned}
T_2(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) &= \left\{ \left[(x_{j_1} + x_{j_2})/2 - z_{(j_1)} \right]^{-1} - \left[(x_{j_1} + x_{j_2})/2 \right]^{-1} \right\} \\
&\times \left\{ \left[(y_{k_1} + y_{k_2})/2 - z_{(k_1)} \right]^{-1} - \left[(y_{k_1} + y_{k_2})/2 \right]^{-1} \right\}.
\end{aligned}$$

From the marginal densities (3.11) and (3.12), the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{\Gamma[n_1 + n_2 - 2]}{\Gamma[n_1 - 1]\Gamma[n_2 - 1]} \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} ME \left[\frac{T_1(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})}{T_2(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})} \right]. \qquad (3.14)$$

4. Numerical studies

In order to assess the developed Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of (μ_1, σ_1), (μ_2, σ_2) and (n_1, n_2). In particular, for fixed (μ_1, σ_1) and (μ_2, σ_2), we take 2,000 independent random samples of X_i and Y_i with sample size n_1 and n_2 from the model (1.1), respectively. We want to test the hypotheses $H_1 : \sigma_1 = \sigma_2$ versus $H_2 : \sigma_1 \neq \sigma_2$. The posterior probabilities of the hypothesis H_1 being true are computed assuming equal prior probabilities. Table 4.1 shows the results of the averages and the standard deviations of posterior probabilities. In Table 4.1, $P^F(\cdot)$,$P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of H_1 being true based on FBF, AIBF and MIBF, respectively. From results of Table 4.1, the FBF, the AIBF and the MIBF give fairly reasonable answers for all configurations, and indicate that for values of σ_2 that are far from σ_1 they select the hypothesis H_2. Also the FBF, the AIBF and the MIBF give a similar behavior for all sample sizes, and the results of table are not much sensitive to the change of the values of (μ_1, μ_2). But the AIBF and the MIBF slightly favor H_1 than the FBF. The FBF gives exact answer even when the difference between σ_1 and σ_2 is small. The behavior of FBF is more desirable.

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

σ_1	σ_2	(n_1, n_2)	$P^F(H_1\|\mathbf{x},\mathbf{y})$	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
			$\mu_1 = 1.0, \mu_2 = 1.0$		
1.0	1.0	5,5	0.528 (0.116)	0.611 (0.139)	0.594 (0.135)
		5,10	0.566 (0.128)	0.647 (0.139)	0.629 (0.137)
		10,10	0.601 (0.131)	0.696 (0.138)	0.678 (0.138)
		10,20	0.641 (0.140)	0.724 (0.142)	0.707 (0.142)
	2.0	5,5	0.471 (0.151)	0.540 (0.184)	0.530 (0.175)
		5,10	0.469 (0.179)	0.537 (0.201)	0.523 (0.193)
		10,10	0.432 (0.221)	0.509 (0.251)	0.495 (0.245)
		10,20	0.416 (0.234)	0.488 (0.258)	0.473 (0.253)
	3.0	5,5	0.398 (0.178)	0.448 (0.219)	0.444 (0.205)
		5,10	0.355 (0.196)	0.403 (0.225)	0.397 (0.214)
		10,10	0.260 (0.218)	0.309 (0.256)	0.301 (0.248)
		10,20	0.199 (0.197)	0.240 (0.228)	0.231 (0.221)
	5.0	5,5	0.275 (0.185)	0.294 (0.224)	0.302 (0.213)
		5,10	0.201 (0.170)	0.221 (0.195)	0.224 (0.186)
		10,10	0.090 (0.138)	0.105 (0.165)	0.104 (0.160)
		10,20	0.046 (0.088)	0.056 (0.107)	0.055 (0.103)
	7.0	5,5	0.203 (0.167)	0.205 (0.198)	0.219 (0.190)
		5,10	0.119 (0.130)	0.126 (0.147)	0.134 (0.143)
		10,10	0.037 (0.084)	0.041 (0.100)	0.042 (0.097)
		10,20	0.012 (0.035)	0.014 (0.044)	0.014 (0.043)
			$\mu_1 = 1.0, \mu_2 = 5.0$		
1.0	1.0	5,5	0.548 (0.116)	0.625 (0.136)	0.609 (0.132)
		5,10	0.586 (0.125)	0.669 (0.135)	0.651 (0.132)
		10,10	0.627 (0.131)	0.712 (0.134)	0.693 (0.134)
		10,20	0.655 (0.139)	0.739 (0.139)	0.721 (0.140)
	2.0	5,5	0.498 (0.151)	0.552 (0.187)	0.544 (0.176)
		5,10	0.490 (0.182)	0.553 (0.205)	0.542 (0.195)
		10,10	0.468 (0.217)	0.527 (0.245)	0.514 (0.239)
		10,20	0.450 (0.235)	0.518 (0.256)	0.504 (0.250)
	3.0	5,5	0.411 (0.181)	0.442 (0.222)	0.443 (0.208)
		5,10	0.368 (0.202)	0.410 (0.231)	0.407 (0.219)
		10,10	0.278 (0.225)	0.310 (0.256)	0.304 (0.248)
		10,20	0.226 (0.215)	0.264 (0.244)	0.257 (0.236)
	5.0	5,5	0.284 (0.186)	0.286 (0.220)	0.299 (0.209)
		5,10	0.211 (0.180)	0.226 (0.203)	0.234 (0.194)
		10,10	0.101 (0.152)	0.109 (0.172)	0.110 (0.168)
		10,20	0.051 (0.098)	0.059 (0.114)	0.059 (0.111)
	7.0	5,5	0.208 (0.176)	0.198 (0.202)	0.215 (0.195)
		5,10	0.124 (0.137)	0.127 (0.153)	0.138 (0.150)
		10,10	0.039 (0.084)	0.040 (0.093)	0.042 (0.092)
		10,20	0.012 (0.036)	0.013 (0.041)	0.013 (0.041)
			$\mu_1 = 1.0, \mu_2 = 10.0$		
1.0	1.0	5,5	0.565 (0.110)	0.637 (0.129)	0.619 (0.124)
		5,10	0.585 (0.128)	0.670 (0.139)	0.651 (0.136)
		10,10	0.636 (0.138)	0.714 (0.143)	0.696 (0.142)
		10,20	0.657 (0.141)	0.741 (0.141)	0.723 (0.142)
	2.0	5,5	0.513 (0.151)	0.558 (0.188)	0.550 (0.175)
		5,10	0.501 (0.183)	0.564 (0.206)	0.553 (0.196)
		10,10	0.493 (0.220)	0.544 (0.248)	0.531 (0.241)
		10,20	0.445 (0.236)	0.511 (0.258)	0.498 (0.252)
	3.0	5,5	0.438 (0.178)	0.459 (0.220)	0.461 (0.206)
		5,10	0.382 (0.204)	0.423 (0.233)	0.422 (0.220)
		10,10	0.317 (0.232)	0.342 (0.261)	0.337 (0.252)
		10,20	0.232 (0.218)	0.268 (0.248)	0.263 (0.240)
	5.0	5,5	0.294 (0.192)	0.284 (0.224)	0.301 (0.214)
		5,10	0.218 (0.178)	0.230 (0.200)	0.241 (0.192)
		10,10	0.111 (0.157)	0.114 (0.173)	0.115 (0.169)
		10,20	0.052 (0.094)	0.059 (0.110)	0.060 (0.107)
	7.0	5,5	0.226 (0.183)	0.208 (0.207)	0.227 (0.199)
		5,10	0.131 (0.140)	0.132 (0.155)	0.146 (0.153)
		10,10	0.041 (0.095)	0.040 (0.102)	0.042 (0.100)
		10,20	0.014 (0.042)	0.015 (0.048)	0.016 (0.047)

Example 4.1 This example taken from Wu and Wu (2005). In a clinical trial, comparing the duration of remission achieved by four drugs used in the treatment of leukemia, four groups of 20 patients each were used. We used the duration of remission achieved by two drugs, and the data sets are given by

$$
\begin{aligned}
\text{Group 1} \quad &: \quad 1.034, 2.344, 1.266, 1.563, 1.169, 4.118, 1.013, 1.509, 1.109, 1.965, 5.136, 1.533, \\
&\quad\ 1.716, 2.778, 2.546, 2.626, 3.413, 1.929, 2.061, 2.951 \\
\text{Group 2} \quad &: \quad 4.158, 4.025, 5.170, 11.909, 4.912, 4.629, 3.955, 6.735, 3.140, 12.446, 8.777, \\
&\quad\ 6.321, 3.256, 8.250, 3.759, 5.205, 3.071, 3.147, 9.773, 10.218.
\end{aligned}
$$

For this data sets, the maximum likelihood estimates of σ_1 in group 1 is 1.176, and for group 2, the maximum likelihood estimates of σ_1 is 3.072. The estimate of group2 is greater than two times of the estimate of group 1. This fact suggests that there is a strong evidence of favoring H_2.

We want to test the hypotheses $H_1 : \sigma_1 = \sigma_2$ versus $H_2 : \sigma_1 \neq \sigma_2$. The values of the Bayes factors and the posterior probabilities of the hypothesis H_1 are given in Table 4.2. From the results of Table 4.2, the posterior probabilities under various Bayes factors give the same answer, and select the hypothesis H_2. The FBF has the smallest posterior probability of H_1, but the values of three Bayes factors are almost the same.

Table 4.2 Bayes factor and posterior probabilities of $H_1 : \sigma_1 = \sigma_2$

B_{21}^F	$P^F(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{AI}	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
13.079	0.071	10.309	0.088	10.679	0.086

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the equality of the scale parameters in two parameter exponential distributions under the reference priors. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations.

When the difference between scale parameters are small and the sample size is small, the posterior probabilities of H_1 based on the FBF are smaller than 0.5. Based on this fact, we can conclude that the FBF gives more reasonable answer than the IBF. When the hypothesis H_1 is true, the posterior probabilities based on the IBF or FBF are greater than 0.5. But the posterior probabilities based on the FBF is smaller than that of the IBF. This fact suggests that the AIBF and the MIBF slightly favors the hypothesis H_1 than the FBF.

From results of our simulation and example, we recommend the use of the FBF than the AIBF and the MIBF for practical application in view of its simplicity and ease of implementation.

For further study, to develop the intrinsic prior or expected posterior prior in this model will be an interesting topic.

References

Arnold, B. C. (1970). Inadmissibility of the usual scale estimate for a shifted exponential distribution. *Journal of the American Statistical Association*, **65**, 1260-1264.

Bain, L. J. (1978). *Statistical analysis of reliability and life testing.* Marcel Dekker, New York.

Berger, J. O. and Bernardo, J. M. (1989). Estimating a product of means : Bayesian analysis with reference priors. *Journal of the American Statistical Association*, **84**, 200-207.

Berger, J. O. and Bernardo, J. M. (1992). On the development of reference priors (with discussion). In *Bayesian Statistics IV*, edited by J. M. Bernardo, et al., Oxford University Press, Oxford, 35-60.

Berger, J. O. and Pericchi, L. R. (1996). The intrinsic Bayes factor for model selection and prediction. *Journal of the American Statistical Association*, **91**, 109-122.

Berger, J. O. and Pericchi, L. R. (1998). Accurate and stable Bayesian model selection: the median intrinsic Bayes factor. *Sankya B*, **60**, 1-18.

Berger, J. O. and Pericchi, L. R. (2001). Objective Bayesian methods for model selection: introduction and comparison (with discussion). In *Model Selection, Institute of Mathematical Statistics Lecture Notes-Monograph Series*, Vol. 38, edited by P. Lahiri, Beachwood, Ohio, 135-207.

Bobotas, P. and Kourouklis, S. (2011). Improved estimation of the scale parameter, the hazard rate parameter and the ratio of the scale parameters in exponential distributions: An integrated approach. *Journal of Statistical Planning and Inference*, **141**, 2399-2416.

Brewster, J. F. (1974). Alternative estimators for the scale parameter of the exponential distribution with unknown location. *The Annals of Statistics*, **2**, 21-38.

Ghosal, S. (1997). Reference priors in multiparameter nonregular cases. *Test*, **6**, 159-186.

Johnson, N. L. and Kotz, S. (1970). *Continuous univariate distributions*, Vol. 2, Wiley, New York.

Kang, S. G., Kim, D. H. and Lee, W. D. (2008). Reference priors for the location parameter in the exponential distribution. *Journal of Korean Data & Information Science Society*, **19**, 1409-1418.

Kang, S. G., Kim, D. H. and Lee, W. D. (2011). Default Bayesian testing for the bivariate normal correlation coefficient. *Journal of Korean Data & Information Science Society*, **22**, 1007-1016.

Kang, S. G., Kim, D. H. and Lee, W. D. (2012). Default Bayesian testing on the common mean of several normal distributions. *Journal of the Korean Data* & *Information Science Society*, **23**, 605-616.

Kubokawa, T. (1994). A unified approach to improving equivariant estimators. *The Annals of Statistics*, **22**, 290-299.

Lawless, J. F. and Singhal, K. (1980). Analysis of data from life-test experiments under an exponential model. *Naval Research Logistics Quarterly*, **27**, 323-334.

Madi, M. (2008). Improved estimation of an exponential scale ratio based on records. *Statistics and Probability Letters*, **78**, 165-172.

Madi, M. and Tsui, K. W. (1990). Estimation of the ratio of the scale parameters of two exponential distribution with unknown location parameters. *Annals of the Institute of Statistical Mathematics*, **42**, 77-87.

O'Hagan, A. (1995). Fractional Bayes factors for model comparison (with discussion). *Journal of Royal Statistical Society B*, **57**, 99-118.

O'Hagan, A. (1997). Properties of intrinsic and fractional Bayes factors. *Test*, **6**, 101-118.

Petropoulos, C. and Kourouklis, S. (2002). A class of improved estimators for the scale parameter of an exponential distribution with unknown location. *Communications in Statistics-Theory and Methods*, **31**, 325-335.

Spiegelhalter, D. J. and Smith, A. F. M. (1982). Bayes factors for linear and log-linear models with vague prior information. *Journal of Royal Statistical Society B*, **44**, 377-387.

Wu, S. F. and Wu, C. C. (2005). Two stage multiple comparisons with the average for exponential location parameters under heteroscedasticity. *Journal of Statistical Planning and Inference*, **134**, 392-408.

Zelen, M. (1966). Application of exponential models to problems in cancer research. *Journal of Royal Statistical Society A*, **129**, 368-398.

Zidek, J. V. (1973). Estimating the scale parameter of the exponential distribution with unknown location. *The Annals of Statistics*, **1**, 264-278.