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Comparison of accumulate-combine and combine-accumulate methods in multivariate CUSUM charts for mean vector[†]

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Abstract

We compared two basic methods, combine-accumulate method and accumulate-combine method, using the past quality information in multivariate quality control procedure for monitoring mean vector of multivariate normal process. When small or moderate shifts have occurred, accumulate-combine method yields smaller average run length (ARL) and average time to signal (ATS) than combine-accumulate method. On the other hand, we have found from our numerical results that combine-accumulate method has better performances in terms of switching behavior than accumulate-combine method. In industry, a quality engineer could select one of the two method under the comprehensive consideration about the required time to signal, switching behavior, and other physical factors in the production process

Keywords: Accumulate-combine method, combine-accumulate method, Markov chain method, switching behavior.

1. Introduction

Control charts are used for continuously monitoring the production process to quickly detect the shifts that may produce any deterioration in the quality of the product. Usually, the quality of the output is determined by multiple quality variables or characteristics.

Shewhart chart, one of the most widely used control charts, was first proposed by Shewhart in 1931. The Shewhart chart has a good ability to detect quickly large shifts in monitored parameter and is easy to implement the process. However, the Shewhart chart uses only the information from the last sample and so it is insensitive to small or moderate shifts in the production process.

In order to overcome this difficulty, Page (1955) added warning lines within the action lines of the standard Shewhart chart. His additional rule is that if r out of the last N sample means fall between the warning lines and control limits, then the chart signals. He showed that $\overline{X}$ control charts with warning lines are more efficient than the standard $\overline{X}$ charts in detecting small shifts in the process mean.

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Another modification of the chart with adding supplementary runs rules was recommended by Moore (1958) and Page (1962). Champ and Woodall (1987) obtained the exact run length properties of Shewhart charts with supplementary runs rules by using Markov chain method.

Cumulative sum (CUSUM) chart using the past sample information was first proposed by Page (1954). CUSUM charts are usually used instead of standard Shewhart charts when the detection of small shifts in a process is important. As demonstrated by Champ and Woodall (1987), the superiority of the CUSUM chart over the Shewhart chart also holds when the Shewhart chart is augmented with runs rules. Barnard (1959) developed the CUSUM procedure as a sequential likelihood ratio test (SPRT) for testing the hypothesis that the process mean is equal to the target value against the alternative that it is not.

There are two basic methods using the past sample information in multivariate quality control chart, combine-accumulate method and accumulate-combine method. Crosier (1988) proposed a multivariate CUSUM chart with accumulate-combine method, and Pignatiello and Runger (1990) proposed new CUSUM charts with accumulate-combine method. They compared accumulate-combine method and combine-accumulate method in terms of ARL under fixed sampling interval (FSI) scheme. They showed through numerical results that accumulate-combine method is more efficient than combine-accumulate method only in terms of ARL. Multivariate control charts with variable sampling interval (VSI) scheme were studied by Cho (2010), Im and Cho (2009) and Chang and Heo (2010).

In this paper, we compared the efficiency of combine-accumulate method and accumulate-combine method in terms of both required time to signal and switching behavior when one operates VSI CUSUM procedure. In addition, we suggest some criteria that a quality engineer in industry should consider before selecting one of the two methods.

2. Description of some control procedures

Suppose that the production process of interest has p quality characteristics whose distribution is multivariate normal with mean vector $\underline{\mu}$ and dispersion matrix Σ, and $(\underline{\mu}_0, \Sigma_0)$ is the known target process values for $(\underline{\mu}, \Sigma)$. The target $\underline{\mu}_0$ and Σ_0 of p quality characteristics is represented as

$$\mu_0 = \begin{pmatrix} \mu_{10} \\ \mu_{20} \\ \vdots \\ \mu_{p0} \end{pmatrix} \text{ and } \Sigma_0 = \begin{pmatrix} \sigma_{10}^2 & \rho_{120}\sigma_{10}\sigma_{20} & \cdots & \rho_{1p0}\sigma_{10}\sigma_{p0} \\ \rho_{210}\sigma_{10}\sigma_{20} & \sigma_{20}^2 & \cdots & \rho_{2p0}\sigma_{20}\sigma_{p0} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p10}\sigma_{10}\sigma_{p0} & \rho_{p20}\sigma_{20}\sigma_{p0} & \cdots & \sigma_{p0}^2 \end{pmatrix}.$$

For successive samples, multivariate control procedure for monitoring mean vector $\underline{\mu}$ of multivariate normal process can be interpreted as repeated tests of the significance of the form

$$\begin{aligned} H_0 &: \underline{\mu} = \underline{\mu}_0 \\ H_1 &: \underline{\mu} \neq \underline{\mu}_0 \end{aligned} \tag{2.1}$$

For simplicity in this paper, we will assume that $\underline{\mu}_0 = \underline{0}$, and the more general case can be handled easily by translation. At each sampling occasion i $(i = 1, 2, \cdots)$, we take a sequence of random vector $\underline{X}_i' = (\underline{X}_{i1}', \underline{X}_{i2}', \cdots, \underline{X}_{in}')$ where $\underline{X}_{ij}' = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})$. Thus $\underline{X}_i$

is an $np \times 1$ column vector. Then the jkth element X_{ijk} of $\underline{X}_i$ is the jth observation for kth quality characteristic at each sampling occasion i ($j = 1, 2, \cdots, n; k = 1, 2, \cdots, p$). We assume that the sequential observation vectors between and within samples are independent and identically distributed.

To test the hypothesis in (2.1), we can obtain multivariate control statistic for monitoring $\underline{\mu}$ by using the likelihood ratio test (LRT) statistic for testing $H_0 : \underline{\mu} = \underline{\mu}_0$ where Σ_0 is known. Likelihood ratio λ_i at the ith sample can be expressed as $\lambda_i = \exp\left[-n(\overline{\underline{X}}_i - \underline{\mu})'\Sigma_0^{-1}(\overline{\underline{X}}_i - \underline{\mu})/2\right]$. By simple transformation, we can obtain LRT statistic χ_i^2 as

$$\chi_i^2 = n(\overline{\underline{X}}_i - \underline{\mu})'\Sigma_0^{-1}(\overline{\underline{X}}_i - \underline{\mu}). \tag{2.2}$$

Since the LRT statistic χ_i^2 has a chi-square distribution with p degrees of freedom, the null hypothesis should be rejected at time i if $\chi_i^2 > \chi_\alpha(p)$ where $\chi_\alpha(p)$ is the upper 100α percentage point of the χ^2 distribution with p degrees of freedom. The noncentrality parameter associated with χ^2 is $\lambda^2(\underline{\mu}) = n(\underline{\mu} - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{\mu} - \underline{\mu}_0)$ and the scale $\lambda^2(\underline{\mu})$ is often used to represent a measure of the distance of $\underline{\mu}$ from $\underline{\mu}_0$. This measure of distance is called the statistical distance by Johnson and Wichern (1988).

Shewhart χ^2 chart signals whenever $\chi_i^2 > \chi_\alpha(p)$, and the ARL of this chart can be calculated as $1/p$ where p denotes the probability that the chart statistic χ_i^2 exceeds the upper control limit (UCL) $\chi_\alpha(p)$. Therefore, the ARL values of the multivariate Shewhart χ^2 chart depend on $\underline{\mu}$ and Σ only through the statistical distance

$$d = \sqrt{n(\underline{\mu} - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{\mu} - \underline{\mu}_0)}.$$

Therefore, it is possible to consider the ARL as a function of d. And we will henceforth call d as distance, which means the square root of the noncentrality parameter $\lambda^2(\underline{\mu})$.

Traditional practice in using a control chart is to take samples from the process at FSI and the properties of control charts have been developed when the sampling interval between samples is fixed. In recent years, application of VSI control charts has become quite frequent and several papers have been published about them in which the sampling interval is varied as a function of what is observed from the process. Cui and Reynolds (1988) considered VSI Shewhart $\overline{X}$ -chart with runs rules using Markov chain method and Reynolds et al. (1990) considered the properties of VSI CUSUM charts.

In the theoretical and numerical comparisons between FSI and VSI procedures, many researchers showed that the VSI schemes are substantially more efficient than the FSI schemes in terms of the required time to signal when the process has shifted.

For the VSI χ^2 Shewhart chart based on LRT statistic, suppose that the two sampling interval

$$\begin{aligned} d_1 &\text{ is used when } \chi^2 \in (g_S, h_S] \\ d_2 &\text{ is used when } \chi^2 \in (0, g_S], \end{aligned}$$

where $d_1 < d_2$. The parameters g_S, h_S can be obtained from chi-square distribution to guarantee a desired ARL and ATS.

One disadvantage of VSI procedure in industry is that frequent switching between short and long sampling intervals requires more cost and effort to administer the process than corresponding FSI procedure. Amin and Letsinger (1991) described general procedures for VSI scheme and presented that the average number of switches to signal (ANSW) of the CUSUM and EWMA procedures exists far fewer than the Shewhart procedure. Bai and Lee (2002) investigated three switching rules to the $\overline{X}$ control and the expressions of the ATS and ANSW were derived with Markov chain approach.

Since ARL and ATS do not provide any switching information between short and long sampling intervals of VSI schemes, it is necessary to define the number of switches (NSW) as the number of switches made from the start of the process until the chart signals, and let ANSW be the expected value of the NSW. The ANSW can be obtained by using Wald's identity as follows

$$ANSW = (ARL - 1) \cdot P(\text{switch}) \tag{2.3}$$

where the ARL can be approximated by a Markov chain approach for the multivariate CUSUM procedures in (3.2). The probability of switch is given by

$$P(\text{switch}) = P(d_1) \cdot P(d_2|d_1) + P(d_2) \cdot P(d_1|d_2) \tag{2.4}$$

where $P(d_i)$ is the probability of using sampling interval d_i, and $P(d_i|d_j)$ is the conditional probability of using sampling interval d_i in the current sample given that the sampling interval d_j $(d_i \neq d_j)$ was used in the previous sample.

To evaluate $P(\text{switch})$ in VSI Shewhart χ^2 scheme, we devide the in control region $C = (0, h_S)$ into r states. The region $C_1 = (0, g_S]$ using long sampling interval d_2 is divided into m states $E_1, E_2, \cdots, E_m$ and the region $C_2 = (g_S, h_S]$ using short sampling interval d_1 is divided into $(r - m)$ states $E_{m+1}, E_{m+2}, \cdots, E_r$.

Then the probability of switch $P(\text{switch})$ can be expressed as

$$\begin{aligned} P(\text{switch}) = & \sum_{i=m+1}^{r} P(\chi_k^2 \in E_i) \cdot \left\{ \sum_{j=1}^{m} P(\chi_{k+1}^2 \in E_j | \chi_k^2 \in E_i) \right\} \\ & + \sum_{i=1}^{m} P(\chi_k^2 \in E_i) \cdot \left\{ \sum_{j=m+1}^{r} P(\chi_k + 1^2 \in E_j | \chi_k^2 \in E_i) \right\}. \end{aligned} \tag{2.5}$$

Because the Shewhart χ^2 chart uses only the information from the last sample and the successive observation vectors are independent, the conditional probabilities in (2.5) can be expressed as $\sum_{j=1}^{m} P(\chi_{k+1}^2 \in E_j | \chi_k^2 \in E_i) = \sum_{j=m+1}^{r} P(\chi_{k+1}^2 \in E_j)$.

Then the transition probability p_{ij} is as follows. For $i = 1, 2, \cdots, m$

$$P[\chi_i^2 \in E_i] = F(jw) - F[(j-1)w]$$

and

$$p_{ij} = F[(g + (j-m)v] - F[g - (j-m-1)v], (j = m+1, m+2, \cdots, r).$$

And, for $i = m+1, m+2, \cdots, r$

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and

$$p_{ij} = F(jw) - F[(j-1)w], (j = 1, 2, \cdots, m)$$

where $F(\cdot)$ is the distribution function of control statistic, $v = (h-g)/(r-m)$ and $\omega = g/m$. In our computation using Markov chain method, P(switch) tends to be stabilized when the number of states r is greater than 100.

For comparison of the switching behavior of the considered charts, we also define $N(d_i \to d_j)$ as the number of switching from d_i to d_j $(i \neq j)$. Since frequent switching between different sampling intervals is a complicating factor in the application of VSI procedure, the small scale of $N(d_i \to d_j)$ $(i \neq j)$ is desirable in terms of operating control chart.

3. CUSUM chart with combine-accumulate method

The basic Shewhart chart, although simple to understand and apply, uses only the information in the current sample and is thus relatively inefficient in detecting small shifts. On the other hand, CUSUM chart directly incorporates all of the information in the sequence of samples by using a control statistic which is a cumulative sum of statistics computed from each sample. When the detection of small or moderate shifts in the process parameters is important, CUSUM chart is a good alternative to the Shewhart chart.

Woodall and Ncube (1985) extended the univariate CUSUM procedure to the multivariate case to detect a shift in the mean vector of a p-variate normal distribution, operate p one-side or two-sided CUSUM schemes simultaneously and evaluate the performance of the collection of schemes.

We can consider control procedures as a sequence of independent tests where each test is actually equivalent to a SPRT for testing $H_0 : \underline{\mu} = \underline{\mu}_0$ vs $H_1 : \underline{\mu} \neq \underline{\mu}_0$. This sequence of SPRT's is equivalent to using the CUSUM statistic

$$\widehat{Y}_i = \max\left\{\widehat{Y}_{i-1} + (\chi_i^2 - k), 0\right\}, \tag{3.1}$$

where $\chi_i^2 = n(\underline{\bar{X}}_i - \underline{\mu})'\Sigma_0^{-1}(\underline{\bar{X}}_i - \underline{\mu})$, $\widehat{Y}_0 = \omega$ $(\omega \geq 0)$ is a constant, and the reference value k (≥ 0). The parameter k is usually determined by the scale of shift which the CUSUM is designed to detect. This chart signals whenever $\widehat{Y}_i > h_{c1}$.

For the purpose of using Markov chain method and adding VSI feature to the CUSUM chart in (3.1), Reynolds et al. (1990) proposed a modified CUSUM statistic

$$Y_i = \max\{Y_{i-1}, 0\} + (\chi_i^2 - k), \tag{3.2}$$

where $Y_0 = \omega$ $(\omega \geq 0)$ is a constant. The difference between $\widehat{Y}_i$ and Y_i is that $\widehat{Y}_i$ immediately resets negative CUSUM value to 0, and Y_i records the negative CUSUM value and then starts the cumulation from 0 for the next sample. Except for recording of negative CUSUM values, $\widehat{Y}_i$ and Y_i are equivalent. The reason for recording negative CUSUM values is that these negative values may be needed to specify the sampling intervals for VSI procedures.

For the VSI CUSUM chart, suppose that the sampling interval

$$d_1 \text{ is used when } Y_i \in (g_{C1}, h_{C1}],$$
$$d_2 \text{ is used when } Y_i \in (-k, g_{C1}],$$

where $g_{C1} \leq h_{C1}$.

In this study, we obtained the design parameters g_{C1} and h_{C1} by Markov chain method. Under the process in control and out of control states, the numerical performances and switching behaviors of the combine-accumulate method based on (3.1) or (3.2) were evaluated by using Markov chain method or simulation with 10,000 iterations. Markov chain method for multivariate EWMA chart can be referred from Chang *et al.* (2003).

4. CUSUM chart with accumulate-combine method

For multivariate quality control procedures, there are two basic methods using the past sample information. The first method, called combine-accumulate method, combines the multivariate data into a univariate statistic and then accumulate over past samples. The second method, called accumulate-combine method, accumulates the past sample information for each process parameter to be monitored and then either uses separate charts for each process parameter or combines the separate accumulations into a univariate statistic.

Under the accumulate-combine method, after the past sample information is accumulated for each process parameter, we can either use separate CUSUM chart and look at joint properties of the separate charts, or form a univariate chart statistic from the multivariate accumulations.

Up to the present, most of the studies in multivariate CUSUM charts with accumulate-combine technique have been concerned only for controlling the mean vector $\underline{\mu}$ of a multivariate normal process $N(\underline{\mu}, \Sigma)$. In this section, our concern on accumulate-combine technique is restricted only on the shifts of mean vector of a multivariate normal process.

Crosier (1988) proposed a multivariate CUSUM chart which accumulates the $\underline{\bar{X}}_i$ vectors before producing the quadratic forms. This chart is based on the statistics

$$C_i = \left\{ n(\underline{S}_{i-1} - \underline{\bar{X}}_i - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{S}_{i-1} + \underline{\bar{X}}_i - \underline{\mu}_0) \right\}^{\frac{1}{2}}$$

and

$$\underline{S}_i = \begin{cases} \underline{0} & \text{if } C_i \leq k_1 \\ (\underline{S}_{i-1} + \underline{\bar{X}}_i - \underline{\mu}_0)(1 - k_1/C_i) & \text{if } C_i > k_1, \end{cases}$$

where $i = 1, 2, \cdots, \underline{S}_0 = \underline{0}$ and $k_1 > 0$. This multivariate CUSUM scheme signals when

$$CY_i = \left\{ n(\underline{S}_i'\Sigma_0^{-1}\underline{S}_i^{\frac{1}{2}} \right\} > h_i \tag{4.1}$$

where $h_1 > 0$. He gave the proof that the distribution of the multivariate CUSUM chart statistic CY_i depends only on the value of noncentrality parameter.

Pignatiello and Runger (1990) proposed a new multivariate CUSUM procedure based on accumulate-combine approach for controlling mean vector of the multivariate normal process. This new CUSUM procedure is based on a quadratic form of the mean vector. This chart, called MC1, is based on the following vector of multivariate sum

$$\underline{MC}_i = \sum_{j=i-l_i+1}^{i} (\underline{\bar{X}}_j - \underline{\mu}_0).$$

The norm of $\underline{MC}_i$, $||\underline{MC}_i|| = \sqrt{n\underline{MC}_i'\Sigma_0^{-1}\underline{MC}_i}$, is a measure of the distance between the process mean vector and the target mean vector for the process. They constructed MC1 chart as

$$\underline{MC}_i = \max\left\{0, (n\underline{MC}_i'\Sigma_0^{-1}\underline{MC}_i)^{\frac{1}{2}} - k_2 l_i\right\} \tag{4.2}$$

where $k_2 > 0$,

$$l_i = \begin{cases} l_{i-1} + 1 & \text{if } MC1_{i-1} > 0 \\ 1 & \text{otherwise} \end{cases}$$

and l_i can be interpreted as the number of subgroups since the most recent renewal of the CUSUM has occurred ($i = 1, 2, \cdots$). The MC1 chart operates by plotting $MC1_i$ on a control chart and an out of control signal is given as soon as $MC1_i > h_{C2}$ where $h_{C2} > 0$.

Pignatiello and Runger (1990) found that the $MC1$ chart in (4.2) has a superior ARL performance than a multivariate CUSUM chart based on the combine-accumulate approach. And they also proved that the ARL performances of the $MC1$ chart depends only on the noncentrality parameter. The multivariate CUSUM chart proposed by Crosier (1988) is similar to MC1 chart. Croiser's multivariate CUSUM chart is somewhat more complicated than MC1 chart, but it has a similar ARL performance.

For VSI CUSUM chart based on $MC1$, we suppose the sampling interval

$$d_1 \text{ is used when MC } 1_i \in (g_{C2}, h_{C2}],$$
$$d_2 \text{ is used when MC } 1_i \in (0, g_{C2}],$$

where $g_{C2} \le h_{C2}$. The ARL performances and switching behaviors of the FSI or VSI MC1 charts based on $MC1_i$ cannot be modeled as a simple stationary Markov chain as described in Brook and Evans (1972). For this reason, we obtained the design parameters h_{C2}, g_{C2} satisfying the desired ARL and ATS through Monte Carlo simulation.

5. Concluding remarks and conclusion

Comparisons between combine-accumulate method and accumulate-combine method of multivariate CUSUM chart for monitoring mean vectorare performed. And Shewhart χ^2 chart which does not use the past sample information is also compared.

In order to compare the ARL performances and switching behaviors of the considered charts, some kinds of standards for comparison are necessary. For simplicity in our computation, we assume that target mean vector $\underline{\mu}_0 = \underline{0}'$, and all diagonal and off-diagonal elements of Σ_0 are 1 and 0.5, respectively. The numerical results were obtained when the ARL and ATS of the in control state was approximately equal to 200, the sampling interval before the first sample $d_0 = 1$, $d_1 = 0.1$, $d_2 = 1.9$, and the sample size n for each characteristic was five for $p = 2$ or 4.

The numerical performances of the considered multivariate charts in (2.2), (3.2) and (4.2) are determined solely by the distance d of the off-target mean $\underline{\mu}$ from the on target mean $\underline{\mu}_0$, not by the particular direction or location of the mean.

Because of direction invariance, we evaluate performances for shifts in the process mean that are of the form $\underline{\mu} = (\mu_1, 0, \cdots, 0)'$ for $p = 2$ or 4. Shifts of these forms were investigated for distances 0.5, 1.0, $\cdots$, 5.0. The values of μ_1 are shown in the Table 5.1 and were computed so that the distance d of $\underline{\mu} = (\mu_1, 0, \cdots, 0)'$ from $\underline{\mu}_0$ is given as in the first column in Table 5.1.

After the reference values of the considered CUSUM charts have been determined, the design parameters $h's$ and $g's$ of the corresponding CUSUM charts were calculated by either Markov chains with the number of transient states $r = 100$ or simulation with 10,000 iterations. And the ARL, ATS and switching behaviors, when the process has on-target or changed, were also obtained by either Markov chains with the number of transient states $r = 100$ or simulation with 10,000 runs.

When large shift of the process has occurred, Shewhart χ^2 chart is effective in terms of both the required time to signal and switching behavior. At distance d over 3.0, we found in Tables 5.2-5.4 and Figure 5.2 that Shewhart χ^2 chart is effective in the performances including ARL, ATS, ANSW, P(switch). However, it is hard to recommend Shewhart χ^2 chart in the case of small or moderate shifts that is key interest in the production process.

From the results in Figure 5.1 and Tables 5.3-5.4, multivariate CUSUM chart with accumulate-combine method appears to be a good control charting device for detecting small or moderate shifts in terms of required time to signal.

On the other hand, multivariate CUSUM chart with combine-accumulate method shows better performance than accumulate-combine method in terms of switching behavior, such as P(switch), ANSW and $N(d_2 \to d_1)$. And we also found that at the small values of the reference value, the required time to signal, ARL and ATS, and switching behaviors, P(switch), ANSW and $N(d_2 \to d_1)$, were reduced in both accumulate-combine and combine-accumulate methods.

When small or moderate shifts have occurred, accumulate-combine method yields smaller ARL and ATS than combine-accumulate method. On the other hand, we have found from the results in Tables 5.3-5.4 and Figures 5.1-5.4 that combine-accumulate method has better performances than accumulate-combine method in terms of switching behavior.

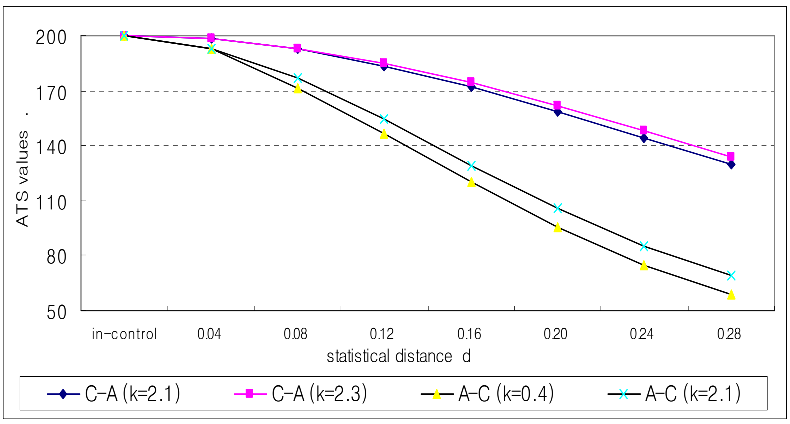

Figure 5.1 ATS performances of the CUSUM charts (p =2)

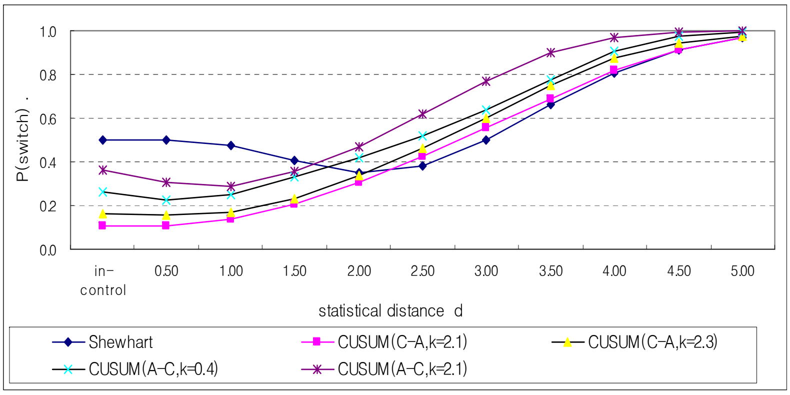

Figure 5.2 P (switch) of the considered charts (p =2)

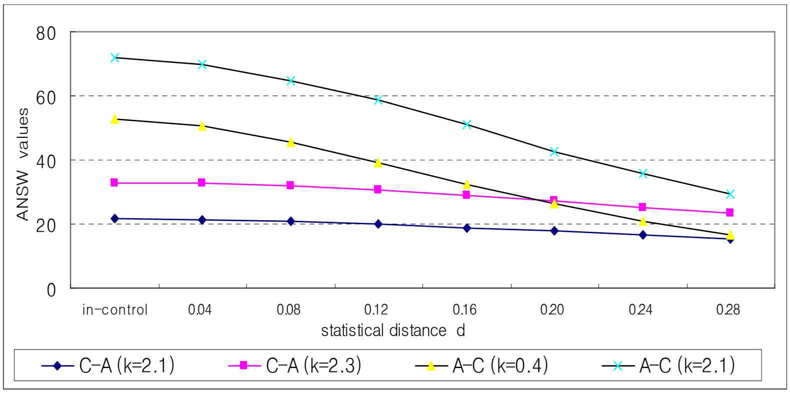

Figure 5.3 ANSW performances of the CUSUM charts (p =2)

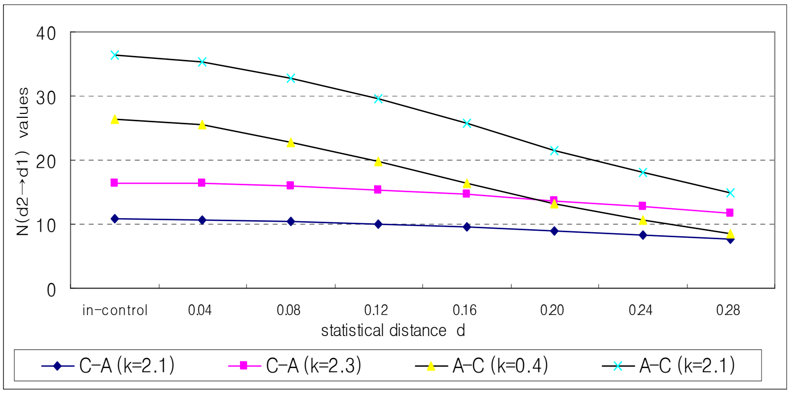

Figure 5.4 N ($d_2 \rightarrow d_1$) performances of the CUSUM charts (p =2)

Table 5.1 The values of μ_1 with different d of $\underline{\mu}$ from $\underline{\mu}_0$ ($\rho_0 = 0.5, n = 5$)

d	$p=2$	$p=4$	d	$p=2$	$p=4$
0.00	0.0000000	0.0000000	0.0	0.0000000	0.0000000
0.04	0.0154920	0.0141421	0.5	0.1936492	0.1767767
0.08	0.0309839	0.0282843	1.0	0.3872983	0.3535534
0.12	0.0464758	0.0424264	1.5	0.5809475	0.5303301
0.16	0.0619677	0.0565686	2.0	0.7745966	0.7071068
0.20	0.0774596	0.0707107	2.5	0.9682459	0.8838835
0.24	0.0929516	0.0848528	3.0	1.1618950	1.0606600
0.28	0.1084436	0.0989949	3.5	1.3555440	1.2374370
0.32	0.1239354	0.1131371	4.0	1.5491930	1.4142140

Table 5.2 Numerical results for Shewhart χ^2 chart ($p = 4$)

d	ARL	ATS	ANSW	P(switch)	$N(d_1 \to d_2)$	$N(d_2 \to d_1)$
on-target	200.00	199.99	98.51	0.493	49.25	49.25
0.5	138.15	130.48	67.32	0.487	33.66	31.78
1.0	60.96	48.70	27.51	0.451	13.75	10.94
1.5	24.62	15.23	8.75	0.355	4.37	2.64
2.0	10.63	5.04	2.31	0.217	1.15	0.48
2.5	5.20	2.18	0.49	0.095	0.25	0.07
3.0	2.93	1.37	0.08	0.028	0.04	0.01
3.5	1.91	1.13	0.01	0.005	0.00	0.00
4.0	1.42	1.05	0.00	0.001	0.00	0.00

Table 5.3 Numerical results for CUSUM chart with combine-accumulate method ($p = 4, k = 4.2$)

d	ARL	ATS	ANSW	P(switch)	$N(d_1 \to d_2)$	$N(d_2 \to d_1)$
on-target	200.00	200.00	24.22	0.121	12.14	12.08
0.5	99.24	83.06	11.95	0.120	6.01	5.94
1.0	32.19	20.94	4.07	0.126	2.09	1.97
1.5	14.50	9.02	2.37	0.164	1.29	1.08
2.0	8.29	5.33	1.84	0.222	1.09	0.75
2.5	5.44	3.73	1.54	0.283	1.02	0.52
3.0	3.90	2.90	1.31	0.336	0.99	0.32
3.5	2.98	2.44	1.13	0.379	0.96	0.18
4.0	2.39	2.19	0.97	0.406	0.89	0.08

Up to the present, some studies showed through numerical results that accumulate-combine method is better than combine-accumulate method only in terms of the required time to signal. In our numerical results, we also have the same results with them in terms of the time required to signal.

However, we have found that when the switching behavior is considered along with the time required to signal, it is hard to conclude that accumulate-combine method has always better performances than combine-accumulate method.

Therefore, when a quality engineer in industry selects one of two methods, accumulate-combine method and combine-accumulate method, he/she could select one of them under the comprehensive consideration about the required time to signal, switching behavior, and additional efforts and costs in the process of operating VSI chart. In addition, he/she could also consider physical time interval between short sampling interval d_1 and long sampling interval d_2 before selecting one of them.

Table 5.4 Numerical results for CUSUM chart with accumulate-combine method ($p = 4, k = 0.5$)

d	ARL	ATS	ANSW	P(switch)	$N(d_1 \to d_2)$	$N(d_2 \to d_1)$
on-target	200.01	200.00	47.80	0.239	24.05	23.75
0.5	47.07	35.37	10.04	0.213	5.18	4.86
1.0	13.45	6.98	2.72	0.202	1.56	1.16
1.5	7.36	3.56	1.80	0.244	1.14	0.65
2.0	5.11	2.37	1.46	0.285	1.04	0.42
2.5	3.99	1.83	1.28	0.320	1.01	0.26
3.0	3.30	1.52	1.16	0.351	1.00	0.15
3.5	2.83	1.33	1.08	0.381	1.00	0.08
4.0	2.49	1.21	1.03	0.415	1.00	0.04

References

Amin, R. W. and Letsinger, W. C. (1991). Improved switching rules in control procedures using variable sampling intervals. *Communications in Statistics-Simulation and Computation*, **20**, 205-230.

Bai, D. S. and Lee, K. T. (2002). Variable sampling interval $\overline{X}$ control charts with an improved switching rule. *International Journal of Production Economics*, **76**, 189-199.

Barnard, G. A. (1959). Control charts and stochastic process. *Journal of Royal Statistical Society B*, **2**, 239-257.

Brook, D. and Evans, D. A. (1972). An approach to the probability distribution of CUSUM run length. *Biometrika*,**59**, 539-549.

Champ, C. W. and Woodall, W. H. (1987). Exact results for Shewhart control charts with supplementary runs rules. *Technometrics*, **29**, 393-399.

Chang, D. J. and Heo, S. (2010). Properties of variable sampling interval control charts. *Journal of the Korean Data & Information Science Society*, **21**, 819-829.

Chang, D. J., Kwon, Y. M. and Hong, Y. W. (2003). Markovian EWMA control chart for several correlated quality characteristics. *Journal of the Korean Data & Information Science Society*, **14**, 1045-1053.

Cho, G. Y. (2010). Multivariate Shewhart control charts with variable sampling intervals. *Journal of the Korean Data & Information Science Society*, **21**, 999-1008.

Crosier, R. B. (1988). Multivariate generalizations of cumulative sum quality-control schemes, *Technometrics*, **30**, 291-303.

Cui, R. Q. and Reynolds, M. R., Jr. (1988). $\overline{X}$ charts with runs rules and variable sampling intervals. *Communications in Statistics-Simulation and Computation*, **17**, 1073-1093.

Im, C. D. and Cho, G. Y. (2009). Multiparameter CUSUM charts with variable sampling intervals. *Journal of the Korean Data & Information Science Society*, **20**, 593-599.

Johnson, R. A. and Wichern, D. W. (1988). *Applied multivariate statistical analysis*, 2nd ed. Prentice Hall. Englewood Cliffs, NY.

Moore, P. G. (1958). Some properties of runs in quality control procedures. *Biometrika*, **45**, 89-95.

Page, E. S. (1954). Continuous inspection schemes. *Biometrika*, **41**, 100-114.

Page, E. S. (1955). Control charts with warning lines. *Biometrika*, **42**, 243-260.

Page, E. S. (1962). A modified control chart with warning lines. *Biometrika*, **49**, 171-176.

Pignatiello, J. J., Jr. and Runger, G. C. (1990). Comparisons of multivariate CUSUM charts, *Journal of Quality Technology*, **22**, 173-186.

Reynolds, M. R., Jr., Amin, R. W. and Arnold, J. C. (1990). CUSUM chart with variable sampling intervals. *Technometrics*, **32**, 371-396.

Woodall, W. H. and Ncube, M. M. (1985). Multivariate CUSUM quality control procedure. *Technometrics*, **27**, 285-292.