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Small sample likelihood based inference for the normal variance ratio[†]

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Abstract

This study deals with the small sample likelihood based inference for the ratio of two normal variances. The small sample likelihood inference is an approximation method. The signed log-likelihood ratio statistic and the modified signed log-likelihood ratio statistic, which converge to standard normal distribution, are proposed for the normal variance ratio. Through the simulation study, the coverage probabilities of confidence interval and power of the exact, the signed log-likelihood and the modified signed log-likelihood ratio statistic will be compared. A real data example will be provided.

Keywords: Likelihood based inference, modified signed log-likelihood ratio statistic, normal variance ratio, signed log-likelihood ratio statistic.

1. Introduction

The normal distribution plays an important role in statistical inference, and there are so many studies about this distribution. The statistical inference for the ratio of two normal variances arises in the areas for comparing the precision of two independent normal populations. About the ratio of two normal variances, there also exists an exact statistical inference with F statistic. This problem is simple, obvious and important.

When comparing the dispersion of two normal population, one can use the distribution table of F statistic. Since F distribution depends on degrees of freedom, the distribution table of F distribution is more than several pages. For the degrees of freedom exceed 30, the table does not have the value. In this case, one must use an interpolation to obtaining the quantile.

An approximation of a statistic to standard normal distribution has been developed in many statistical models. Since the percentile of standard normal distribution is well known, a statistic, which distributes as standard normal distribution asymptotically, may be very useful. But these statistics have the error rate depending on sample size. When the sample size is small, these approximation to standard normal distribution is quite inaccurate. For example, a signed log-likelihood ratio statistic converges to standard normal distribution with an error of $O(n^{-1/2})$, when the sample size is small, this statistic is inaccurate.

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A modified signed log-likelihood ratio statistic of Barndorff-Nielsen and Cox (1994) converges to standard normal distribution with an error of $O(n^{-3/2})$. This statistic converges to standard normal distribution fastly, so even in small sample size, it gives accurate approximation.

Wu *et al.* (2002) developed signed log-likelihood statistic and modified signed log-likelihood statistic for the ratio of means in two independent log-normal distributions. And they showed that the modified signed log-likelihood statistic worked well in a small sample size.

Wu and Jiang (2007) studied confidence interval of effect size of paired study in normal distribution, and they developed signed log-likelihood statistic and modified signed log-likelihood statistic. They compared the length and coverage of confidence interval based on those statistics to the other approximate confidence interval.

Lee *et al.* (2006) studied a confidence intervals for the common scale parameer in the inverse Gaussian distributions. Lee and Lee (2008) developed a likelihood based inference for the shape parameter of Pareto distribution. Lee *et al.* (2008) proposed a likelihood based inference for the shape parameter of inverse Gaussian distribution. They developed modified signed log-likelihood statistic and showed that this statistic performed well when the sample size is small. Kang *et al.* (2012) developed a likelihood based inference for the ratio of parameters in two Maxwell distributions.

Wong and Wu (2009) considered interval estimation of stress-strength reliability based on likelihood based inference in generalized exponential distribution. They proposed likelihood based statistic for reliability and compared coverage probabilities of confidence interval based on likelihood based statistic with exact confidence interval.

When the parameter of interest is normal variance ratio, this paper devotes to develop a signed log-likelihood statistic and a modified signed log-likelihood statistic for the parameter of interest. Even though there exists an exact statistic for this inference, to develop the highly accurate statistic which converges to standard normal distibution has a practical meaning.

This paper is arranged as follows. In Section 2, the signed log-likelihood statistic and the modified signed log-likelihood statistic for normal variance ratio are developed. In Section 3, through simulation study, the coverage probabilities of proposed statistics are compared. And a real data example is given. Concluding remarks are given in Section 4.

2. Likelihood based statistics for normal variance ratio

Let $X_1, X_2, \cdots, X_m$ be a random sample of size m from $N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from $N(\mu_2, \sigma_2^2)$. Assume that X_i and Y_j are independent.

The parameter of interest is σ_2^2/σ_1^2. Statistical inference for this parameter of interest is well known. The F statistic is used for interval estimation or testing.

Based on observations $\mathbf{x} = (x_1, \cdots, x_m)$ and $\mathbf{y} = (y_1, \cdots, y_n)$, the likelihood function for μ_1, μ_2, σ_1^2 and σ_2^2 is given by

$$L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-m} \sigma_2^{-n} \times exp \left\{ -\frac{1}{2\sigma_1^2} \sum_{i=1}^{m} (x_i - \mu_1)^2 - \frac{1}{2\sigma_2^2} \sum_{j=1}^{n} (y_j - \mu_2)^2 \right\}. \quad (2.1)$$

To develop a statistic for normal variance ratio which converges to standard normal distribution, let

$$\theta_1 = \frac{\sigma_2^2}{\sigma_1^2},\ \theta_2 = \sigma_1^2,\ \theta_3 = \mu_1,\ \theta_4 = \mu_2$$

and θ=$(\theta_1, \theta_2, \theta_3, \theta_4)$=$(\psi, \lambda)$, where $\psi = \theta_1$ and $\lambda = (\theta_2, \theta_3, \theta_4)$. Then $\psi = \theta_1$ is a parameter of interest and $\lambda = (\theta_2, \theta_3, \theta_4)$ is a nuisance parameter. The likelihood function for θ is given by

$$L(\theta) \propto \theta_1^{-\frac{n}{2}} \theta_2^{-\frac{N}{2}} exp \left\{ -\frac{t_1 - 2\theta_3 t_3 + m\theta_3^2}{2\theta_2} - \frac{t_2 - 2\theta_4 t_4 + n\theta_4^2}{2\theta_1\theta_2} \right\}, \tag{2.2}$$

where $N = m+n$, $t_1 = \sum_{i=1}^m x_i^2$, $t_2 = \sum_{j=1}^n y_j^2$, $t_3 = \sum_{i=1}^m x_i$ and $t_4 = \sum_{j=1}^n y_j$. From (2.2), the log-likelihood function for θ is given by

$$l(\theta) \equiv l(\theta; t) \propto -\frac{n}{2} log(\theta_1) - \frac{N}{2} log(\theta_2) - \frac{t_1}{2\theta_2} + \frac{\theta_3 t_3}{\theta_2} - \frac{m\theta_3^2}{2\theta_2} - \frac{t_2}{2\theta_1\theta_2} + \frac{\theta_4 t_4}{\theta_1\theta_2} - \frac{n\theta_4^2}{2\theta_1\theta_2}. \tag{2.3}$$

In the above log-likelihood (2.3), $t = (t_1, t_2, t_3, t_4)$ is a minimal sufficient statistic for θ.

From now on, some notations for developing likelihood based statistic is introduced. Let for $i, j = 1, 2, 3, 4$

$$l_{\theta_i}(\theta) = \frac{\partial l(\theta)}{\partial \theta_i},\ l_{\theta_i\theta_j}(\theta) = \frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j},$$

the sample space derivatives

$$l_{;t}(\theta) = \frac{\partial}{\partial t} l(\theta),$$

the mixed derivatives

$$l_{\theta;t}(\theta) = \frac{\partial}{\partial \theta} l_{;t}(\theta),$$

for $i, j = 1, 2, 3, 4$

$$j(\theta) = -\frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j} = -l_{\theta_i\theta_j}(\theta),$$

and for $i, j = 2, 3, 4$

$$j_{\lambda\lambda}(\theta) = -\frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j}.$$

The signed log-likelihood ratio statistic of Cox and Hinkely (1974) is given by

$$r \equiv r(\psi) = sgn(\widehat{\psi} - \psi) \left\{ 2 \left[l(\widehat{\psi}, \widehat{\lambda}) - l(\psi, \widehat{\lambda}_\psi) \right] \right\}^{1/2}, \tag{2.4}$$

where $\widehat{\theta} = (\widehat{\psi}, \widehat{\lambda})$ is the maximum likelihood estimator of θ, which is obvious, $\widehat{\lambda}_\psi$ is the constrained maximum likelihood estimator of $\lambda = (\theta_2, \theta_3, \theta_4)$ for a fixed ψ. The elements of constrained maximum likelihood estimator of λ with fixed ψ is

$$\widetilde{\theta}_2 = \frac{2}{N} \left\{ \frac{t_1}{2} - \widetilde{\theta}_3 t_3 + \frac{m\widetilde{\theta}_3^2}{2} + \frac{t_2}{2\theta_1} - \frac{\widetilde{\theta}_4 t_4}{\theta_1} + \frac{n\widetilde{\theta}_4^2}{2\theta_1} \right\},$$

$\widetilde{\theta}_3 = \widehat{\theta}_3 = \frac{t_3}{m}$ and $\widetilde{\theta}_4 = \widehat{\theta}_4 = \frac{t_4}{n}$. That is $\widehat{\lambda}_\psi$=$(\widetilde{\theta}_2, \widetilde{\theta}_3, \widetilde{\theta}_4)$.

This signed log-likelihood ratio statistic r distributes as standard normal distribution asymptotically. According to Cox and Hinkely (1974), r converges to standard normal distribution with the rate of $O(n^{-1/2})$. For testing the null hypothesis of $H_0 : \psi = \psi_0$ with known value of ψ_0, the null hypothesis is rejected if $|r(\psi_0)| \geq z_{\alpha/2}$, where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of standard normal distribution. A two-sided p-value for testing H_0 is approximately $2\{1 - \Phi(|r(\psi_0)|)\}$, where $\Phi(\cdot)$ is the distribution function of standard normal distribution. And the approximate $100(1-\alpha)\%$ confidence interval for ψ can be obtained from

$$\left\{\psi : |r(\psi)| \leq z_{\alpha/2}\right\}. \tag{2.5}$$

The advantage of r is that this statistic is invariant under reparametrization of ψ. However, r does not give accurate approximation to standard normal distribution, especially when the sample size is small.

There exist various ways to improve the accuracy of this approximation by adjusting the signed log-likelihood ratio statistic. From now on, the modified signed log-likelihood ratio statistic, known as the r^*, introduced by Barndorff-Nielsen (1986, 1991) is considered. The modified signed log-likelihood ratio statistic, r^*, has the form

$$r^* \equiv r^*(\psi) = r(\psi) + r(\psi)^{-1} \log\left\{\frac{u(\psi)}{r(\psi)}\right\}, \tag{2.6}$$

where $r(\psi)$ is given in (2.4) and

$$u(\psi) = \frac{\left| l_{;\widehat{\theta}}(\widehat{\theta}) - l_{;\widehat{\theta}}(\psi, \widehat{\lambda}_\psi) \quad l_{\lambda;\widehat{\theta}}(\psi, \widehat{\lambda}_\psi) \right|}{\left\{ \left| j(\widehat{\theta}) \right| \left| j_{\lambda\lambda}(\psi, \widehat{\lambda}_\psi) \right| \right\}^{1/2}}. \tag{2.7}$$

In the above $u(\psi)$, following Barndorff-Neilsen (1991), the sample-space derivatives are defined as

$$l_{;\widehat{\theta}}(\theta) = \frac{\partial}{\partial \widehat{\theta}} l(\theta; \widehat{\theta}),$$

the mixed derivatives as

$$l_{\lambda;\widehat{\theta}}(\theta) = \frac{\partial}{\partial \lambda} l_{;\widehat{\theta}}(\theta),$$

and $j(\widehat{\theta})$ is the observed information matrix and $j_{\lambda\lambda}(\psi, \widehat{\lambda}_\psi)$ is the observed nuisance information matrix with ψ and constrained maximum likelihood estimate, $\widehat{\lambda}_\psi$.

From the likelihood function of θ given in (2.2), we know that this model is a full rank exponential model. Also, the log-likelihood function based on data $(\mathbf{x}, \mathbf{y})$, given in (2.3) is only related to a minimum sufficient statistic t. There is a one-to-one transformation between $\widehat{\theta}$ and t, and the Jacobian matrix of this transformation is $\partial\widehat{\theta}/\partial t$. Therefore, the sample space derivatives with respect to $\widehat{\theta}$ in (2.7) can be derived based on the sample-space derivatives with respect to t. By using the fact that $j(\widehat{\theta}) = l_{\theta;\widehat{\theta}}(\widehat{\theta})$ and by canceling the determinant of the transformation Jacobian matrix, u in (2.7) reduces to the following form:

$$u(\psi) = \frac{\left| l_{;t}(\widehat{\theta}) - l_{;t}(\psi, \widehat{\lambda}_\psi) \; l_{\lambda;t}(\psi, \widehat{\lambda}_\psi) \right|}{\left| l_{\theta;t}(\widehat{\theta}) \right|} \left\{ \frac{\left| j(\widehat{\theta}) \right|}{\left| j_{\lambda\lambda}(\psi, \widehat{\lambda}_\psi) \right|} \right\}^{1/2},$$

where the sample-space derivatives $l_{;t}(\theta)$ is a column vector with the following elements :

$$l_{;t_1}(\theta) = -\frac{1}{2\theta_2}, l_{;t_2}(\theta) = -\frac{1}{2\theta_1\theta_2}, l_{;t_3}(\theta) = \frac{\theta_3}{\theta_2}, l_{;t_4}(\theta) = \frac{\theta_4}{\theta_1\theta_2}.$$

The mixed derivatives $l_{\lambda;t}(\theta)$ and $l_{\theta;t}(\theta)$ are given by

$$l_{\lambda;t}(\theta) = \begin{bmatrix} \frac{1}{2\theta_2^2} & 0 & 0 \\ \frac{1}{2\theta_1\theta_2^2} & 0 & 0 \\ -\frac{\theta_3}{\theta_2^2} & \frac{1}{\theta_2} & 0 \\ -\frac{\theta_4}{\theta_1\theta_2^2} & 0 & \frac{1}{\theta_1\theta_2} \end{bmatrix},$$

and

$$l_{\theta;t}(\theta) = \begin{bmatrix} 0 & \frac{1}{2\theta_2^2} & 0 & 0 \\ \frac{1}{2\theta_1^2\theta_2} & \frac{1}{2\theta_1\theta_2^2} & 0 & 0 \\ 0 & -\frac{\theta_3}{\theta_2^2} & \frac{1}{\theta_2} & 0 \\ -\frac{\theta_4}{\theta_1^2\theta_2} & -\frac{\theta_4}{\theta_1\theta_2^2} & 0 & \frac{1}{\theta_1\theta_2} \end{bmatrix},$$

respectively. The elements of observed information matrix $j(\theta)=[-l_{\theta_i\theta_j}]_{4\times4}, i,j = 1,2,3,4$ and the observed nuisance information matrix $j_{\lambda\lambda}(\theta) = [-l_{\theta_i\theta_j}]_{3\times3}, i,j = 2,3,4$ are given by

$$l_{\theta_1\theta_1} = \frac{n}{2\theta_1^2} - \frac{t_2}{\theta_1^3\theta_2} + \frac{2\theta_4 t_4}{\theta_1^3\theta_2} - \frac{n\theta_4^2}{\theta_1^3\theta_2},\ l_{\theta_1\theta_2} = l_{\theta_2\theta_1} = -\frac{t_2}{2\theta_1^2\theta_2^2} + \frac{\theta_4 t_4}{\theta_1^2\theta_2^2} - \frac{n\theta_4^2}{2\theta_1^2\theta_2^2},\ l_{\theta_1\theta_3} = l_{\theta_3\theta_1} = 0,$$

$$l_{\theta_1\theta_4} = l_{\theta_4\theta_1} = -\frac{t_4}{\theta_1^2\theta_2} + \frac{n\theta_4}{\theta_1^2\theta_2},\ l_{\theta_2\theta_2} = \frac{N}{2\theta_2^2} - \frac{t_1}{\theta_2^3} + \frac{2\theta_3 t_3}{\theta_2^3} - \frac{m\theta_3^2}{\theta_2^3} - \frac{t_2}{\theta_1\theta_2^3} + \frac{2\theta_4 t_4}{\theta_1\theta_2^3} - \frac{n\theta_4^2}{\theta_1\theta_2^3},$$

$$l_{\theta_2\theta_3} = l_{\theta_3\theta_2} = -\frac{t_3}{\theta_2^2} + \frac{m\theta_3}{\theta_2^2},\ l_{\theta_2\theta_4} = l_{\theta_4\theta_2} = -\frac{t_4}{\theta_1\theta_2^2} + \frac{n\theta_4}{\theta_1\theta_2^2},\ l_{\theta_3\theta_3} = -\frac{m}{\theta_2},\ l_{\theta_3\theta_4} = l_{\theta_4\theta_3} = 0,$$

and

$$l_{\theta_4\theta_4} = -\frac{n}{\theta_1\theta_2}.$$

Since $u(\psi)$ can be calculated from the above results, one can obtain $r^*(\psi)$.

According to Barndorff-Neilsen (1986, 1991), r^* is approximately distributed as a standard normal distribution to the third order. Hence the p-value and confidence intervals based on r^* are highly accurate. The two-sided p-value for testing $H_0 : \psi = \psi_0$ is approximately $2\{1 - \Phi(|r^*(\psi_0)|)\}$. The $100(1-\alpha)\%$ confidence interval for ψ can be obtained from

$$\left\{\psi : |r^*(\psi)| \le z_{\alpha/2}\right\}. \tag{2.8}$$

3. Simulation studies and example

Simulation studies were performed to investigate the performance of the proposed methods for small sample sizes. The aim of simulation is to assess the coverage probabilities of the confidence intervals based on r, r^* and exact F statistic. Moreover, the probability of type I error and powers of these three tests are also investigated.

The estimated coverage probabilities of confidence intervals produced by r, r^* and F are given in Table 3.1. In Table 3.1, the sample size (m, n) is assumed to $(3, 3)$, $(5, 5)$, $(5, 7)$, $(7, 5)$, $(10, 10)$, $(30, 20)$, $(20, 30)$ and $(40, 40)$. This design of sample sizes has intention to know the performance of proposed statistics when the sample size is small or large. The value of parameter of interest is assumed to $\theta_1 = 0.25, 1, 4, 9$. For each of possible combinations of sample size and parameter values, 10,000 random samples from two independent normal distributions were generated to calculate estimated coverage probabilities of 90% and 95% confidence interval for θ_1 with equal tail probabilities based on r, r^* and F.

When the sample size is large $(m, n) = (40, 40)$, given nominal levels $0.025, 0.05, 0.95$ and 0.975, the estimated coverage probabilities of r, r^* and F are close to nominal levels. But when the sample size is small, the estimated coverage probabilities based on r do not achieve nominal levels. This means r is inaccurate when the sample size is small. This phenomenon is continued until the sample size reaches to (30, 30). In contrast, the coverage probabilities of F and r^* are close to nominal level even in small sample sizes such as $(3, 3)$ or $(5, 5)$. In some cases, the coverage probabilities of r^* is closer to nominal probabilities than that of the exact F statistic. These results do not depend on the assumed values of θ_1 or nuisance parameters.

About the null hypothesis $H_0 : \theta_1 = 1$, the probability of the type I error and the power of the test based on F, r and r^* are given in Table 3.2. In this table, the significance level $\alpha = 0.05$. When the value of parameter of interest differs from 1 or the sample size become large, the power of the test increases constantly.

The power of r is the largest of them all. But the probability of the type I error of r when the sample size is small is serious. Until the sample size increases from $(3, 3)$ to $(30, 20)$, the probability of type I error of r is too big. Specially, when $(m, n) = (3, 3), (5, 5), (5, 7), (7, 5)$, the probability of type I error is almost two times of significance level.

But the behaviors of type I error and power based on F and r^* are almost similar. The probability of type I error of r^*, when the sample size $(m, n) = (3, 3)$ or $(5, 5)$, differs only two decimal points below from given significance level. Conclusively, one can use r^* instead of F even in small sample size.

The real data example is a bioavailability study of parallel-group experiment of 20 subjects to compare a new test formulation $(m = 10)$ with a reference formulation $(n = 10)$ of a drug product. This data is called as $C_{\max}$ data and was analyzed by Wu *et al.* (2002). They assumed the distribution of the data as a lognormal distribution. The $C_{\max}$ data is given below.

New	732.89	1371.97	614.62	557.24	821.39	363.94	430.95	401.42	436.16	951.46
Reference	1053.63	1351.54	197.95	1204.72	447.20	3357.66	567.36	668.48	842.19	284.86

For the above data, the Shapiro-Wilk tests for the nomality on the log-transformed data give a p-value of 0.595 for the "New" group and a p-value of 0.983 for "Reference" group.

The exact F test for equality of variances of the log-transformed data between two groups results in acception of $H_0 : \theta_1 = 1$ with a p-value of 0.068. The signed log-likelihood ratio statistic for the equality of two variances is $r = 1.943439$ and the p-value is 0.052. So there is no evidence for rejection of H_0. The modified log-likelihood ratio statistic is $r^* = 1.716538$ and the p-value is 0.086. So, the three tests give a same result for testing $H_0 : \theta_1 = 1$.

Table 3.1 The estimated coverage probabilities of F, r^* and r

		0.025			0.05			0.95			0.975		
m	n	F	r^*	r	F	r^*	r	F	r^*	r	F	r^*	r
					$\mu_1 = 1, \mu_2 = 2, \sigma_1 = 1, \sigma_2 = 1, \theta_1 = 1$								
3	3	.0235	.0286	.0717	.0478	.0529	.1138	.9523	.9459	.8872	.9754	.9726	.9279
5	5	.0237	.0232	.0461	.0468	.0484	.0804	.9532	.9511	.9210	.9782	.9749	.9542
5	7	.0220	.0231	.0318	.0464	.0476	.0577	.9513	.9505	.9067	.9767	.9763	.9446
7	5	.0269	.0258	.0556	.0494	.0509	.0951	.9517	.9516	.9417	.9757	.9732	.9664
10	10	.0240	.0255	.0350	.0464	.0506	.0640	.9495	.9534	.9377	.9748	.9758	.9671
30	20	.0237	.0256	.0346	.0482	.0497	.0628	.9505	.9519	.9539	.9746	.9764	.9768
20	30	.0253	.0259	.0258	.0498	.0489	.0477	.9509	.9500	.9349	.9740	.9745	.9664
40	40	.0237	.0266	.0287	.0461	.0500	.0546	.9500	.9539	.9514	.9734	.9763	.9735
					$\mu_1 = 1, \mu_2 = 2, \sigma_1 = 1, \sigma_2 = 2, \theta_1 = 4$								
3	3	.0266	.0271	.0718	.0507	.0520	.1110	.9535	.9432	.8861	.9766	.9691	.9236
5	5	.0248	.0261	.0489	.0505	.0511	.0808	.9504	.9480	.9139	.9753	.9739	.9502
5	7	.0269	.0272	.0354	.0515	.0502	.0604	.9487	.9445	.9015	.9726	.9708	.9400
7	5	.0222	.0295	.0588	.0498	.0538	.0977	.9491	.9511	.9395	.9717	.9781	.9690
10	10	.0235	.0232	.0317	.0459	.0463	.0597	.9541	.9540	.9409	.9771	.9764	.9668
30	20	.0244	.0243	.0338	.0482	.0499	.0632	.9505	.9521	.9537	.9758	.9756	.9761
20	30	.0246	.0253	.0247	.0497	.0497	.0484	.9501	.9499	.9370	.9747	.9753	.9665
40	40	.0247	.0237	.0256	.0505	.0504	.0538	.9497	.9495	.9457	.9764	.9753	.9729
					$\mu_1 = 1, \mu_2 = 2, \sigma_1 = 1, \sigma_2 = 3, \theta_1 = 9$								
3	3	.0284	.0271	.0701	.0532	.0518	.1078	.9545	.9419	.8782	.9764	.9682	.9214
5	5	.0260	.0258	.0498	.0526	.0530	.0841	.9493	.9463	.9153	.9755	.9724	.9484
5	7	.0234	.0236	.0315	.0472	.0471	.0556	.9519	.9493	.9078	.9761	.9753	.9444
7	5	.0246	.0260	.0544	.0517	.0501	.0933	.9525	.9486	.9390	.9767	.9756	.9665
10	10	.0273	.0236	.0341	.0542	.0481	.0627	.9524	.9457	.9320	.9767	.9724	.9611
30	20	.0258	.0253	.0342	.0531	.0498	.0654	.9502	.9471	.9498	.9748	.9742	.9742
20	30	.0246	.0249	.0245	.0492	.0499	.0484	.9501	.9507	.9356	.9751	.9753	.9674
40	40	.0262	.0228	.0245	.0474	.0463	.0495	.9537	.9526	.9503	.9772	.9738	.9722
					$\mu_1 = 1, \mu_2 = 2, \sigma_1 = 2, \sigma_2 = 1, \theta_1 = 0.25$								
3	3	.0273	.0290	.0730	.0540	.0522	.1146	.9529	.9407	.8860	.9754	.9685	.9215
5	5	.0270	.0275	.0498	.0535	.0534	.0863	.9486	.9441	.9139	.9739	.9719	.9473
5	7	.0267	.0230	.0313	.0512	.0474	.0559	.9513	.9455	.9021	.9764	.9710	.9387
7	5	.0241	.0299	.0609	.0496	.0554	.0993	.9481	.9509	.9388	.9735	.9761	.9682
10	10	.0238	.0250	.0354	.0487	.0494	.0629	.9509	.9509	.9363	.9752	.9757	.9650
30	20	.0237	.0230	.0320	.0495	.0470	.0605	.9536	.9507	.9523	.9771	.9764	.9769
20	30	.0232	.0259	.0256	.0463	.0513	.0499	.9484	.9534	.9368	.9739	.9767	.9701
40	40	.0232	.0233	.0252	.0473	.0477	.0504	.9524	.9527	.9496	.9768	.9768	.9747

Table 3.2 Type I error and the power of test when $H_0 : \theta_1 = 1$

m	n	F	r^*	r	F	r^*	r	F	r^*	r
			$\theta_1 = 1$			$\theta_1 = 0.5$			$\theta_1 = 2$	
3	3	.0497	.0574	.1482	.0603	.0698	.1776	.0656	.0756	.1785
5	5	.0539	.0569	.1033	.0896	.0937	.1615	.0879	.0906	.1523
5	7	.0487	.0515	.0965	.0869	.0932	.1750	.1094	.1087	.1464
7	5	.0495	.0520	.0926	.1045	.1040	.1402	.0821	.0874	.1723
10	10	.0512	.0517	.0722	.1642	.1658	.2076	.1561	.1570	.2021
30	20	.0487	.0487	.0563	.3783	.3776	.3749	.3401	.3413	.3961
20	30	.0497	.0502	.0583	.3412	.3424	.3952	.3848	.3841	.3805
40	40	.0495	.0496	.0534	.5650	.5650	.5807	.5698	.5701	.5840
			$\theta_1 = 5$			$\theta_1 = 7$			$\theta_1 = 9$	
3	3	.1171	.1313	.3011	.1565	.1779	.3755	.1891	.2148	.4352
5	5	.2705	.2789	.4053	.3928	.4025	.5390	.4709	.4811	.6178
5	7	.3809	.3788	.4336	.5210	.5189	.5703	.6251	.6232	.6724
7	5	.2875	.3031	.4757	.4133	.4289	.6132	.5086	.5283	.7068
10	10	.6243	.6262	.6889	.7924	.7936	.8363	.8752	.8762	.9071
30	20	.9634	.9638	.9737	.9948	.9949	.9966	.9990	.9990	.9993
20	30	.9639	.9639	.9633	.9936	.9936	.9934	.9988	.9988	.9988
40	40	.9987	.9987	.9988	.9999	.9999	.9999	1.0000	1.0000	1.0000

4. Conclusions

Two small sample likelihood based inference methods for testing equality of two independent normal variances are proposed. When the parameter of interest is the ratio of two variances, the signed log-likelihood ratio statistic, r, and the modified signed log-likelihood ratio statistic, r^*, are developed.

The estimated coverage probabilities of conficence intervals, type I error and power of testing $H_0 : \theta_1 = 1$ based on r, r^* and F statistic are obtained. Simulation results show that the modified signed log-likelihood ratio statistic gives exact coverage probability and is almost an exact test even for small sample. As a real data example, C_{max} data is analyzed using the proposed test statistics.

Conclusively, the modified signed log-likelihood ratio statistic is comparable to the F statistic. This fact suggests that normal approximation like r^* is a good alternative instead of using F. Using quantile of standard normal distribution, one can perform the test of equality of two normal variances without resorting F test.

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