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A Functional Central Limit Theorem for an ARMA(p, q) Process with Markov Switching

Oesook Lee[1,a]

[a]Department of Statistics, Ewha Womans University

Abstract

In this paper, we give a tractable sufficient condition for functional central limit theorem to hold in Markov switching ARMA (p, q) model.

Keywords: Functional central limit theorem, Markov switching ARMA (p, q) process, φ-mixing.

1. Introduction

Since the seminal work of Hamilton (1989), nonlinear time series models subject to Markov switching are widely used for modelling dynamics in different fields, especially in econometric studies (Krolzig, 1997; Hamilton and Raj, 2002). Markov switching autoregressive moving average(MSARMA)(p, q) model, in which a hidden Markov process governs the behavior of an observable time series, exhibits structural breaks and local linearity. When we consider a time series model as a data generating process, one of the important properties to show is the (functional) central limit theorem. Functional central limit theorem(FCLT) is applied for statistical inference in time series to establish the asymptotics of various statistics concerning, for example a test for stability such as CUCUM or MOSUM and unit root testing. Probabilistic properties of MSARMA(p, q) models have been studied in, *e.g.*, Francq and Zakoïan (2001), Yao and Attali (2000), Yang (2000), Lee (2005), and Stelzer (2009).

The purpose of this paper is to find a sufficient condition under which FCLT holds for the partial sums processes of the given MSARMA(p, q) models. The typical approach to obtaining the FCLT for nonlinear time series models is to show that a specific dependence property such as various mixing condition, L_p-NED(near-epoch dependent), association or $\theta, \mathcal{L}$ or ψ-weak dependence holds. However, restrictive conditions such as distributional assumptions on errors and higher order moment are required in order to prove such dependence properties (Ango Nze and Doukhan, 2004; De Jong and Davidson, 2000; Davidson, 2002; Dedecker *et al.*, 2007; Doukhan and Wintenberger, 2007; Harrndorf, 1984).

Our proof is based on Theorem 21.1 of Billingsley (1968) that is an extension of the results of Ibragimov (1962). The proof is short and relies on a second moment assumption.

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[1] Professor, Department of Statistics, Ewha Womans University, Ewhayeodaegil 52, 120-750 Seoul, Korea.
E-mail: oslee@ewha.ac.kr

2. Results

We consider the following Markov switching autoregressive moving average (MSARMA)(p, q) model, where ARMA coefficients are allowed to change over time according to a Markov chain:

$$x_t = \sum_{i=1}^{p} \phi_i(u_{t-1})x_{t-i} + \sum_{j=1}^{q} \theta_j(u_{t-1})e_{t-j} + e_t, \quad t \in Z, \tag{2.1}$$

where $p \geq 1, q \geq 0$, $\{u_t\}$ is an irreducible aperiodic Markov chain on a finite state space E with n-step transition probability matrix $P^{(n)} = (p^{(n)}_{u,v})_{u,v\in E}$ and the stationary distribution π, $\phi_i(u)$ and $\theta_j(u)(i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, q,\ u \in E)$ are constants and $\{e_t\}$ is a sequence of independent and identically distributed random variables with mean 0 and finite variance σ^2. We assume that $\{u_t\}$ and $\{e_t\}$ are independent.

Assuming without loss of generality that $p = q$ and letting $X_t = (x_t, x_{t-1}, \ldots, x_{t-p+1})'$ and $\varepsilon_t = (e_t, e_{t-1}, \ldots, e_{t-p+1})'$, define $p \times p$ matrix

$$\Phi_t = \Phi(u_t) = \begin{bmatrix} \phi_1(u_t) & \phi_2(u_t) & \cdots & \phi_{p-1}(u_t) & \phi_p(u_t) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and

$$\Theta_t = \Theta(u_t) = \begin{bmatrix} \theta_1(u_t) & \theta_2(u_t) & \cdots & \theta_{p-1}(u_t) & \theta_p(u_t) \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Then

$$X_t = \Phi_{t-1}X_{t-1} + \Theta_{t-1}\varepsilon_{t-1} + \varepsilon_t \tag{2.2}$$

and $x_t = c'X_t$, $c' = (1, 0, \ldots, 0)$.

Applying recursively the Equation (2.2), after m-step we have that

$$X_t = \Pi_{j=1}^{m}\Phi_{t-j}X_{t-m} + \sum_{k=1}^{m-1} \Pi_{j=1}^{k-1}\Phi_{t-j}(\Phi_{t-k} + \Theta_{t-k})\varepsilon_{t-k} + \Pi_{j=1}^{m-1}\Phi_{t-j}\Theta_{t-m}\varepsilon_{t-m} + \varepsilon_t \tag{2.3}$$

(assume $\Pi_{j=1}^{0}\Phi_{t-j} = I$).

We make the assumptions.

(A1) $\rho := \sum_{i=1}^{p} \alpha_i < 1$ with $\alpha_i = \sup_{u\in E} \left(E(\phi_i^2(u_1)|u_0 = u)\right)^{\frac{1}{2}}$.

Note that the assumption (A1) depends only on the autoregressive coefficients $\phi_i(u_t)$ of the Equation (2.1). Next theorem is for the existence of a strictly stationary solution to the Equation (2.1).

Theorem 1. *Suppose the assumption (A1) holds. Then there is a unique non-anticipative strictly stationary solution x_t to (2.1) with $E(x_t^2) < \infty$.*

Proof: Define

$$X_t^* = \sum_{k=1}^{\infty} \left(\Pi_{j=1}^{k-1} \Phi_{t-j}\right) (\Phi_{t-k} + \Theta_{t-k})\, \varepsilon_{t-k} + \varepsilon_t. \tag{2.4}$$

Then according to Minkowski inequality, we have that

$$\left(E(c' X_t^*)^2\right)^{\frac{1}{2}} = \left\|c' X_t^*\right\|_2 \le \sum_{k=1}^{\infty} \left\|c' \Pi_{j=1}^{k-1} \Phi_{t-j} (\Phi_{t-k} + \Theta_{t-k}) \varepsilon_{t-k}\right\|_2 + \sigma^2.$$

Let $\beta_i := (E_\pi(\phi_i(u_t) + \theta_i(u_t))^2)^{1/2} < \infty$. Apply the independence of $\{u_t\}$ and $\{e_t\}$ and Minkowski inequality and use the assumption (A1) to obtain, after some tedious calculations, that

$$\left\|c' \Pi_{j=1}^{k-1} \Phi_{t-j}\, (\Phi_{t-k} + \Theta_{t-k})\, \varepsilon_{t-k}\right\|_2 \le C(k-1), \quad k = 1, 2, 3, \ldots,$$

where

$$\begin{aligned} C := C(0) &= \sigma\left(\sum_{i=1}^{p} \beta_i\right), \\ C(1) &= \alpha_1 C \le \rho C, \\ C(2) &= \left(\alpha_1^2 + \alpha_2\right) \le \rho C, \\ &\vdots \\ C(p) &= \alpha_1 C(p-1) + \alpha_2 C(p-2) + \cdots + \alpha_p C \le \rho C. \end{aligned}$$

Moreover,

$$\begin{aligned} C(p+1) &= \alpha_1 C(p) + \alpha_2 C(p-1) + \cdots + \alpha_p C(1) \le \rho^2 C, \\ C(p+2) &= \alpha_1 C(p+1) + \alpha_2 C(p) + \cdots + \alpha_p C(2) \le \rho^2 C, \\ &\vdots \end{aligned}$$

In general, for $1 \le r \le p$, $m \ge 0$,

$$C(mp + r) = \alpha_1 C(mp + r - 1) + \alpha_2 C(mp + r - 2) + \cdots + \alpha_p C((m-1)p + r) \le \rho^{m+1} C.$$

Therefore,

$$\sum_{k=1}^{\infty} \left\|c' \Pi_{j=1}^{k-1} \Phi_{t-j} (\Phi_{t-k} + \Theta_{t-k}) \varepsilon_{t-k}\right\|_2 \le C \sum_{k=1}^{\infty} \rho^{\left[\frac{(k-1)}{p}\right]+1} < \infty. \tag{2.5}$$

Hence we have that $E(c' X_t^*)^2 < \infty$ and $x_t^* = c' X_t^* < \infty$ a.e. One can easily verify that X_t^* in (2.4) is a strictly stationary sequence satisfying (2.2).

To prove the uniqueness, assume that $\hat{X}_t$ is any non-anticipative, strictly stationary solution to (2.2) with $E(c'\hat{X}_t)^2 < \infty$. Obviously $\hat{X}_t$ satisfies (2.3) for every $m \geq 1$ and thus we obtain that $X_t^* - \hat{X}_t = (\Pi_{j=1}^m \Phi_{t-j})(X_{t-m}^* - \hat{X}_{t-m})$.

$$\begin{aligned}
E\left|x_t^* - \hat{x}_t\right| &= E\left|c'\left(X_t^* - \hat{X}_t\right)\right| \\
&= E\left|c'\Pi_{j=1}^m \Phi_{t-j}\left(X_{t-m}^* - \hat{X}_{t-m}\right)\right| \\
&\leq \sum_{i=1}^{p} E\left|\left(x_{t-m-i+1}^* - \hat{x}_{t-m-i+1}\right) c'\left(\Pi_{j=1}^m \Phi_{t-j}\right) \nu_i\right| \\
&\leq \sum_{i=1}^{p} \left\|x_{t-m-i+1}^* - \hat{x}_{t-m-i+1}\right\|_2 \left\|c'\Pi_{j=1}^m \Phi_{t-j}\nu_i\right\|_2,
\end{aligned}$$

where $\nu_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, 0 except the i^{th} entry which is 1. Adopt the similar method used to derive the inequality (2.5), then we have that $\|c'\Pi_{j=1}^m \Phi_{t-j}\nu_i\|_2 \leq M\rho^{[(m-1)/p]} \to 0$ as $m \to 0$, where $M = \sum_{i=1}^p E_\pi|\phi_i(u_t)| < \infty$. Thus we have that $E|x_t^* - \hat{x}_t| = 0$, and $x_t^* = \hat{x}_t$ a.e. Take $x_t = x_t^*$ to get the conclusion. □

A stationary process $(x_t)_{t\in Z}$ is called φ- mixing if

$$\varphi_n = \sup\left\{|P(E_2|E_1) - P(E_2)| : E_1 \in \mathcal{F}_{-\infty}^k, E_2 \in \mathcal{F}_{k+n}^\infty\right\} \to 0$$

as $n \to \infty$, where for $s < t$, $\mathcal{F}_s^t := \sigma(x_s, x_{s+1}, \ldots, x_t)$.

Let $[x]$ denote the largest integer not exceeding x and let " $\xrightarrow{\mathcal{D}}$ " denote convergence in distribution. Let B denote standard Brownian motion on $[0, 1]$. Following theorem is our main result.

Theorem 2. *Suppose the assumption (A1) holds. Then for a strictly stationary solution x_t of (2.1) with $E(x_t^2) < \infty$, $\tau^2 = Var(x_0) + 2\sum_{1\leq k<\infty} Cov(x_0, x_t)$ is convergent and as $n \to \infty$,*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{[n\xi]} x_t \xrightarrow{\mathcal{D}} \tau B(\xi), \quad 0 \leq \xi \leq 1.$$

Proof: Note that $v_t := (u_{t-1}, e_t, e_{t-1}, \ldots, e_{t-p})$ is an aperiodic $E \times R^{p+1}$-valued Markov chain. Since $\{u_t\}$ is a positive recurrent Markov chain with finite state space, $\{u_t\}$ is exponentially φ-mixing. By independence of $\{u_t\}$ and $\{e_t\}$, $\{v_t\}$ is also exponentially $\varphi-$mixing with $\sum \varphi_n^{1/2} < \infty$. By the above Theorem 1,

$$x_t = c' \sum_{k=1}^{\infty} \left(\Pi_{j=1}^{k-1} \Phi_{t-j}\right) (\Phi_{t-k} + \Theta_{t-k})\, \varepsilon_{t-k} + c'\varepsilon_t,$$

and take

$$x_{t,l} = c' \sum_{k=1}^{l} \left(\Pi_{j=1}^{k-1} \Phi_{t-j}\right) (\Phi_{t-k} + \Theta_{t-k})\, \varepsilon_{t-k} + c'\varepsilon_t.$$

$\{x_{t,l}\}$ is stationary for each l. For some measurable functions f and f_l, we can write

$$x_t = f(v_t, v_{t-1}, v_{t-2}, \dots),$$

and

$$x_{t,l} = f_l(v_t, v_{t-1}, v_{t-2}, \dots, v_{t-l+1}).$$

Then we have that

$$\begin{aligned}\left\|x_t - x_{t,l}\right\|_2 &= \left\|c' \sum_{k=l+1}^{\infty} \left(\Pi_{j=1}^{k-1}\Phi_{t-j}\right)(\Phi_{t-k} + \Theta_{t-k})\,\varepsilon_{t-k}\right\|_2 \\ &\le \sum_{k=l+1}^{\infty} \left\|c' \left(\Pi_{j=1}^{k-1}\Phi_{t-j}\right)(\Phi_{t-k} + \Theta_{t-k})\,\varepsilon_{t-k}\right\|_2 \\ &\le C \sum_{k=l+1}^{\infty} \rho^{\left[\frac{(k-1)}{p}\right]+1}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{l=1}^{\infty} \left\|x_t - x_{t,l}\right\|_2 &\le C \sum_{l=1}^{\infty} \sum_{k=l+1}^{\infty} \rho^{\left[\frac{(k-1)}{p}\right]+1} \\ &\le Cp^2 \sum_{l=1}^{\infty} \sum_{i=0}^{\infty} \rho^{i+l} \\ &= Cp^2 \frac{\rho}{(1-\rho)^2} \\ &< \infty.\end{aligned}$$

Applying Theorem 21.1 in Billingsley (1968) yields the conclusion. □

Remark 1. For x_t in (2.1), define a process $Y_t = A(u_t)Y_{t-1} + \eta_t$, with $Y_t = (x_{t-1}, \dots, x_{t-p}, e_t, \dots, e_{t-q+1})$ for properly defined $(p+q) \times (p+q)$ matrix $A(u_t)$ and $(p+q) \times 1$ vector η_t. If we take $W_t = (u_t, Y_t')'$, then W_t is a Markov chain. One way to prove the FCLT is to use the Markovian structure of the model to obtain mixing properties. If W_t is ϕ-irreducible weak Feller and top Lyapunov exponent of $A(u_t)$ is negative, then the process Y_t is geometrically ergodic and β-mixing and the FCLT for x_t is obtained (Lee, 2005). The stationarity and geometric ergodicity for vector valued MSARMA(p, q) process with a general state space parameter chain are examined by Stelzer (2009).

Remark 2. It is known that if the largest modulus of the eigenvalues λ of the set $\Phi(u)$, $u \in E$ is less than 1, then the FCLT holds for x_t (Davidson, 2002, p.256).

Following is a simple example, which shows that $\lambda > 1$, but the FCLT holds due to Theorem 2.

Example 1. Consider a MSARMA$(3, 2)$ process given by

$$x_t = \sum_{i=1}^{3} \phi_i(u_{t-1})x_{t-i} + \sum_{j=1}^{2} \theta_j(u_{t-1})e_{t-j} + e_t, \tag{2.6}$$

where $\{u_t\}$ is a Markov chain with state space $E = \{1, 2, 3\}$ and one step transition probability matrix P is given by

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.25 & 0.3 & 0.45 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}.$$

Take

$$\begin{aligned} (\phi_1(1), \phi_2(1), \phi_3(1)) &= (0.1, 0.3, 0.01), \\ (\phi_1(2), \phi_2(2), \phi_3(2)) &= (1.1, 0.2, 0.2), \\ (\phi_1(3), \phi_2(3), \phi_3(3)) &= (0.2, 0.03, 0.2), \end{aligned}$$

and

$$\Phi(u) = \begin{bmatrix} \phi_1(u) & \phi_2(u) & \phi_3(u) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad u \in E.$$

For the above example, the largest modulus of the eigenvalues λ of the set $\Phi(1)$, $\Phi(2)$ and $\Phi(3)$ is larger than 1 but $\sum_{i=1}^{3} \sup_{u\in E}(E(\phi_i^2(u_1)|u_0 = u))^{1/2} < 1$. Hence Theorem 1 and 2 guarantee the existence of a strictly stationary solution x_t of (2.6) and the FCLT for $\{x_t\}$.

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