

A two-stage service policy for an $M/G/1$ queueing system

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Abstract

We introduce the $P_{\lambda,\tau}^M$ service policy, as a generalized two-stage service policy of the P_{λ}^M policy of Bae *et al.* (2002) for an $M/G/1$ queueing system. By using the level crossing theory and solving the corresponding integral equations, we obtain the explicit expression for the stationary distribution of the workload in the system.

Keywords: $M/G/1$ queue, $P_{\lambda,\tau}^M$ service policy, stationary distribution.

1. Introduction

Bae *et al.* (2002) introduced the P_{λ}^M service policy for an $M/G/1$ queueing system; when a customer arrives, the server starts to work with service speed 1, that is, the workload of the server starts to decrease by one per unit time, until the workload in the system exceeds the threshold $\lambda > 0$. As soon as this occurs, the server increases his/her service speed to $M \geq 1$ instantaneously, and the decrease in the workload becomes M per unit time until the system becomes empty. Under this policy, Bae *et al.* (2002) obtained the stationary distribution of the workload process. After assigning the corresponding costs, Kim *et al.* (2006) showed the existence of the optimal service speed M which minimizes the long-run average cost per unit time. Recently, Kim *et al.* (2011) derived the long-run average cost per unit time under the P_{λ}^M policy in an infinite dam with exponential inputs.

In this paper, we generalize the P_{λ}^M service policy by adopting the $P_{\lambda,\tau}^M$ releasing policy for a dam model. The $P_{\lambda,\tau}^M$ policy was introduced by Yeh (1985) as a generalized releasing policy of the P_{λ}^M policy of Faddy (1974) for a dam with input formed by a Wiener process; the release rate is kept at 0 until the level of water exceeds the threshold λ and, as soon as this occurs, water is released at rate $M > 0$ not until the dam is empty but until the level of water reaches the threshold τ with $0 < \tau < \lambda$. Abdel-Hameed (2000) considered the optimal control of an infinite dam using $P_{\lambda,\tau}^M$ policies when the input process is a compound Poisson process with positive drift. Bae *et al.* (2003) determined the long-run average cost per unit time under the $P_{\lambda,\tau}^M$ policy in a finite dam with a compound Poisson input.

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We modify this $P_{\lambda, \tau}^M$ policy and introduce it as a generalized two-stage service policy of the P_{λ}^M policy of Bae *et al.* (2002) for the $M/G/1$ queueing system. A server is initially idle and starts to work, if a customer arrives, with ordinary service speed 1. The customers arrive according to a Poisson process of rate $\nu > 0$ and each customer brings a job consisting of an amount of work to be processed, that is independent and identically distributed with a distribution function G and a mean $m > 0$, and is also independent of the arrival process. If the workload exceeds the threshold $\lambda > 0$, the server increases the service speed to $M \geq 1$ instantaneously and continues to follow that service speed until the workload level reaches τ with $0 \leq \tau < \lambda$. When the workload reaches level τ , the service speed is changed again to 1 instantaneously. The service speed 1 is kept until the level up-crosses λ again. For the stability of the system, we assume that $\rho := \nu m < 1$. Clearly, the $P_{\lambda, \tau}^M$ policy coincides with the P_{λ}^M policy in case $\tau = 0$. When $M = 1$, both policies are reduced to the ordinary service policy for the $M/G/1$ queueing system. Lee and Kim (2006) studied the similar two-stage service policy for the workload-dependent $M/G/1$ queueing systems and Bar-Lev and Perry (1993) considered the two-stage release rule procedure in a regenerative dam. Both dealt with the case that the service rate or the release rate is dependent on the current state of the system, which results in the complicated expression for the stationary distribution in terms of a certain positive kernel.

In section 2, we derive the stationary distribution of the workload of this system in the simpler form by using the level crossing arguments of Cohen (1977). In section 3, as an example we obtain an explicit distribution for $M/M/1$ queueing system under the $P_{\lambda, \tau}^M$ service policy.

2. The stationary distribution of the workload

Let $X(t)$ denote the workload of the system at time t under the $P_{\lambda, \tau}^M$ service policy. If we define, for $n = 1, 2, \dots$,

$$T_n^\lambda := \inf\{t > T_{n-1}^\tau \mid X(t) > \lambda\} \quad \text{and} \quad T_n^\tau := \inf\{t > T_n^\lambda \mid X(t) = \tau\},$$

with $T_0^\tau = 0$, then $\{X(t), t \geq 0\}$ is a delayed regenerative process having $T_1^\tau, T_2^\tau, \dots$ as regeneration points. Since $\{X(t), t \geq 0\}$ is not a Markov process, to analyze the process $\{X(t), t \geq 0\}$, we decompose it into two Markov processes. Let $\{X_1(t), t \geq 0\}$ be a process obtained from $\{X(t), t \geq 0\}$ by deleting the time periods from T_n^λ to T_n^τ , for all $n = 1, 2, \dots$, and by gluing together the remaining periods. Note that in the process $\{X_1(t), t \geq 0\}$ the system operates with service speed 1. Let $\{X_2(t), t \geq 0\}$ be formed similarly by separating and connecting the periods which start at T_n^λ and end at T_n^τ , for all $n = 1, 2, \dots$. Then, clearly the process $\{X_2(t), t \geq 0\}$ has the service speed M . We note that the idle periods in $\{X(t), t \geq 0\}$ are exactly those of $\{X_1(t), t \geq 0\}$ and that no idle periods exist in $\{X_2(t), t \geq 0\}$.

Let C be the cycle of the process $\{X(t), t \geq 0\}$ and C_i of the process $\{X_i(t), t \geq 0\}$ for $i = 1, 2$. Then obviously $C = C_1 + C_2$. Because $\{X(t), t \geq 0\}$ and $\{X_i(t), t \geq 0\}$, for $i = 1, 2$, are regenerative processes with finite mean cycles, each process has its stationary distribution function. Let F be the stationary distribution function of $\{X(t), t \geq 0\}$, and F_i of $\{X_i(t), t \geq 0\}$, for $i = 1, 2$. Then it follows that

$$F(x) = bF_1(x) + (1 - b)F_2(x), \quad 0 \leq x < \infty,$$

where $b := E[C_1]/E[C]$. Notice that F_1 has a jump at zero. If we denote the jump size of F_1 at zero by a , then F can be rewritten as

$$F(x) = ab + (1 - a)bF_1^{ac}(x) + (1 - b)F_2(x), \quad 0 \leq x < \infty,$$

where F_1^{ac} denotes the absolutely continuous part of F_1 .

2.1. Level crossing equations

For $i = 1, 2$, let $D_i(x)$ and $U_i(x)$ be the numbers of down- and up-crossings of level x by the process $\{X_i(t), t \geq 0\}$ during C_i , respectively, and N_i the number of arrivals during C_i . By convention the arrival that causes $\{X(t), t \geq 0\}$ to up-cross level λ for the first time during C is counted only in N_1 .

Because the process $\{X(t), t \geq 0\}$ is the delayed regenerative process having the same level τ at all regeneration points $T_n^\tau, n = 1, 2, \dots$, the number of up-crossings of level x equals the number of down-crossings of that level during C . Observing the sample paths of two processes $\{X_i(t), t \geq 0\}$, for $i = 1, 2$, we have that

$$D_1(x) \stackrel{\text{a.s.}}{=} \begin{cases} U_1(x), & 0 < x < \tau, \\ U_1(x) - 1, & \tau \leq x < \lambda, \end{cases} \tag{2.1}$$

and

$$D_2(x) \stackrel{\text{a.s.}}{=} \begin{cases} U_2(x) + 1, & \tau < x < \lambda, \\ U_2(x) + U_1(x), & x \geq \lambda, \end{cases} \tag{2.2}$$

where $U_1(x)$ in (2.2) means the number of arrivals during C_1 that cause the process $\{X_1(t), t \geq 0\}$ to up-cross both level λ and level $x (\geq \lambda)$ simultaneously, in other words, for $x \geq \lambda$,

$$U_1(x) = \begin{cases} 1, & X(T_n^\lambda) > x, \\ 0, & \text{otherwise.} \end{cases}$$

By using the level crossing theory (Cohen, 1977; Lee, 2008), for the number of down-crossings we have that

$$E[D_1(x)] = E[C_1](1 - a)f_1^{ac}(x), \quad 0 < x < \lambda, \tag{2.3}$$

$$E[D_2(x)] = ME[C_2]f_2(x), \quad \tau < x < \infty, \tag{2.4}$$

where f_1^{ac} and f_2 are densities corresponding to F_1^{ac} and F_2 , respectively. Let X_1 and X_2 be the generic random variables with distributions F_1 and F_2 , respectively, and S be the random variable representing the amount of work that each arriving customer carries to the system. Then, we also have that

$$E[U_1(x)] = E[N_1]E[1_{\{X_1 \leq x\}} - 1_{\{X_1 + S \leq x\}}], \quad 0 < x < \lambda, \tag{2.5}$$

$$E[U_2(x)] = E[N_2]E[1_{\{X_2 \leq x\}} - 1_{\{X_2 + S \leq x\}}], \quad \tau < x < \infty. \tag{2.6}$$

Taking expectations in (2.1) and (2.2) and using the relations (2.3), (2.4), (2.5), and (2.6) yield the following level crossing equations:

$$(1 - a)f_1^{ac}(x) = \begin{cases} \nu \int_0^x (1 - G(x - y))dF_1(y), & 0 < x < \tau, \\ \nu \int_0^x (1 - G(x - y))dF_1(y) - \frac{1}{E[C_1]}, & \tau \leq x < \lambda, \end{cases} \tag{2.7}$$

and

$$Mf_2(x) = \begin{cases} \nu \int_{\tau}^x (1 - G(x - y))f_2(y)dy + \frac{1}{E[C_2]}, & \tau < x < \lambda, \\ \nu \int_{\tau}^x (1 - G(x - y))f_2(y)dy + \frac{E[U_1(x)]}{E[C_2]}, & x \geq \lambda, \end{cases} \tag{2.8}$$

since $E[N_i]/E[C_i] = \nu$, for $i = 1, 2$.

Let T_x be the exit time of the process $\{X_1(t), t \geq 0\}$, starting at x , from $(0, \lambda]$, namely,

$$T_x := \inf\{t \geq 0 \mid X_1(t) \notin (0, \lambda], X_1(0) = x\}, \quad 0 \leq x \leq \lambda,$$

and define

$$P(l, x) := \Pr\{X_1(T_x) > \lambda + l\}, \quad l \geq 0, 0 \leq x \leq \lambda.$$

Bae *et al.* (2002) obtained

$$P(l, x) = \frac{(H_1 * \tilde{G}_l)(\lambda)}{H_1(\lambda)} H_1(\lambda - x) - (H_1 * \tilde{G}_l)(\lambda - x), \quad l \geq 0, 0 \leq x \leq \lambda,$$

where $\tilde{G}_l(x) := \rho(G_e(x + l) - G_e(l))$, $G_e(x) := (1/m) \int_0^x (1 - G(y))dy$, the equilibrium distribution of G , and

$$H_1(x) := \begin{cases} \sum_{n=0}^{\infty} \rho^n G_e^{*n}(x), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

the renewal function and G_e^{*n} denotes the n th convolution power of G_e with G_e^{*0} begin the Heaviside function.

Now, we observe the excess amount over λ at the first exit time from $[0, \lambda]$ for the process $\{X(t), t \geq 0\}$. Note that it is the same as the excess amount over λ at the end of the cycle of the process $\{X_1(t), t \geq 0\}$. Bae *et al.* (2003) derived the distribution of the excess amount over λ , denoted by L_x , starting with x , in terms of $P(l, x)$ and it is given by

$$\begin{aligned} P_x(l) &:= \Pr\{L_x \leq l\} \\ &= 1 - P(l, x) - \frac{\nu H_1(\lambda - x)}{H_1'(\lambda)} \left(1 - G(\lambda + l) + \int_0^{\lambda} P(l, y)dG(y) \right). \end{aligned} \tag{2.9}$$

2.2. The stationary distribution

Since $D_1(x) \leq N_1$ with probability 1 for $0 < x < \lambda$, and N_1 is integrable, it follows, from the dominated convergence theorem, that $\lim_{x \rightarrow 0^+} E[D_1(x)] = E[D_1(0^+)]$ exists. From the Markovian property of the process $\{X_1(t), t \geq 0\}$, we can show that $D_1(0^+)$ has the following distribution:

$$\Pr\{D_1(0^+) = n\} = \begin{cases} P(0, \tau), & n = 0, \\ (1 - P(0, \tau))P_{\lambda}(1 - P_{\lambda})^{n-1}, & n \geq 1, \end{cases}$$

where P_{λ} is the probability that the process $\{X_1(t), t \geq 0\}$, leaving from 0, up-crosses level λ before returning to 0, which is given by

$$P_{\lambda} = 1 - G(\lambda) + \int_0^{\lambda} P(0, x)dG(x) = \frac{H_1'(\lambda)}{\nu H_1(\lambda)}.$$

Therefore, we have

$$E[D_1(0^+)] = \frac{\nu H_1(\lambda - \tau)}{H_1'(\lambda)}. \tag{2.10}$$

Letting $x \rightarrow 0^+$ in the first equation of (2.7) gives

$$(1 - a)f_1^{ac}(0^+) = a\nu. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.3) yields

$$E[C_1] = \frac{H_1(\lambda - \tau)}{aH_1'(\lambda)}$$

and from the relation $b = E[C_1]/(E[C_1] + E[C_2])$, it follows that

$$E[C_2] = \frac{(1 - b)H_1(\lambda - \tau)}{abH_1'(\lambda)}.$$

The expectation $E[U_1(x)]$ for $x \geq \lambda$ is equal to the probability that the excess amount over λ for the process $\{X_1(t), t \geq 0\}$, starting at τ , is greater than $x - \lambda$. That is exactly $1 - P_\tau(x - \lambda)$ of (2.9). Thus (2.7) and (2.8) can be rewritten as follows:

$$f_1^{ac}(x) = \begin{cases} \frac{a}{1-a}\rho g_e(x) + \int_0^x \rho g_e(x-y)f_1^{ac}(y)dy, & 0 < x < \tau, \\ \frac{a}{1-a}\left(\rho g_e(x) - \frac{H_1'(\lambda)}{H_1(\lambda - \tau)}\right) + \int_0^x \rho g_e(x-y)f_1^{ac}(y)dy, & \tau \leq x < \lambda, \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

and

$$f_2(x) = \begin{cases} \frac{abH_1'(\lambda)}{(1-b)MH_1(\lambda - \tau)} + \int_\tau^x \frac{\rho}{M}g_e(x-y)f_2(y)dy, & \tau < x < \lambda, \\ \frac{abH_1'(\lambda)(1 - P_\tau(x - \lambda))}{(1-b)MH_1(\lambda - \tau)} + \int_\tau^x \frac{\rho}{M}g_e(x-y)f_2(y)dy, & x \geq \lambda, \\ 0, & \text{otherwise,} \end{cases} \tag{2.13}$$

where $g_e(x) := G_e'(x) = (1/m)(1 - G(x))$. Therefore, we have the following theorem:

Theorem 2.1 The stationary densities f_1^{ac} and f_2 are given, respectively, by

$$f_1^{ac}(x) = \begin{cases} \frac{aH_1'(x)}{1-a}, & 0 < x < \tau, \\ \frac{a}{1-a}\left(H_1'(x) - \frac{H_1'(\lambda)H_1(x - \tau)}{H_1(\lambda - \tau)}\right), & \tau \leq x < \lambda, \\ 0, & \text{otherwise,} \end{cases} \tag{2.14}$$

and

$$f_2(x) = \begin{cases} \frac{abH_1'(\lambda)H_M(x - \tau)}{(1-b)MH_1(\lambda - \tau)}, & \tau < x < \lambda, \\ \frac{abH_1'(\lambda)}{(1-b)MH_1(\lambda - \tau)}(H_M(x - \tau) - P_\tau(x - \lambda) - \int_\lambda^x H_M'(x-y)P_\tau(y - \lambda)dy), & x \geq \lambda, \\ 0, & \text{otherwise,} \end{cases} \tag{2.15}$$

where

$$H_M(x) := \begin{cases} \sum_{n=0}^{\infty} \left(\frac{\rho}{M}\right)^n G_e^{*n}(x), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and finally a and b are determined by two normalizing conditions

$$a + (1 - a) \int_0^\lambda f_1^{ac}(x) dx = 1$$

and

$$\int_\tau^\infty f_2(x) dx = 1.$$

Proof: Using the bound $g_e(x) \leq 1/m$, it follows easily by induction that

$$g_e^{*n}(x) = \int_0^x g_e^{*(n-1)}(x-y)g_e(y)dy \leq \frac{x^{n-1}}{m^n(n-1)!}, \quad x \geq 0,$$

for all $n \geq 1$ and hence $H'_1(x) = \sum_{n=1}^\infty \rho^n g_e^{*n}(x)$ is well-defined. Iterating the first equation of (2.12) and using the dominated convergence theorem give

$$f_1^{ac}(x) = \frac{a}{1-a} H'_1(x).$$

The second equation of (2.12) can be restated as

$$\begin{aligned} f_1^{ac}(x) &= \frac{a}{1-a} \left(\rho g_e(x) - \frac{H'_1(\lambda)}{H_1(\lambda-\tau)} + \int_0^\tau \rho g_e(x-y)H'_1(y)dy \right) \\ &\quad + \int_\tau^x \rho g_e(x-y)f_1^{ac}(y)dy \\ &= \frac{a}{1-a} \left(H'_1(x) - \frac{H'_1(\lambda)}{H_1(\lambda-\tau)} \right) + \int_\tau^x \rho g_e(x-y) \left(f_1^{ac}(y) - \frac{a}{1-a} H'_1(y) \right) dy, \end{aligned}$$

and hence, for $\tau \leq x < \lambda$,

$$\begin{aligned} f_1^{ac}(x) - \frac{a}{1-a} H'_1(x) &= - \frac{a}{1-a} \frac{H'_1(\lambda)}{H_1(\lambda-\tau)} \\ &\quad + \int_\tau^x \rho g_e(x-y) \left(f_1^{ac}(y) - \frac{a}{1-a} H'_1(y) \right) dy. \end{aligned}$$

Iterating the above equation and using the dominated convergence theorem give the second equation of (2.14). Similarly, iterating (2.13) gives (2.15), since $H'_M(x) = \sum_{n=1}^\infty (\rho/M)^n g_e^{*n}(x)$ is well-defined. □

3. The case of exponential jumps

In this section, we consider the special case of exponential jumps, that is, $G(x) = 1 - e^{-x/m}$, for all $x \geq 0$. By the memoryless property of the exponential random variable, for all starting level x ($0 < x \leq \lambda$), the probability $P_x(l)$ of (2.9) is given by

$$P_x(l) = 1 - e^{-l/m}, \quad l \geq 0,$$

which is independent of starting level x . Let $\theta_1 = 1/m - \nu$ and $\theta_M = 1/m - \nu/M$, then

$$H_1(x) = \frac{1 - \rho e^{-\theta_1 x}}{1 - \rho} \quad \text{and} \quad H_M(x) = \frac{M - \rho e^{-\theta_M x}}{M - \rho}.$$

We can obtain that

$$P(l, x) = \frac{\rho e^{-l/m} (e^{-\theta_1(\lambda-x)} - e^{-\theta_1 \lambda})}{1 - \rho e^{-\theta_1 \lambda}}, \quad l \geq 0, 0 \leq x \leq \lambda,$$

and

$$P_\lambda = \frac{(1 - \rho)e^{-\theta_1 \lambda}}{1 - \rho e^{-\theta_1 \lambda}}.$$

Therefore the densities can be derived by

$$f_1^{ac}(x) = \begin{cases} \frac{a\nu e^{-\theta_1 x}}{1 - a}, & 0 < x < \tau, \\ \frac{a\nu(e^{-\theta_1 x} - e^{-\theta_1 \lambda})}{(1 - a)(1 - \rho e^{-\theta_1(\lambda-\tau)})}, & \tau \leq x < \lambda, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{ab\nu(1 - \rho)e^{-\theta_1 \lambda}(M - \rho e^{-\theta_M(x-\tau)})}{M(1 - b)(M - \rho)(1 - \rho e^{-\theta_1(\lambda-\tau)})}, & \tau < x < \lambda, \\ \frac{ab\nu(1 - \rho)e^{-\theta_1 \lambda}(M e^{-\theta_M(x-\lambda)} - \rho e^{-\theta_M(x-\tau)})}{M(1 - b)(M - \rho)(1 - \rho e^{-\theta_1(\lambda-\tau)})}, & x \geq \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Applying the normalizing conditions, we have

$$a = \frac{(1 - \rho)(1 - \rho e^{-\theta_1(\lambda-\tau)})}{1 - \rho e^{-\theta_1(\lambda-\tau)} - \nu(1 - \rho)(m + \lambda - \tau)e^{-\theta_1 \lambda}}$$

and

$$b = \frac{M - \rho}{M - 1 + a}.$$

It coincides with the probability density for the case that $r_1 = 1$ and $r_2 = M$ in Lee and Kim (2006).

Remark 3.1 If τ decreases to 0, this stationary distribution of the workload process coincides with that of the $M/M/1$ queue under the P_λ^M policy obtained in Bae *et al.* (2002).

References

Abdel-Hameed, M. (2000). Optimal control of a dam using $P_{\lambda, \tau}^M$ policies and penalty cost when the input process is a compound Poisson process with positive drift. *Journal of Applied Probability*, **37**, 408-416.
 Bae, J., Kim, S. and Lee, E. Y. (2002). A P_λ^M -policy for an $M/G/1$ queueing system. *Applied Mathematical Modelling*, **26**, 929-939.

- Bae, J., Kim, S. and Lee, E. Y. (2003). Average cost under the $P_{\lambda, \tau}^M$ policy in a finite dam with compound Poisson inputs. *Journal of Applied Probability*, **40**, 519-526.
- Bar-Lev, S. K. and Perry, D. (1993). Two-stage release rule procedure in a regenerative dam. *Probability in the Engineering and Informational Sciences*, **7**, 571-588.
- Cohen, J. W. (1977). On up- and downcrossings. *Journal of Applied Probability*, **14**, 405-410.
- Faddy, M. J. (1974). Optimal control of finite dams: Discrete(2-stage) output procedure. *Journal of Applied Probability*, **11**, 111-121.
- Kim, M. H., Baek, J. S., Choi, S. K. and Lee, E. Y. (2011). An optimal policy for an infinite dam with exponential inputs of water. *Journal of the Korean Data & Information Science Society*, **22**, 1089-1096.
- Kim, J., Bae, J. and Lee, E. Y. (2006). An optimal P_{λ}^M -service policy for an $M/G/1$ queueing system. *Applied Mathematical Modelling*, **30**, 38-48.
- Lee, E. Y. (2008). A new approach to an inventory with constant demand. *Journal of the Korean Data & Information Science Society*, **19**, 1345-1352.
- Lee, J. and Kim, J. (2006). A workload-dependent $M/G/1$ queue under a two-stage service policy. *Operations Research Letters*, **34**, 531-538.
- Yeh, L. (1985). Optimal control of a finite dam: Average-cost case. *Journal of Applied Probability*, **22**, 480-484.